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Note

A Completion of Lu's Determination of the Spectrum for Large Sets of Disjoint Steiner Triple Systems

LUC TEIRLINCK*

Department of Algebra, Combinatorics, and Analysis, Auburn University, Alabama 36849

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Lu [6, 7, 8] proved that large sets of disjoint $S(2, 3, v)$ exist for all $v \equiv 1$ or 3 (mod 6), $v \neq 7$. However, due to the death of the author, [8] remained unfinished and crucial parts of the proof for six values were lost. As Lu's result can be used in many other combinatorial problems, the fact that nobody actually could provide a construction for these values was very annoying. It is the purpose of this paper to complete the proof for these six values. Our method is different from the procedure suggested in [8], but uses the same combinatorial structures, namely $LD(n)$. Our new construction for $LD(n)$ also allows us to substantially shorten Lu's proof.

If S is a set, we denote by $P(S)$ the set of all subsets of S and by $P_k(S)$ the set of all k-subsets of S. A *pairwise balanced design* $S(2, K, v)$, $K \subset \mathbb{N}$, $v \in \mathbb{N}$, is a pair (S, β), where S is a v-set and β is a set of subsets of S, called *blocks*, such that any two distinct elements of S are contained in exactly one block and such that $|B| \in K$ for all $B \in \beta$. If $x, y \in S$, $x \neq y$, we denote the unique block through x and y by xy. We write $S(2, k, v)$ instead of $S(2, \{k\}, v)$. An $S(2, 3, v)$ is called a *Steiner triple system*. A necessary and sufficient condition for the existence of an $S(2, 3, v)$ is $v \equiv 1$ or 3 (mod 6) or $v = 0$ [5]. A *large set of disjoint* $S(2, k, v)$, briefly $LS(2, k, v)$, is a partition of $P_k(S)$ into $S(2, k, v)$. The only known $LS(2, k, v)$ with $3 < k < v$ are the $LS(2, 4, 13)$ constructed by Chouinard [2]. No $LS(2, 3, 7)$ exists [1]. Lu [6, 7, 8] proved that $LS(2, 3, v)$ exist for all $v \equiv 1$ or 3 (mod 6), $v \neq 7$. However, he died before being able to finish [8], so that no complete proof for the cases $v \in \{141, 283, 501, 789, 1501, 2365\}$ is known. A scheme for

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constructing $LS(2, 3, v)$ for these six values is outlined in [8], but, to our knowledge, nobody has been able to fully carry out the described procedure. As Lu's result can be used in many other combinatorial problems, the ambiguous situation for these six values was very annoying. It is the purpose of this paper to provide a complete proof for the six problematic values. Our proof does not completely follow the scheme outlined in [8], but uses the same combinatorial structures, namely $LD(n)$. An obstacle in adapting Lu's proof to other structures or in using it to construct $LS(2, 3, v)$ with additional properties, is the fact that it is very lengthy and complicated. Our new construction for $LD(n)$ allows to substantially shorten Lu's proof, although certain parts of the proof remain complicated.

An *orthogonal array* $OA(2, k, v)$ is a subset $\mathscr{L}$ of S^k, S a v-set, such that for any 2-subset $\{i_1, i_2\}$ of $\{1, ..., k\}$ and for any pair x_1, x_2 of (not necessarily distinct) elements of S, there is exactly one element $(y_1, ..., y_k)$ of $\mathscr{L}$ with $y_{i_1} = x_1$ and $y_{i_2} = x_2$. An $LD(n)$ is a quadruple $(X, \mathscr{L}^1, \mathscr{L}^2, (\mathscr{L}_x)_{x \in X})$, where

(i) X is an n-set.

(ii) $\mathscr{L}^1$ and $\mathscr{L}^2$ are two $OA(2, 4, n)$ on X.

(iii) there is an element c_0 of X such that $(x, x, x, c_0) \in \mathscr{L}^1 \cap \mathscr{L}^2$ for all $x \in X$.

(iv) for each $x \in X$, $\mathscr{L}_x$ is an $OA(2, 3, n-1)$ on $X - \{x\}$.

