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Braid groups of non-orientable surfaces and the Fadell–Neuwirth short exact sequence

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ABSTRACT

Let M be a compact, connected non-orientable surface without boundary and of genus $g \geqslant 3$. We investigate the pure braid groups $P_n(M)$ of M, and in particular the possible splitting of the Fadell–Neuwirth short exact sequence

$$1 \longrightarrow P_m(M \setminus \{x_1, \ldots, x_n\}) \hookrightarrow P_{n+m}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1,$$

where $m, n \geqslant 1$, and p_* is the homomorphism which corresponds geometrically to forgetting the last m strings. This problem is equivalent to that of the existence of a section for the associated fibration $p: F_{n+m}(M) \longrightarrow F_n(M)$ of configuration spaces, defined by $p((x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})) = (x_1, \ldots, x_n)$. We show that p and p_* admit a section if and only if $n = 1$. Together with previous results, this completes the resolution of the splitting problem for surface pure braid groups.

1. Introduction

Braid groups of the plane were defined by Artin in 1925 [1], and further studied in [2,3]. Braid groups of surfaces were studied by Zariski [4], and were later generalised using the following definition due to Fox [5]. Let M be a compact, connected surface, and let $n \in \mathbb{N}$. We denote the set of all ordered n-tuples of distinct points of M, known as the nth *configuration space of M*, by:

$$F_n(M) = \{(p_1, \ldots, p_n) \mid p_i \in M \text{ and } p_i \neq p_j \text{ if } i \neq j\}.$$

Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [6,7] for example.

The symmetric group S_n on n letters acts freely on $F_n(M)$ by permuting coordinates. The corresponding quotient space will be denoted by $D_n(M)$. Notice that $F_n(M)$ is a regular covering of $D_n(M)$. The nth *pure braid group* $P_n(M)$ (respectively the nth *braid group* $B_n(M)$) is defined to be the fundamental group of $F_n(M)$ (respectively of $D_n(M)$). If $m \in \mathbb{N}$, then we may define a homomorphism $p_*: P_{n+m}(M) \longrightarrow P_n(M)$ induced by the projection $p: F_{n+m}(M) \longrightarrow F_n(M)$ defined by $p((x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})) = (x_1, \ldots, x_n)$. Representing $P_{n+m}(M)$ geometrically as a collection of $n + m$ strings, p_* corresponds to forgetting the last m strings. **We adopt the convention, that unless explicitly stated, all homomorphisms** $P_{n+m}(M) \longrightarrow P_n(M)$ **in the text will be this one**. If M is the 2-disc (or the plane $\mathbb{R}^2$), $B_n(M)$ and $P_n(M)$ are respectively the classical Artin braid group B_n and pure braid group P_n [8].

If M is without boundary, Fadell and Neuwirth study the map p, and show [9, Theorem 3] that it is a locally-trivial fibration. The fibre over a point $(x_1, \ldots, x_n)$ of the base space is $F_m(M \setminus \{x_1, \ldots, x_n\})$ which we interpret as a subspace of the total

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space via the map $i: F_m(M \setminus \{x_1, \ldots, x_n\}) \longrightarrow F_n(M)$ defined by

$$i((y_1, \ldots, y_m)) = (x_1, \ldots, x_n, y_1, \ldots, y_m).$$

Applying the associated long exact sequence in homotopy, we obtain the *pure braid group short exact sequence of Fadell and Neuwirth*:

$$1 \longrightarrow P_m(M \setminus \{x_1, \ldots, x_n\}) \xrightarrow{i_*} P_{n+m}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1, \qquad \text{(PBS)}$$

where $n \geqslant 3$ if M is the sphere $\mathbb{S}^2$ [10,8], $n \geqslant 2$ if M is the real projective plane $\mathbb{R}P^2$ [11], and $n \geqslant 1$ otherwise [9], and where i_* and p_* are the homomorphisms induced by the maps i and p respectively. The short exact sequence (**PBS**) has been widely studied, and may be employed for example to determine presentations of $P_n(M)$ (see Section 2), its centre, and possible torsion. It was also used in recent work on the structure of the mapping class groups [12] and on Vassiliev invariants for surface braids [13].

In the case of P_n, and taking $m = 1$, Ker (p_*) is a free group of rank n. The short exact sequence (**PBS**) splits for all $n \geqslant 1$, and so P_n may be described as a repeated semi-direct product of free groups. This decomposition, known as the 'combing' operation, is the principal result of Artin's classical theory of braid groups [2], and yields normal forms and a solution to the word problem in B_n. More recently, it was used by Falk and Randell to study the lower central series and the residual nilpotence of P_n [14], and by Rolfsen and Zhu to prove that P_n is bi-orderable [15].

The problem of deciding whether such a decomposition exists for braid groups of surfaces is thus fundamental. This was indeed a recurrent and central question during the foundation of the theory and its subsequent development during the 1960's [10,9,8,11,16]. If the fibre of the fibration is an Eilenberg–MacLane space then the existence of a section for p_* is equivalent to that of a cross-section for p [17,18] (cf. [19]). But with the exception of the construction of sections in certain cases (for $\mathbb{S}^2$ [10] and the 2-torus $\mathbb{T}^2$ [16]), no progress on the possible splitting of (**PBS**) was recorded for nearly forty years. In the case of orientable surfaces without boundary of genus at least two, the question of the splitting of (**PBS**) which was posed explicitly by Birman in 1969 [16], was finally resolved by the authors, the answer being positive if and only if $n = 1$ [20].

