Global and Local Refinement Techniques Yielding Nonobtuse Tetrahedral Partitions

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Abstract—Preservation of basic qualitative properties (for example, the maximum principle) of the solution of partial differential equations by its finite-element approximations is an important goal in mathematical modelling and simulation. Nonobtuse tetrahedral partitions and linear finite elements guarantee the validity of the discrete analogues of the maximum principle for a wide class of parabolic and elliptic problems. In order to get more accurate approximation, we often need to refine the used partitions globally or locally. In this paper, we first propose two variants of global refinement techniques, which produce nonobtuse face-to-face tetrahedral partitions. Second, we present a new local refinement technique which generates nonobtuse face-to-face tetrahedral partitions in a neighbourhood of a given vertex. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Besides an obligatory requirement of the convergence of finite-element approximations to the exact solution of a mathematical model, the finite-element approximations are naturally required to mirror the most basic qualitative properties of the exact solution in order to be reliable and useful in numerical simulation. Mathematical models, described by second order elliptic equations, e.g., with Dirichlet boundary conditions obey the maximum principle; the models described by parabolic equations with initial and boundary conditions satisfy the maximum principle, non-negativity property, monotonicity, etc., see [1–3]. The corresponding discrete analogues of the above properties with computational schemes producing approximations satisfying these discrete

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analyses, are formulated, e.g., in [1,4–6]. Already in the first papers devoted to the preservation of basis qualitative properties in numerical approximations by the finite-element method, it was noticed that such a preservation essentially depends on geometric properties of the discretization of the solution domain [4,7,8]. In particular, the formulations of the discrete maximum principle for the problems of elliptic and parabolic types were shown to be valid if the triangular or tetrahedral partitions used are of the nonobtuse type, it means that all the triangles and tetrahedra have no obtuse interior angles.

Obviously, the first (rough) finite-element approximations cannot be sufficiently close to the the exact solution, in general, and therefore more exact approximations, on more refined finite-element partitions, are needed. Thus, a natural question arises—how to refine the initial partition satisfying the required geometric conditions so that the next (globally or locally refined) partition has the same properties. It is relatively simple to do that in 2D, but much more difficult in 3D.

Throughout the paper, we deal with standard tetrahedral “face-to-face” partitions of a given polyhedral domain \( \Omega \) (see, e.g., [9]), which everywhere will be simply called only partitions.

A tetrahedron is said to be nonobtuse, if all its six dihedral angles between faces are nonobtuse. A partition is said to be nonobtuse, if it only consists of nonobtuse tetrahedra. Such partitions play an important role in the finite-element analysis for boundary value problems, since they yield irreducible and diagonally dominant stiffness matrices and guarantee the validity of the discrete maximum principle (see [4,6,10–12]), when using finite elements to solve linear and nonlinear second order elliptic boundary value problems. In particular, for the Poisson equation

\[
-\Delta u = f,
\]

with the homogeneous Dirichlet boundary conditions, the discrete maximum principle states (cf. [14])

\[
f \leq 0 \implies \max_{x \in \Omega} u_h(x) = 0, \tag{1}
\]

where \( u_h \) is a piecewise linear Galerkin approximation of \( u \). This principle for parabolic problems has the same form provided the initial conditions are nonnegative (see [8]).

In [11], we prove that the discrete maximum principle (1) is still valid if some of the dihedral angles of tetrahedral elements are slightly bigger than \( \pi/2 \). In this case, certain off-diagonal entries of the stiffness matrix might become positive, but the matrix remains monotone.