(v) any $(x_1, x_2, x_3) \in X^3$ is either contained in an $\mathscr{L}_x$, $x \in X$, or in $\overline{\mathscr{L}^1}$ or $\overline{\mathscr{L}^2}$, where $\overline{\mathscr{L}^j} = \{(x_1, x_2, x_3);$ there is an $x \in X$ with $(x_1, x_2, x_3, x) \in \mathscr{L}^j\}, j = 1, 2$.

An $LD^*(n)$ is an $LD(n)$ such that the set $\mathscr{L}^0 = \{(x_0, x_1, x_2, x_3); (x_0, x_2, x_3) \in \overline{\mathscr{L}^1}$ and $(x_1, x_2, x_3) \in \overline{\mathscr{L}^2}\}$ is an $OA(2, 4, n)$. The notions of $LD(n)$ and $LD^*(n)$ are due to Lu [6], who proved the following.

PROPOSITION 1 [6]. *If an $LD(n)$ and an $LS(2, 3, n+2)$ both exist, then an $LS(2, 3, 3n)$ exists.*

In this paper, we will need one further notion. An $LD^i(n)$ is defined in the same way as an $LD(n)$, except that condition (iii) is replaced by the condition

(iii′) for all $x \in X$, we have $(x, x, x, x) \in \mathscr{L}^1 \cap \mathscr{L}^2$.

If $(X, \mathscr{L}^1, \mathscr{L}^2, (\mathscr{L}_x)_{x \in X})$ is an $LD^*(n)$, then we can construct an $LD^i(n)(X, \mathscr{L}^{1'}, \mathscr{L}^{2'}, (\mathscr{L}_x)_{x \in X})$, where $\mathscr{L}^{1'} = \{(x_1, x_2, x_3, x_4); (x_1, x_2, x_3) \in \overline{\mathscr{L}^1}, (x_4, x_2, x_3) \in \overline{\mathscr{L}^2}\}$ and $\mathscr{L}^{2'} = \{(x_1, x_2, x_3, x_4); (x_1, x_2, x_3) \in \overline{\mathscr{L}^2}$ and $(x_4, x_2, x_3) \in \overline{\mathscr{L}^1}\}$. Our main tool will be the following.

PROPOSITION 2. *Let* (X, β) *be an* $S(2, K, n)$, $\{0, 1, 2\} \cap K = \emptyset$. *Let* $c_0 \in X$. *If for each* $B \in \beta$ *with* $c_0 \in B$ *there is an* $LD(|B|)$ *and for each* $B \in \beta$ *with* $c_0 \notin B$ *there is an* $LD^i(|B|)$, *then there is an* $LD(n)$.

Proof. For each $B \in \beta$ with $c_0 \in B$, let $(B, \mathscr{L}^1_B, \mathscr{L}^2_B, (\mathscr{L}_{Bx})_{x \in B})$ be an $LD(|B|)$ such that $(x, x, x, c_0) \in \mathscr{L}^1_B \cap \mathscr{L}^2_B$ for all $x \in B$. For each $B \in \beta$ with $c_0 \notin B$, let $(B, \mathscr{L}^1_{Bo}, \mathscr{L}^2_{Bo}, (\mathscr{L}_{Bx})_{x \in B})$ be an $LD^i(|B|)$. Put $\mathscr{L}^j_B = \mathscr{L}^j_{Bo} - \{(x, x, x, x);\ x \in B\}$, $j = 1, 2$. As $2 \notin K$, we can construct, for each $B \in \beta$, an idempotent quasigroup $(B, \cdot_B)$. It is easy to check that $(X, \mathscr{L}^1, \mathscr{L}^2, (\mathscr{L}_x)_{x \in X})$ is an $LD(n)$, where $\mathscr{L}^1 = \bigcup_{B \in \beta} \mathscr{L}^1_B$, $\mathscr{L}^2 = \bigcup_{B \in \beta} \mathscr{L}^2_B$ and $\mathscr{L}_x = (\bigcup_{x \in B \in \beta} \mathscr{L}_{Bx}) \cup \{(x_1, x_2, x_3) \in (X - \{x\})^3;\ \{x_1, x_2, x_3\}$ is not contained in an element of β and $x_1 \cdot_{x_1 x_2} x_2 = x_3 \cdot_{x_3 x} x\}$. ∎