As for the non-orientable case, the braid groups of $\mathbb{R}P^2$ were first studied by Van Buskirk [11], and more recently by Wang [21] and the authors [19,22,23]. For $n = 1$, we have $P_1(\mathbb{R}P^2) = B_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$. Van Buskirk showed that for all $n \geqslant 2$, neither the fibration $p: F_n(\mathbb{R}P^2) \longrightarrow F_1(\mathbb{R}P^2)$ nor the homomorphism $p_*: P_n(\mathbb{R}P^2) \longrightarrow P_1(\mathbb{R}P^2)$ admit a cross-section (for p, this is a manifestation of the fixed point property of $\mathbb{R}P^2$), but that the fibration $p: F_3(\mathbb{R}P^2) \longrightarrow F_2(\mathbb{R}P^2)$ admits a cross-section, and hence so does the corresponding homomorphism p_*. Using coincidence theory, we showed that for $n = 2, 3$ and $m \geqslant 4 - n$, neither the fibration nor the short exact sequence (**PBS**) admit a section [19]. In [22], we gave a complete answer to the splitting problem for $\mathbb{R}P^2$: if $m, n \in \mathbb{N}$, the homomorphism $p_*: P_{n+m}(\mathbb{R}P^2) \longrightarrow P_n(\mathbb{R}P^2)$ and the fibration $p: F_{n+m}(\mathbb{R}P^2) \longrightarrow F_n(\mathbb{R}P^2)$ admit a section if and only if $n = 2$ and $m = 1$. In other words, Van Buskirk's values ($n = 2$ and $m = 1$) are the only ones for which a section exists (both on the geometric and the algebraic level).

In this paper, we study the splitting problem for compact, connected non-orientable surfaces without boundary and of genus $g \geqslant 3$ (every non-orientable compact surface M without boundary is homeomorphic to the connected sum of g copies of $\mathbb{R}P^2$, $g \in \mathbb{N}$ being the *genus* of M). In the case of the Klein bottle $\mathbb{K}^2$ ($g = 2$), the existence of a non-vanishing vector field implies that there always exists a section, both geometric and algebraic (cf. [9]). Our main result is:

Theorem 1. *Let M be a compact, connected, non-orientable surface without boundary of genus $g \geqslant 3$, and let $m, n \in \mathbb{N}$. Then the homomorphism $p_*: P_{n+m}(M) \longrightarrow P_n(M)$ and the fibration $p: F_{n+m}(M) \longrightarrow F_n(M)$ admit a section if and only if $n = 1$.*

Applying Theorem 1 and the results of [20,22], we may solve completely the splitting problem for surface pure braid groups:

Theorem 2. *Let $m, n \in \mathbb{N}$ and $r \geqslant 0$. Let N be a compact, connected surface possibly with boundary, let $\{x_1, \ldots x_r\}$ be a finite subset in the interior of N, let $M = N \setminus \{x_1, \ldots x_r\}$, and let $p_*: P_{n+m}(M) \longrightarrow P_n(M)$ be the standard projection.*

(a) *If $r > 0$ or if M has non-empty boundary then p_* admits a section for all m and n.*
(b) *Suppose that $r = 0$ and that M is without boundary. Then p_* admits a section if and only if one of the following conditions holds:*
  (i) *M is $\mathbb{S}^2$, the 2-torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}^2$ (for all m and n).*
  (ii) *$M = \mathbb{R}P^2$, $n = 2$ and $m = 1$.*
  (iii) *$M \neq \mathbb{R}P^2, \mathbb{S}^2, \mathbb{T}^2, \mathbb{K}^2$ and $n = 1$.*

The rest of the paper is organised as follows. In Section 2, we determine a presentation of $P_n(M)$ (Theorem 3). In Section 3, we study the consequences of the existence of a section in the case $m = 1$ and $n \geqslant 2$, *i.e.* $p_*: P_{n+1}(M) \longrightarrow P_n(M)$. The general strategy of the proof of Theorem 1 is based on the following remark. Suppose that (**PBS**) splits. If H is any normal subgroup of $P_{n+1}(M)$ contained in Ker (p_*), the quotiented short exact sequence $1 \longrightarrow \text{Ker}(p_*)/H \hookrightarrow P_{n+1}(M)/H \longrightarrow P_n(M) \longrightarrow 1$ must also split. In order to obtain a contradiction, we seek such a subgroup H for which this short exact sequence does *not* split. However the choice of H needed to achieve this may be somewhat delicate: if H is too 'small', the structure of the quotient $P_{n+1}(\mathbb{R}P^2)/H$ remains complicated; on the other hand, if H is too 'large', we lose too much information and cannot reach a conclusion. In Section 4, we first show that we may reduce to the case $m = 1$, and then go on to prove Theorem 1 using the analysis of Section 3. As we shall see in Section 4, it suffices to take H to be Abelianisation of Ker (p_*), in which case the quotient Ker $(p_*)/H$ is a free Abelian group. We will then deduce Theorem 2.

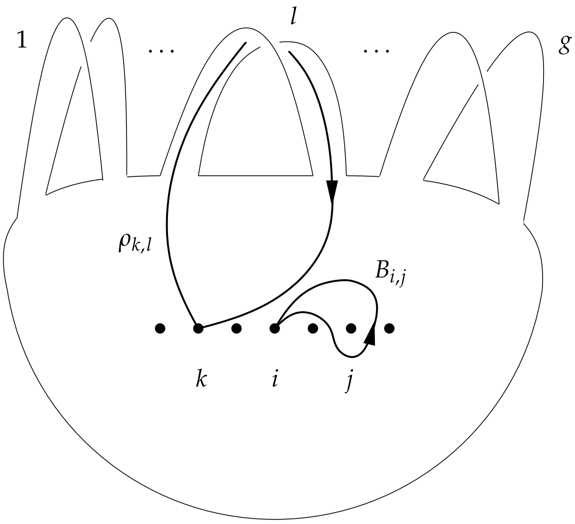

Fig. 1. The generators $B_{i,j}$ and $\rho_{k,l}$ of $P_n(M)$, represented geometrically by loops lying in M minus a disc.

2. A presentation of $P_n(M)$

Let $M = M_g$ be a compact, connected, non-orientable surface without boundary of genus $g \geqslant 2$. If $n \in \mathbb{N}$ and $\mathbb{D}^2 \subseteq M$ is a topological disc, the inclusion induces a homomorphism $\iota: B_n(\mathbb{D}^2) \longrightarrow B_n(M)$. If $\beta \in B_n(\mathbb{D}^2)$ then we shall denote its image $\iota(\beta)$ simply by β. For $1 \leqslant i < j \leqslant n$, we consider the following elements of $P_n(M)$:

$$B_{i,j} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_{j-2} \cdots \sigma_i,$$

where $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of $B_n(\mathbb{D}^2)$. The geometric braid corresponding to $B_{i,j}$ takes the ith string once around the jth string in the positive sense, with all other strings remaining vertical. For each $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant g$, we define a generator $\rho_{k,l}$ which is represented geometrically by a loop based at the kth point and which goes round the lth twisted handle. These elements are illustrated in Fig. 1 that represents M minus a disc.