When using the trilinear block elements then, similarly, some off-diagonal entries of the stiffness matrix may be positive (see [15, p. 68]). In this case, the discrete maximum principle (1) may be violated and the numerical flux may have a converse direction than the continuous flux, or we may get a negative concentration, mass, pressure, etc. On the other hand, for nonobtuse tetrahedral partitions such a pathological situation cannot happen, since (1) is valid.
In Figure 1, the standard 3D red refinement technique (see [16,17]) applied to a tetrahedron is presented. This technique, which uses midlines of faces, partitiones the tetrahedron into eight smaller subtetrahedra. The four “exterior” subtetrahedra are similar to the original tetrahedron, whereas the remaining four “interior” subtetrahedra are not, in general. The four interior subtetrahedra share a common edge, which means that their four dihedral angles, related to this common edge, are either right (which is very unprobable in a general case), or at least one of them is obtuse. This implies that the 3D red refinement technique cannot be generally recommended to get nonobtuse partition refinements (except for some special cases—see, e.g., Lemma 2.3). Instead, in the next section, we present the so-called 3D yellow refinement technique [2], which always produces nonobtuse tetrahedra.

We note that in 2D case, the situation with the red-type refinement is much simpler, since the red refinement (by midlines) of a nonobtuse triangle obviously gives four congruent nonobtuse (sub)triangles similar to the original one.

In practical problems, we have to generate local refinements near those points, where singularities or oscillations of the solution of initial-boundary value problems occur. These points are usually situated at vertices of the domain in question. Locally refined partitions enable us to reduce the global discretization error, which can be evaluated by various a posteriori error estimation techniques, see [18].

In Figure 2, we see a local refinement technique proposed by Guo (see [19]) for a refinement near a vertex of the domain. We observe that there are always interior faces that bisect the angle $\pi$ such that one angle is acute and the other one is obtuse. Thus, Guo’s local refinement technique never produces only nonobtuse tetrahedra.

The other local refinement techniques, using bisection [20,21], or Delaunay triangulations [22], have the same disadvantage.

![Figure 2. B. Guo's local refinement technique.](image)

2. GLOBAL NONOBTUSE PARTITION REFINEMENT TECHNIQUES

A tetrahedron is said to be a path tetrahedron if its three edges, which do not meet at the same vertex, are mutually orthogonal. In [2, p. 728], we prove that such a tetrahedron is nonobtuse. For instance, Figure 3 shows a path tetrahedron $ABCD$ whose edges $AB$, $BC$, and $CD$ are mutually orthogonal.

A set $\mathcal{F}$ of partitions of a polyhedron $\bar{\Omega}$ is called a family of partitions of $\bar{\Omega}$, if for every $\varepsilon > 0$ there exists a partition from $\mathcal{F}$ whose norm is less than $\varepsilon$.

**Theorem 2.1.** Let $T$ be an arbitrary nonobtuse tetrahedron such that its circumcentre belongs to $T$. Then there exists a family of tetrahedral partitions of $T$ containing only path tetrahedra.
A

Figure 3. Partition of a path tetrahedron $ABCD$ into three path tetrahedra.

For the proof see [2]. Theorem 2.1 can be slightly generalized as follows.

**Theorem 2.2.** Let $T$ be an arbitrary tetrahedron such that its circumcentre belongs to $T$, and let the faces of $T$ be nonobtuse triangles. Then there exists a family of tetrahedral partitions of $T$ containing only path tetrahedra.

For the proof see [23]. It is easy to show (cf. [14, p. 66]) that the tetrahedron $T$ from Theorem 2.1 satisfies the assumptions of Theorem 2.2. On the other hand, the tetrahedron with vertices $A = (-2, 0, 0)$, $B = (2, 0, 0)$, $C = (0, -2, 1)$, and $D = (0, 2, 1)$ satisfies the assumptions of Theorem 2.2, even though the dihedral angle at the edge $AB$ is obtuse (it is greater than $126^\circ$). Therefore, Theorem 2.1 cannot be employed in this case.

We shall now briefly describe a construction of nonobtuse partitions of a tetrahedron $T$ guaranteed by the previous theorems. First, we divide each face $F$ of $T$ into four or six right triangles whose common vertex $Z$ is the circumcentre of $F$ (see Figure 4). This kind of refinement in 2D is called yellow (see [2]). If the circumcentre of $T$ (let us denote it by $G$) lies in $T$ then we can define the path subtetrahedra as the convex hull of $G$ and particular right subtriangles contained in the faces of $T$ (see Figure 5). In this way, we can decompose the tetrahedron $T$ into 24, 22, 20, 18, 16, 14, 12, ten, or eight path subtetrahedra. Such a 3D partition is called yellow.