In [6] $LD^*(4)$, $LD^*(5)$ and $LD^*(7)$ are constructed. As mentioned before, the existence of an $LD^*(v)$ implies the existence of an $LD^i(v)$. Thus, by Proposition 2, $LD(n)$ exist for all n for which an $S(2, \{4, 5, 7\}, n)$ exists. A *parallel class* of an $S(2, k, v)(S, \beta)$ is a partition of S into blocks. An $S(2, k, v)(S, \beta)$ is called *resolvable* if β can be partitioned into parallel classes. If $n \equiv 11 \pmod{12}$, $n \geqslant 35$, then there exists a resolvable $S(2, 4, n-7)(S, \beta)$ with at least seven parallel classes [4]. Choose seven parallel classes $\gamma_1, ..., \gamma_7$ and seven distinct objects $\infty_1, ..., \infty_7$ such that $S \cap \{\infty_1, ..., \infty_7\} = \emptyset$. Then $(S \cup \{\infty_1, ..., \infty_7\}, \beta')$ is an $S(2, \{4, 5, 7\}, n)$, where the blocks of β' are the blocks of $\beta - (\gamma_1 \cup \cdots \cup \gamma_7)$, the sets $B \cup \{\infty_i\}$, $B \in \gamma_i$, $i = 1, ..., 7$, and the set $\{\infty_1, ..., \infty_7\}$. Hence, an $LD(n)$ exists for all $n \equiv 11 \pmod{12}$, $n \geqslant 35$. In particular, $LD(47)$, $LD(167)$, and $LD(263)$ all exist. By Proposition 1, this implies the existence of $LS(2, 3, 141)$, $LS(2, 3, 501)$, and $LS(2, 3, 789)$. Rosa [9] proved that the existence of an $LS(2, 3, v)$ with $v > 3$ implies the existence of an $LS(2, 3, 2v+1)$. Thus, the existence of an $LS(2, 3, 141)$ implies the existence of an $LS(2, 3, 283)$. Lu [7] proved that the existence of an $LS(2, 3, 1+4v)$ implies the existence of an $LS(2, 3, 1+12v)$. Thus, the existence of an $LS(2, 3, 501)$ implies the existence of an $LS(2, 3, 1501)$ and the existence of an $LS(2, 3, 789)$ implies the existence of an $LS(2, 3, 2365)$. This completes the proof of Lu's result that $LS(2, 3, v)$ exist for all $v \equiv 1$ or $3 \pmod 6$, $v \neq 7$.

In [10] a relatively easy construction of an $LS(2, 3, 3n)$ from an $LS(2, 3, n)$ is given. This means that Proposition 1 is mainly useful for $n \equiv 5 \pmod 6$ and $n = 7$. Lu [6, 7, 8] constructed $LD(n)$ for most, but not all of these values. For instance, for $n \equiv 11 \pmod{12}$, the important case in Lu's proof, there were 12 open cases left. Proposition 2, together with the $LD^*(4)$, $LD^*(5)$, $LD^*(7)$, and $LD^*(11)$ constructed in [6], can be used to construct $LD(n)$ for all $n \equiv 5 \pmod 6$, $n \neq 23$. Indeed, we already proved that $LD(n)$ exist for all $n \equiv 11 \pmod{12}$, $n \geqslant 35$. Adding one point to a resolvable $S(2, 4, n-1)$ produces an $S(2, \{4, 5\}, n)$ and thus an $LD(n)$, for all $n \equiv 5 \pmod{12}$. The existence of an $LD(23)$ remains open. (An $LS(2, 3, 69)$ is constructed in [3] by other methods.)

A substantial part of Lu's proof was concerned with showing that $LD(n)$ exist for all $n \equiv 11 \pmod{12}$, with at most 12 exceptions. In fact, the lack of an easy construction for $LD(n)$ with $n \equiv 11 \pmod{12}$ was one of the two main reasons why Lu's proof was so lengthy and complicated. This problem is now eliminated. The other difficult part of Lu's proof concerns the inductive construction of $LS(2, 3, v)$ for $v \equiv 13 \pmod{36}$. This requires, for the moment, several very complicated constructions [6, 7]. A unified simple construction for this case, or even better for all $v \equiv 1 \pmod{12}$, would further shorten and simplify Lu's proof. We were unable to produce such a construction, however.

As a final remark, we mention that Proposition 2 is a particular case of a much more general construction method for combinatorial structures of various types, described in [11]. Actually, [11, Proposition 2] was implicitly used in Proposition 2.

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