A presentation of the braid groups of non-orientable surfaces was originally given by Scott [24]. Other presentations were later obtained in [25,26]. In the following theorem, we derive another presentation of $P_n(M)$.

Theorem 3. *Let M be a compact, connected, non-orientable surface without boundary of genus $g \geqslant 2$, and let $n \in \mathbb{N}$. The following constitutes a presentation of the pure braid group $P_n(M)$:*

generators: $B_{i,j}$, $1 \leqslant i < j \leqslant n$, *and* $\rho_{k,l}$, *where* $1 \leqslant k \leqslant n$ *and* $1 \leqslant l \leqslant g$.

relations: (a) *the Artin relations between the $B_{i,j}$ emanating from those of $P_n(\mathbb{D}^2)$:*

$$B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j \\ B_{i,j}^{-1}B_{r,j}^{-1}B_{i,j}B_{r,j}B_{i,j} & \text{if } r<i=s<j \\ B_{s,j}^{-1}B_{i,j}B_{s,j} & \text{if } i=r<s<j \\ B_{s,j}^{-1}B_{r,j}^{-1}B_{s,j}B_{r,j}B_{i,j}B_{r,j}^{-1}B_{s,j}^{-1}B_{r,j}B_{s,j} & \text{if } r<i<s<j. \end{cases} \quad (1)$$

(b) *for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant k, l \leqslant g$,*

$$\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k<l \\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^2 & \text{if } k=l \\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k} & \text{if } k>l. \end{cases} \quad (2)$$

(c) *for all $1 \leqslant i \leqslant n$, the 'surface relations' $\prod_{l=1}^{g} \rho_{i,l}^2 = B_{1,i} \cdots B_{i-1,i}B_{i,i+1} \cdots B_{i,n}$.*

(d) *for all $1 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant n$, $k \neq j$, and $1 \leqslant l \leqslant g$,*

$$\rho_{k,l}B_{i,j}\rho_{k,l}^{-1} = \begin{cases} B_{i,j} & \text{if } k<i \text{ or } j<k \\ \rho_{j,l}^{-1}B_{i,j}^{-1}\rho_{j,l} & \text{if } k=i \\ \rho_{j,l}^{-1}B_{k,j}^{-1}\rho_{j,l}B_{k,j}^{-1}B_{i,j}B_{k,j}\rho_{j,l}^{-1}B_{k,j}\rho_{j,l} & \text{if } i<k<j. \end{cases} \quad (3)$$

Proof. We apply induction and standard results concerning the presentation of an extension (see [27, Theorem 1, Chapter 13]). The proof generalises that of [22] for $\mathbb{R}P^2$, and is similar in spirit to that of [24].

First note that the given presentation is correct for $n = 1$ ($P_1(M) = \pi_1(M)$ has a presentation $\langle \rho_{1,1}, \ldots, \rho_{1,g} \mid \prod_{l=1}^{g} \rho_{1,l}^2 = 1 \rangle$). So let $n \geqslant 1$, and suppose that $P_n(M)$ has the given presentation. Taking $m = 1$ in (**PBS**), we have a short exact sequence:

$$1 \longrightarrow \pi_1 \left(M \setminus \{x_1, \ldots, x_n\}, x_{n+1}\right) \longrightarrow P_{n+1}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1.$$

In order to retain the symmetry of the presentation, we take the free group Ker (p_*) to have the following one-relator presentation:

$$\left\langle \rho_{n+1,1}, \dots \rho_{n+1,g}, B_{1,n+1}, \dots, B_{n,n+1} \;\middle|\; \prod_{l=1}^{g} \rho_{n+1,l}^2 = B_{1,n+1} \cdots B_{n,n+1} \right\rangle.$$

Together with these generators of Ker (p_*), the elements $B_{i,j}$, $1 \leqslant i < j \leqslant n$, and $\rho_{k,l}$, $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant g$, of $P_{n+1}(M)$ (which are coset representatives of the generators of $P_n(M)$) form the given generating set of $P_{n+1}(M)$.

There are three classes of relations of $P_{n+1}(M)$ which are obtained as follows. The first consists of the single relation $\prod_{l=1}^{g} \rho_{n+1,l}^2 = B_{1,n+1} \cdots B_{n,n+1}$ of Ker (p_*). The second class is obtained by rewriting the relators of the quotient in terms of the coset representatives, and expressing the corresponding element as a word in the generators of Ker (p_*). In this way, all of the relations of $P_n(M)$ lift directly to relations of $P_{n+1}(M)$, with the exception of the surface relations which become

$$\prod_{l=1}^{g} \rho_{i,l}^2 = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n} B_{i,n+1} \quad \text{for all } 1 \leqslant i \leqslant n.$$

Along with the relation of Ker (p_*), we obtain the complete set of surface relations (relations (c) for $P_{n+1}(M)$.

The third class of relations is obtained by rewriting the conjugates of the generators of Ker (p_*) by the coset representatives in terms of the generators of Ker (p_*):

(i) For all $1 \leqslant i < j \leqslant n$ and $1 \leqslant l \leqslant n$,

$$B_{i,j} B_{l,n+1} B_{i,j}^{-1} = \begin{cases} B_{l,n+1} & \text{if } l < i \text{ or } j < l \\ B_{l,n+1}^{-1} B_{i,n+1}^{-1} B_{l,n+1} B_{i,n+1} B_{l,n+1} & \text{if } l = j \\ B_{j,n+1}^{-1} B_{l,n+1} B_{j,n+1} & \text{if } l = i \\ B_{j,n+1}^{-1} B_{i,n+1}^{-1} B_{j,n+1} B_{i,n+1} B_{l,n+1} B_{i,n+1}^{-1} B_{j,n+1}^{-1} B_{i,n+1} B_{j,n+1} & \text{if } i < l < j. \end{cases}$$

(ii) $B_{i,j} \rho_{n+1,l} B_{i,j}^{-1} = \rho_{n+1,l}$ for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant l \leqslant g$.