Each of the path subtetrahedra can be further divided into exactly eight smaller path tetrahedra, since its faces are the right triangles (see Figure 5b, where the thick emphasized lines indicate the edges of the subtetrahedra lying on the three mutually perpendicular edges of the original tetrahedron).

Figure 4. 2D yellow refinements.
Let now the initial partition of $\Omega$ be such that each of its tetrahedra satisfies the assumptions of Theorem 2.2. Since faces of adjacent tetrahedra are divided in the same manner (see Figure 4), we obviously obtain face-to-face partitions.

Another way to obtain global nonobtuse partition refinements is to apply the 3D red refinement technique, after we have previously obtained a partition into path tetrahedra. This observation follows from the following lemma.

**Lemma 2.3.** Let $ABCD$ be an arbitrary path tetrahedron with $AB \perp BC \perp CD \perp AB$. Consider the 3D red refinement of $ABCD$ such that the four interior subtetrahedra share a common edge $MN$, where $M$ is the midpoint of $AC$ and $N$ is the midpoint of $BD$. Then all eight obtained subtetrahedra are path.

**Proof.** Without loss of generality we may assume that $A = (0, 0, a), B = (0, 0, 0), C = (c, 0, 0),$ and $D = (c, d, 0)$ for arbitrary real positive numbers $a, c,$ and $d$ (cf. Figure 3). Then $M = (c/2, 0, a/2)$ and $N = (c/2, d/2, 0)$.

The four exterior subtetrahedra containing the vertices $A, B, C,$ and $D$ are obviously path, since each face of $ABCD$ is divided by midlines, i.e., these subtetrahedra are similar to the original tetrahedron $ABCD$.

Consider now the interior subtetrahedron $KLMN$, where $K = (c/2, 0, 0)$ and $L = (c, d/2, 0)$ are the midpoints of $BC$ and $CD$, respectively. We observe that

$$MK \perp KN \perp NL \perp MK,$$

and therefore, $KLMN$ is a path tetrahedron and its dihedral angle at the interior edge $MN$ is right. The remaining three "interior" tetrahedra can be treated similarly.

**Remark 2.4.** After the first 3D yellow refinement step, we can choose whether for the next refinement step we still apply the 3D yellow refinement or the 3D red refinement (keeping in mind a possible degeneracy discovered by Zhang in [17]). In [2], we used only yellow refinement technique. However, the above Lemma 2.3 enables us to alternate the use of the yellow and red refinement techniques.

### 3. LOCAL NONOBTUSE PARTITION REFINEMENT TECHNIQUE

The main idea of a local nonobtuse tetrahedral partition refinement technique is exposed in the following theorem, whose proof is constructive.

**Theorem 3.1.** Let $ABCD$ be a path tetrahedron whose edges $AB, BC,$ and $CD$ are mutually orthogonal. Then there exists an infinite family of nonobtuse partitions of it into path tetrahedra that locally refine $ABCD$ in a neighbourhood of the vertex $A$. 
PROOF. Let $P$ be the orthogonal projection of the point $B$ onto the line $AC$. Obviously, $P$ lies in the interior of the line segment $AC$, since $ABC$ is the right triangle.

Further, let $Q$ be the orthogonal projection of the point $P$ onto the line $AD$. Since $ACD$ is a right triangle, $APD$ is an obtuse triangle, and thus the point $Q$ lies in the interior of the segment $AD$.