(iii) for all $1 \leqslant i \leqslant n$ and $1 \leqslant k, l \leqslant g$,

$$\rho_{i,k} \rho_{n+1,l} \rho_{i,k}^{-1} = \begin{cases} \rho_{n+1,l} & \text{if } k < l \\ \rho_{n+1,k}^{-1} B_{i,n+1}^{-1} \rho_{n+1,k}^2 & \text{if } k = l \\ \rho_{n+1,k}^{-1} B_{i,n+1}^{-1} \rho_{n+1,k} B_{i,n+1}^{-1} \rho_{n+1,l} B_{i,n+1} \rho_{n+1,k}^{-1} B_{i,n+1} \rho_{n+1,k} & \text{if } k > l. \end{cases}$$

(iv) For all $1 \leqslant i, k \leqslant n$ and $1 \leqslant l \leqslant g$,

$$\rho_{k,l} B_{i,n+1} \rho_{k,l}^{-1} = \begin{cases} B_{i,n+1} & \text{if } k < i \\ \rho_{n+1,l}^{-1} B_{i,n+1}^{-1} \rho_{n+1,l} & \text{if } k = i \\ \rho_{n+1,l}^{-1} B_{k,n+1}^{-1} \rho_{n+1,l} B_{k,n+1}^{-1} B_{i,n+1} B_{k,n+1} \rho_{n+1,l}^{-1} B_{k,n+1} \rho_{n+1,l} & \text{if } i < k. \end{cases}$$

Then relations (a) for $P_{n+1}(M)$ are obtained from relations (a) for $P_n(M)$ and relations (i), relations (b) for $P_{n+1}(M)$ are obtained from relations (b) for $P_n(M)$ and relations (iii), and relations (d) for $P_{n+1}(M)$ are obtained from relations (d) for $P_n(M)$, and relations (ii) and (iv). □

3. Analysis of the case $P_{n+1}(M_g) \longrightarrow P_n(M_g)$, $n \geqslant 2$

For the whole of this section, we suppose that $g \geqslant 3$ and $n \geqslant 2$. By Theorem 3, $P_n(M_g)$ is generated by the union of the $B_{i,j}$, $1 \leqslant i < j \leqslant n$, and of the $\rho_{k,l}$, $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant g$. Let us consider the homomorphism $p_* \colon P_{n+1}(M_g) \longrightarrow P_n(M_g)$. In this section, we suppose that p_* admits a section, denoted by s_*. Applying (**PBS**), we thus have a split short exact sequence

$$1 \longrightarrow K \longrightarrow P_{n+1}(M_g) \underset{s_*}{\overset{p_*}{\rightleftarrows}} P_n(M_g) \longrightarrow 1, \tag{4}$$

where $K = \text{Ker}\,(p_*) = \pi_1(M_g \setminus \{x_1, \dots, x_n\}, x_{n+1})$ is a free group of rank $n + g - 1$, generated by $\{B_{1,n+1}, \dots, B_{n,n+1}, \rho_{n+1,1}, \dots, \rho_{n+1,g}\}$, and subject to the relation

$$B_{1,n+1} \cdots B_{n,n+1} = \rho_{n+1,1}^2 \cdots \rho_{n+1,g}^2.$$

Let $H = [K, K]$ be the commutator subgroup of K. Then K/H is a free Abelian group of rank $n + g - 1$. In what follows, we shall not distinguish notationally between the elements of K and those of K/H. The quotient group K/H thus has a basis

$$\mathcal{B} = \left\{B_{1,n+1}, \dots, B_{n-1,n+1}, \rho_{n+1,1}, \dots, \rho_{n+1,g}\right\}, \tag{5}$$

and the relation

$$B_{n,n+1} = B_{1,n+1}^{-1} \cdots B_{n-1,n+1}^{-1} \rho_{n+1,1}^2 \cdots \rho_{n+1,g}^2 \tag{6}$$

holds in the Abelian group K/H. Since H is normal in $P_{n+1}(M_g)$ and p_* admits a section, it follows from Eq. (4) that we have a split short exact sequence

$$1 \longrightarrow K/H \longrightarrow P_{n+1}(M_g)/H \underset{\bar{s}}{\overset{\bar{p}}{\rightleftarrows}} P_n(M_g) \longrightarrow 1,$$

where $\bar{p}$ is the homomorphism induced by p_*, and $\bar{s}$ is the induced section.

Consider the subset

$$\Gamma = \left\{ B_{i,j}, \rho_{k,l} \;\middle|\; 1 \leqslant i < j \leqslant n,\ 1 \leqslant k \leqslant n,\ 1 \leqslant l \leqslant g \right\}$$

of $P_{n+1}(M_g)/H$. If $g \in \Gamma$ then $\bar{p}(g) = g \in P_n(M_g)$, and so $g^{-1}.\bar{s}(\bar{p}(g)) \in \operatorname{Ker}(\bar{p}) = K/H$. Then the integer coefficients $\alpha_{i,j,r}, \beta_{i,j,q}, \gamma_{k,l,r}, \eta_{k,l,q}$, where $1 \leqslant r \leqslant g$ and $1 \leqslant q \leqslant n-1$, are (uniquely) defined by the equations:

$$\begin{cases} \bar{s}\left(B_{i,j}\right) = B_{i,j}\rho_{n+1,1}^{\alpha_{i,j,1}} \cdots \rho_{n+1,g}^{\alpha_{i,j,g}} B_{1,n+1}^{\beta_{i,j,1}} \cdots B_{n-1,n+1}^{\beta_{i,j,n-1}} \\ \bar{s}\left(\rho_{k,l}\right) = \rho_{k,l}\rho_{n+1,1}^{\gamma_{k,l,1}} \cdots \rho_{n+1,g}^{\gamma_{k,l,g}} B_{1,n+1}^{\eta_{k,l,1}} \cdots B_{n-1,n+1}^{\eta_{k,l,n-1}}. \end{cases} \tag{7}$$

There is an equation for each element of Γ. Most of the elements of Γ commute with the elements of the basis $\mathcal{B}$ of K/H given in Eq. (5). We record the list of conjugates of such elements for later use. In what follows, $1 \leqslant i < j \leqslant n$, $1 \leqslant k, m \leqslant n$ and $1 \leqslant l, r \leqslant g$. In K/H, we have

$$B_{i,j}B_{m,n+1}B_{i,j}^{-1} = B_{m,n+1}$$

(this follows from Eq. (1) and the fact that the elements $B_{q,n+1}$, $1 \leqslant q \leqslant n$, belong to K/H and so commute pairwise), and

$$B_{i,j}\rho_{n+1,l}B_{i,j}^{-1} = \rho_{n+1,l}$$

by Eq. (3). Thus $B_{i,j}$ belongs to the centraliser of K/H in $P_{n+1}(M_g)/H$. Also by Eq. (3), we have

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = \begin{cases} B_{m,n+1} & \text{if } k < m \\ \rho_{n+1,l}^{-1}B_{m,n+1}^{-1}\rho_{n+1,l} = B_{m,n+1}^{-1} & \text{if } k = m \\ \rho_{n+1,l}^{-1}B_{k,n+1}^{-1}\rho_{n+1,l}B_{k,n+1}^{-1}B_{m,n+1}B_{k,n+1}\rho_{n+1,l}^{-1}B_{k,n+1}\rho_{n+1,l} = B_{m,n+1} & \text{if } k > m, \end{cases}$$

so

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = B_{m,n+1}^{1-2\delta_{k,m}}, \tag{8}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker delta. By Eq. (2), we have

$$\rho_{k,l}\rho_{n+1,r}\rho_{k,l}^{-1} = \begin{cases} \rho_{n+1,r} & \text{if } l < r \\ \rho_{n+1,l}^{-1}B_{k,n+1}^{-1}\rho_{n+1,l}^{2} = \rho_{n+1,l}B_{k,n+1}^{-1} & \text{if } l = r \\ \rho_{n+1,l}^{-1}B_{k,n+1}^{-1}\rho_{n+1,l}B_{k,n+1}^{-1}\rho_{n+1,r}B_{k,n+1}\rho_{n+1,l}^{-1}B_{k,n+1}\rho_{n+1,l} = \rho_{n+1,r} & \text{if } l > r, \end{cases}$$

so

$$\rho_{k,l}\rho_{n+1,r}\rho_{k,l}^{-1} = \rho_{n+1,r}B_{k,n+1}^{-\delta_{l,r}}. \tag{9}$$

Combining Eqs. (8) and (9), we obtain

$$\rho_{k,r}^{2}\rho_{n+1,r}\rho_{k,r}^{-2} = \rho_{k,r}\rho_{n+1,r}B_{k,n+1}^{-1}\rho_{k,r}^{-1} = \rho_{n+1,r}B_{k,n+1}^{-1}B_{k,n+1} = \rho_{n+1,r},$$

so

$$\rho_{k,r}\rho_{n+1,r}\rho_{k,r}^{-1} = \rho_{k,r}^{-1}\rho_{n+1,r}\rho_{k,r}. \tag{10}$$

Furthermore, by Eq. (8), $\rho_{k,l}^2$ commutes with $B_{m,n+1}$, and therefore

$$\rho_{k,l}B_{m,n+1}\rho_{k,l}^{-1} = \rho_{k,l}^{-1}B_{m,n+1}\rho_{k,l}. \tag{11}$$

Hence $\rho_{k,l}^2$ also belongs to the centraliser of K/H in $P_{n+1}(M_g)/H$. From Eqs. (8) and (9), we obtain the following relations:

$$\rho_{n+1,1}^{\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}} \cdot \rho_{j,l} = \rho_{j,l} \cdot B_{j,n+1}^{-\gamma_{i,k,l}} \rho_{n+1,1}^{\gamma_{i,k,1}} \cdots \rho_{n+1,g}^{\gamma_{i,k,g}} \quad \text{for all } 1 \leqslant j \leqslant n, \tag{12}$$

and

$$B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}}\rho_{j,l} = \begin{cases} \rho_{j,l}B_{j,n+1}^{-2\eta_{i,k,j}}B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} & \text{if } 1 \leqslant j \leqslant n-1 \\ \rho_{j,l}B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} & \text{if } j = n. \end{cases}$$

Setting $\eta_{i,k,n} = 0$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant g$ yields:

$$B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \cdot \rho_{j,l} = \rho_{j,l} \cdot B_{j,n+1}^{-2\eta_{i,k,j}} B_{1,n+1}^{\eta_{i,k,1}} \cdots B_{n-1,n+1}^{\eta_{i,k,n-1}} \quad \text{for all } 1 \leqslant j \leqslant n. \tag{13}$$

Eqs. (12) and (13) will be employed repeatedly in the ensuing calculations.

We now investigate the images under $\bar{s}$ of some of the relations (b)–(d) of Theorem 3 (it turns out that the analysis of the other relations, including (a), will not be necessary for our purposes).

(a) Let $1 \leqslant i < j \leqslant n$ and $1 \leqslant k, l \leqslant g$. We examine the three possible cases of Eq. (7) (relation (b) of Theorem 3).