We observe that the line segment $BP$ is perpendicular to the face $ACD$. Therefore, $BP$ is perpendicular to any line which is contained in the plane $ACD$. From this property we easily find that the original tetrahedron $ABCD$ can be decomposed into the following three path tetrahedra (see Figure 3):

$$
BPCD \text{ with } BP \perp PC \perp CD \perp BP,
$$

$$
BPQD \text{ with } BP \perp PQ \perp QD \perp BP,
$$

$$
AQPB \text{ with } AQ \perp QP \perp PB \perp AQ.
$$

![Figure 6. Partition of a path tetrahedron $ABQP$ into three path tetrahedra.](image)

Now we decompose the last path subtetrahedron $AQPB$ into three path subtetrahedra following the same rules as above. Let $S$ be the orthogonal projection of the point $Q$ onto the line $AP$, and let $T$ be the orthogonal projection of the point $S$ onto the line $AB$. Then the path tetrahedron $AQPB$ can be decomposed into the following three path subtetrahedra (see Figure 6):

$$
QSPB \text{ with } QS \perp SP \perp PB \perp QS,
$$

$$
QSTB \text{ with } QS \perp ST \perp TB \perp QS,
$$

$$
ATSQ \text{ with } AT \perp TS \perp SQ \perp AT.
$$

Consequently, the five path subtetrahedra $BPCD$, $BPQD$, $QSPB$, $QSTB$, and $ATSQ$ form a face-to-face partition of the original path tetrahedron $ABCD$ (see Figure 7).

Since $S$ is the orthogonal projection of $Q$ onto the line $AC$, the line segments $QS$ and $DC$ are parallel. Similarly we find that $TS$ and $BC$ are parallel, since $T$ is the orthogonal projection of $S$ onto the line $AB$. From here we conclude that the face $TSQ$ is parallel to $BCD$, and thus, the path subtetrahedron $ATSQ$ is similar to the original tetrahedron $ABCD$.

The subtetrahedron $ATSQ$ can be now decomposed into five subtetrahedra in a similar way as $ABCD$, and thus we can get further refinement near the point $A$. By this recurrence procedure, we obtain the required infinite family of face-to-face tetrahedral partitions. Each partition from this family is nonobtuse, since according to [2, p. 728-729], each path tetrahedron is nonobtuse.

Further, we give sufficient conditions which enable us to generate local refinements, involving only nonobtuse partitions near some vertex.
Let $T_1, \ldots, T_k$ be tetrahedra from the initial partition which share a given vertex $A$. Moreover, we assume that

(i) $T_1$ is a path tetrahedron,
(ii) each $T_i$ is a mirror image of another tetrahedron $T_j$ with respect to their common triangular face $T_i \cap T_j, i, j \in \{1, \ldots, k\}$.

In Figure 8, we observe two examples of clusters of six path tetrahedra satisfying the above Conditions (i) and (ii).

Now let us consider a family of nonobtuse partitions of $T_1$ whose existence is guaranteed by Theorem 3.1. All adjacent tetrahedra to $T_1$ that share the vertex $A$ are mirror images of $T_1$. Therefore, their refinements will be defined as mirror images of refinements of $T_1$. Similarly, we define refinements of all other tetrahedra. Obviously, this construction preserves the overall face-to-face property of partitions.

Applying the above described construction to a particular case of the tetrahedral partition of the cube presented in Figure 8b, we obtain the local refinement algorithm described by us in [24].

Figure 9 shows nonobtuse tetrahedral refinements of $42 = 7 \times 6$ tetrahedra having a common vertex (each of seven subcubes is divided into 6 tetrahedra). The concave polyhedron in Figure 9 is usually called the Fischera domain.
It is worth to mention that also (strictly) acute triangulations and tetrahedral partitions were used to prove the discrete maximum principle for some computational schemes in parabolic case in [8]. There exist several algorithms (see [25–27]) that produce triangulations of a polygonal domain, which consist only of acute triangles. Therefore, it would be natural to examine the following unsolved 3D problem.

**OPEN PROBLEM.** Does there exist for every polyhedron a partition into tetrahedra all of whose dihedral angles are acute?

**REFERENCES**