(i) $k < l$: then $\rho_{i,k}\rho_{j,l} = \rho_{j,l}\rho_{i,k}$ in $P_n(M_g)$. The respective images under $\bar{s}$ are:

$$\begin{aligned}\bar{s}\left(\rho_{i,k}\rho_{j,l}\right) &= \rho_{i,k}\rho_{n+1,1}^{\gamma_{i,k,1}}\cdots\rho_{n+1,g}^{\gamma_{i,k,g}}B_{1,n+1}^{\eta_{i,k,1}}\cdots B_{n-1,n+1}^{\eta_{i,k,n-1}}\rho_{j,l}\rho_{n+1,1}^{\gamma_{j,l,1}}\cdots\rho_{n+1,g}^{\gamma_{j,l,g}}B_{1,n+1}^{\eta_{j,l,1}}\cdots B_{n-1,n+1}^{\eta_{j,l,n-1}}\\ &= \rho_{i,k}\rho_{j,l}B_{j,n+1}^{-\gamma_{i,k,l}-2\eta_{i,k,j}}\rho_{n+1,1}^{\gamma_{i,k,1}+\gamma_{j,l,1}}\cdots\rho_{n+1,g}^{\gamma_{i,k,g}+\gamma_{j,l,g}}B_{1,n+1}^{\eta_{i,k,1}+\eta_{j,l,1}}\cdots B_{n-1,n+1}^{\eta_{i,k,n-1}+\eta_{j,l,n-1}},\end{aligned}$$

and

$$\begin{aligned}\bar{s}\left(\rho_{j,l}\rho_{i,k}\right) &= \rho_{j,l}\rho_{n+1,1}^{\gamma_{j,l,1}}\cdots\rho_{n+1,g}^{\gamma_{j,l,g}}B_{1,n+1}^{\eta_{j,l,1}}\cdots B_{n-1,n+1}^{\eta_{j,l,n-1}}\rho_{i,k}\rho_{n+1,1}^{\gamma_{i,k,1}}\cdots\rho_{n+1,g}^{\gamma_{i,k,g}}B_{1,n+1}^{\eta_{i,k,1}}\cdots B_{n-1,n+1}^{\eta_{i,k,n-1}}\\ &= \rho_{j,l}\rho_{i,k}B_{i,n+1}^{-\gamma_{j,l,k}-2\eta_{j,l,i}}\rho_{n+1,1}^{\gamma_{j,l,1}+\gamma_{i,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,l,g}+\gamma_{i,k,g}}B_{1,n+1}^{\eta_{j,l,1}+\eta_{i,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,l,n-1}+\eta_{i,k,n-1}}.\end{aligned}$$

The relation $\rho_{i,k}\rho_{j,l} = \rho_{j,l}\rho_{i,k}$ in $P_{n+1}(M_g)$ implies that $B_{j,n+1}^{-\gamma_{i,k,l}-2\eta_{i,k,j}} = B_{i,n+1}^{-\gamma_{j,l,k}-2\eta_{j,l,i}}$. Comparing coefficients of the elements of $\mathcal{B}$ in K/H (cf. Eq. (5)), if $j < n$, we have

$$\begin{cases}\gamma_{j,l,k} + 2\eta_{j,l,i} = 0 \quad \text{and}\\ \gamma_{i,k,l} + 2\eta_{i,k,j} = 0,\end{cases} \tag{14}$$

while if $j = n$, applying Eq. (6) yields

$$B_{i,n+1}^{\gamma_{n,l,k}+2\eta_{n,l,i}} = B_{n,n+1}^{\gamma_{i,k,l}+2\eta_{i,k,n}} = B_{1,n+1}^{-(\gamma_{i,k,l}+2\eta_{i,k,n})}\cdots B_{n-1,n+1}^{-(\gamma_{i,k,l}+2\eta_{i,k,n})}\rho_{n+1,1}^{2(\gamma_{i,k,l}+2\eta_{i,k,n})}\cdots\rho_{n+1,g}^{2(\gamma_{i,k,l}+2\eta_{i,k,n})},$$

and thus Eq. (14) also holds for $j = n$. So for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant k < l \leqslant g$,

$$\gamma_{j,l,k} + 2\eta_{j,l,i} = 0 \quad \text{and} \tag{15}$$

$$\gamma_{i,k,l} + 2\eta_{i,k,j} = 0. \tag{16}$$

(ii) $k = l$: then $\rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1} = \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^{2}$ in $P_n(M_g)$ for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant k \leqslant g$. The respective images under $\bar{s}$ are:

$$\begin{aligned}\bar{s}\left(\rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1}\right) &= \rho_{i,k}\rho_{n+1,1}^{\gamma_{i,k,1}}\cdots\rho_{n+1,g}^{\gamma_{i,k,g}}B_{1,n+1}^{\eta_{i,k,1}}\cdots B_{n-1,n+1}^{\eta_{i,k,n-1}}\rho_{j,k}\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}\\ &\quad\times B_{n-1,n+1}^{-\eta_{i,k,n-1}}\cdots B_{1,n+1}^{-\eta_{i,k,1}}\rho_{n+1,g}^{-\gamma_{i,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{i,k,1}}\rho_{i,k}^{-1}\\ &= \rho_{i,k}\rho_{j,k}B_{j,n+1}^{-\gamma_{i,k,k}}\rho_{n+1,1}^{\gamma_{i,k,1}+\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{i,k,g}+\gamma_{j,k,g}}B_{j,n+1}^{-2\eta_{i,k,j}}B_{1,n+1}^{\eta_{i,k,1}+\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{i,k,n-1}+\eta_{j,k,n-1}}\\ &\quad\times\rho_{i,k}^{-1}B_{i,n+1}^{2\eta_{i,k,i}}B_{n-1,n+1}^{-\eta_{i,k,n-1}}\cdots B_{1,n+1}^{-\eta_{i,k,1}}B_{i,n+1}^{\gamma_{i,k,k}}\rho_{n+1,g}^{-\gamma_{i,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{i,k,1}}\\ &= \rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1}B_{j,n+1}^{-\gamma_{i,k,k}}B_{i,n+1}^{-(\gamma_{i,k,k}+\gamma_{j,k,k})}\rho_{n+1,1}^{\gamma_{i,k,1}+\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{i,k,g}+\gamma_{j,k,g}}B_{j,n+1}^{-2\eta_{i,k,j}}B_{i,n+1}^{-2(\eta_{i,k,i}+\eta_{j,k,i})}\\ &\quad\times B_{1,n+1}^{\eta_{i,k,1}+\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{i,k,n-1}+\eta_{j,k,n-1}}B_{i,n+1}^{2\eta_{i,k,i}}B_{n-1,n+1}^{-\eta_{i,k,n-1}}\cdots B_{1,n+1}^{-\eta_{i,k,1}}B_{i,n+1}^{\gamma_{i,k,k}}\rho_{n+1,g}^{-\gamma_{i,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{i,k,1}}\\ &= \rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1}\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}B_{i,n+1}^{-(2\eta_{j,k,i}+\gamma_{j,k,k})}B_{j,n+1}^{-(2\eta_{i,k,j}+\gamma_{i,k,k})}\end{aligned}$$

and

$$\begin{aligned}\bar{s}\left(\rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^{2}\right) &= B_{n-1,n+1}^{-\eta_{j,k,n-1}}\cdots B_{1,n+1}^{-\eta_{j,k,1}}\rho_{n+1,g}^{-\gamma_{j,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{j,k,1}}\rho_{j,k}^{-1}\cdot B_{n-1,n+1}^{-\beta_{i,j,n-1}}\cdots B_{1,n+1}^{-\beta_{i,j,1}}\rho_{n+1,g}^{-\alpha_{i,j,g}}\cdots\rho_{n+1,1}^{-\alpha_{i,j,1}}B_{i,j}^{-1}\\ &\quad\times\rho_{j,k}\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}\rho_{j,k}\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}\\ &= \rho_{j,k}^{-1}B_{i,j}^{-1}B_{j,n+1}^{2\eta_{j,k,j}}B_{n-1,n+1}^{-\eta_{j,k,n-1}}\cdots B_{1,n+1}^{-\eta_{j,k,1}}B_{j,n+1}^{\gamma_{j,k,k}}\rho_{n+1,g}^{-\gamma_{j,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{j,k,1}}B_{n-1,n+1}^{-\beta_{i,j,n-1}}\cdots B_{1,n+1}^{-\beta_{i,j,1}}\\ &\quad\times\rho_{n+1,g}^{-\alpha_{i,j,g}}\cdots\rho_{n+1,1}^{-\alpha_{i,j,1}}\rho_{j,k}^{2}B_{j,n+1}^{-\gamma_{j,k,k}}\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{j,n+1}^{-2\eta_{j,k,j}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}\\ &\quad\times\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}\\ &= \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^{2}\rho_{n+1,1}^{\gamma_{j,k,1}-\alpha_{i,j,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}-\alpha_{i,j,g}}B_{1,n+1}^{\eta_{j,k,1}-\beta_{i,j,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}-\beta_{i,j,n-1}}.\end{aligned}$$

Since $\rho_{i,k}\rho_{j,k}\rho_{i,k}^{-1} = \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^{2}$ in $P_{n+1}(M_g)$, we obtain

$$B_{i,n+1}^{-(2\eta_{j,k,i}+\gamma_{j,k,k})}B_{j,n+1}^{-(2\eta_{i,k,j}+\gamma_{i,k,k})} = \rho_{n+1,1}^{-\alpha_{i,j,1}}\cdots\rho_{n+1,g}^{-\alpha_{i,j,g}}B_{1,n+1}^{-\beta_{i,j,1}}\cdots B_{n-1,n+1}^{-\beta_{i,j,n-1}}. \tag{17}$$

If $j < n$ then all of the terms in Eq. (17) are expressed in terms of the basis $\mathcal{B}$ of K/H of Eq. (5), and so for all $1 \leqslant i < j \leqslant n-1$,

$$\alpha_{i,j,r} = 0 \quad \text{for all } 1 \leqslant r \leqslant g \tag{18}$$

$$\beta_{i,j,s} = 0 \quad \text{for all } 1 \leqslant s \leqslant n-1, s \notin \{i,j\} \tag{19}$$

$$\beta_{i,j,i} = \gamma_{j,k,k} + 2\eta_{j,k,i} \tag{20}$$

$$\beta_{i,j,j} = \gamma_{i,k,k} + 2\eta_{i,k,j}. \tag{21}$$

If $j = n$ then substituting for $B_{n,n+1}$ in Eq. (17) using Eq. (6) and comparing coefficients in K/H of the elements of $\mathcal{B}$ yields

$$2(2\eta_{i,k,n} + \gamma_{i,k,k}) = \alpha_{i,n,r} \quad \text{for all } 1 \leqslant r \leqslant g$$

$$(2\eta_{i,k,n} + \gamma_{i,k,k}) = -\beta_{i,n,s} \quad \text{for all } 1 \leqslant s \leqslant n-1, s \neq i$$

$$2(\eta_{i,k,n} - \eta_{n,k,i}) + (\gamma_{i,k,k} - \gamma_{n,k,k}) = -\beta_{i,n,i}.$$

But $\eta_{i,k,n} = 0$, so for all $1 \leqslant i \leqslant n-1$ and $1 \leqslant k \leqslant g$,

$$\alpha_{i,n,r} = 2\gamma_{i,k,k} \quad \text{for all } 1 \leqslant r \leqslant g \tag{22}$$

$$\beta_{i,n,s} = -\gamma_{i,k,k} \quad \text{for all } 1 \leqslant s \leqslant n-1, s \neq i \tag{23}$$

$$\beta_{i,n,i} = 2\eta_{n,k,i} + (\gamma_{n,k,k} - \gamma_{i,k,k}). \tag{24}$$

(iii) $k > l$: then $\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k}$ in $P_n(M_g)$. The respective images under $\bar{s}$ are:

$$\begin{aligned}\bar{s}\left(\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1}\right) &= \rho_{i,k}\rho_{n+1,1}^{\gamma_{i,k,1}}\cdots\rho_{n+1,g}^{\gamma_{i,k,g}}B_{1,n+1}^{\eta_{i,k,1}}\cdots B_{n-1,n+1}^{\eta_{i,k,n-1}}\rho_{j,l}\rho_{n+1,1}^{\gamma_{j,l,1}}\cdots\rho_{n+1,g}^{\gamma_{j,l,g}}B_{1,n+1}^{\eta_{j,l,1}}\cdots B_{n-1,n+1}^{\eta_{j,l,n-1}}\\ &\quad\times B_{n-1,n+1}^{-\eta_{i,k,n-1}}\cdots B_{1,n+1}^{-\eta_{i,k,1}}\rho_{n+1,g}^{-\gamma_{i,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{i,k,1}}\rho_{i,k}^{-1}\\ &= \rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1}B_{j,n+1}^{-\gamma_{i,k,l}-2\eta_{i,k,j}}B_{i,n+1}^{-\gamma_{j,l,k}-2\eta_{j,l,i}}\rho_{n+1,1}^{\gamma_{j,l,1}}\cdots\rho_{n+1,g}^{\gamma_{j,l,g}}B_{1,n+1}^{\eta_{j,l,1}}\cdots B_{n-1,n+1}^{\eta_{j,l,n-1}}\end{aligned}$$

and

$$\begin{aligned}&\bar{s}\left(\rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k}\right)\\ &= B_{n-1,n+1}^{-\eta_{j,k,n-1}}\cdots B_{1,n+1}^{-\eta_{j,k,1}}\rho_{n+1,g}^{-\gamma_{j,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{j,k,1}}\rho_{j,k}^{-1}\cdot B_{n-1,n+1}^{-\beta_{i,j,n-1}}\cdots B_{1,n+1}^{-\beta_{i,j,1}}\rho_{n+1,g}^{-\alpha_{i,j,g}}\cdots\rho_{n+1,1}^{-\alpha_{i,j,1}}B_{i,j}^{-1}\\ &\quad\times\rho_{j,k}\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}\cdot B_{n-1,n+1}^{-\beta_{i,j,n-1}}\cdots B_{1,n+1}^{-\beta_{i,j,1}}\rho_{n+1,g}^{-\alpha_{i,j,g}}\cdots\rho_{n+1,1}^{-\alpha_{i,j,1}}B_{i,j}^{-1}\\ &\quad\times\rho_{j,l}\rho_{n+1,1}^{\gamma_{j,l,1}}\cdots\rho_{n+1,g}^{\gamma_{j,l,g}}B_{1,n+1}^{\eta_{j,l,1}}\cdots B_{n-1,n+1}^{\eta_{j,l,n-1}}\cdot B_{i,j}\rho_{n+1,1}^{\alpha_{i,j,1}}\cdots\rho_{n+1,g}^{\alpha_{i,j,g}}B_{1,n+1}^{\beta_{i,j,1}}\cdots B_{n-1,n+1}^{\beta_{i,j,n-1}}\\ &\quad\times B_{n-1,n+1}^{-\eta_{j,k,n-1}}\cdots B_{1,n+1}^{-\eta_{j,k,1}}\rho_{n+1,g}^{-\gamma_{j,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{j,k,1}}\rho_{j,k}^{-1}\cdot B_{i,j}\rho_{n+1,1}^{\alpha_{i,j,1}}\cdots\rho_{n+1,g}^{\alpha_{i,j,g}}B_{1,n+1}^{\beta_{i,j,1}}\cdots B_{n-1,n+1}^{\beta_{i,j,n-1}}\\ &\quad\times\rho_{j,k}\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}\\ &= B_{n-1,n+1}^{-\eta_{j,k,n-1}}\cdots B_{1,n+1}^{-\eta_{j,k,1}}\rho_{n+1,g}^{-\gamma_{j,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{j,k,1}}\rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{j,n+1}^{2\beta_{i,j,j}+\alpha_{i,j,k}}\\ &\quad\times\rho_{n+1,1}^{\gamma_{j,k,1}-2\alpha_{i,j,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}-2\alpha_{i,j,g}}B_{1,n+1}^{\eta_{j,k,1}-2\beta_{i,j,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}-2\beta_{i,j,n-1}}\\ &\quad\times B_{i,j}^{-1}\rho_{j,l}\rho_{n+1,1}^{\gamma_{j,l,1}}\cdots\rho_{n+1,g}^{\gamma_{j,l,g}}B_{1,n+1}^{\eta_{j,l,1}}\cdots B_{n-1,n+1}^{\eta_{j,l,n-1}}\cdot B_{i,j}\rho_{n+1,1}^{\alpha_{i,j,1}}\cdots\rho_{n+1,g}^{\alpha_{i,j,g}}B_{1,n+1}^{\beta_{i,j,1}}\cdots B_{n-1,n+1}^{\beta_{i,j,n-1}}\\ &\quad\times B_{n-1,n+1}^{-\eta_{j,k,n-1}}\cdots B_{1,n+1}^{-\eta_{j,k,1}}\rho_{n+1,g}^{-\gamma_{j,k,g}}\cdots\rho_{n+1,1}^{-\gamma_{j,k,1}}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k}B_{j,n+1}^{-\alpha_{i,j,k}-2\beta_{i,j,j}}\\ &\quad\times\rho_{n+1,1}^{\alpha_{i,j,1}}\cdots\rho_{n+1,g}^{\alpha_{i,j,g}}B_{1,n+1}^{\beta_{i,j,1}}\cdots B_{n-1,n+1}^{\beta_{i,j,n-1}}\cdot\rho_{n+1,1}^{\gamma_{j,k,1}}\cdots\rho_{n+1,g}^{\gamma_{j,k,g}}B_{1,n+1}^{\eta_{j,k,1}}\cdots B_{n-1,n+1}^{\eta_{j,k,n-1}}\\ &= \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k}B_{j,n+1}^{2\alpha_{i,j,l}-2\alpha_{i,j,k}}\cdot B_{1,n+1}^{\eta_{j,l,1}}\cdots B_{n-1,n+1}^{\eta_{j,l,n-1}}\rho_{n+1,1}^{\gamma_{j,l,1}}\cdots\rho_{n+1,g}^{\gamma_{j,l,g}}.\end{aligned}$$

Since $\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k}$ in $P_{n+1}(M_g)$, we see that

$$B_{i,n+1}^{-\gamma_{j,l,k}-2\eta_{j,l,i}} = B_{j,n+1}^{2\alpha_{i,j,l}-2\alpha_{i,j,k}+\gamma_{i,k,l}+2\eta_{i,k,j}}.$$

If $j < n$, it follows by comparing coefficients of the elements of $\mathcal{B}$ in K/H that for all $1 \leqslant i < j < n$ and $1 \leqslant l < k \leqslant g$,

$$\begin{cases}\gamma_{j,l,k} + 2\eta_{j,l,i} = 0\\ 2\alpha_{i,j,l} - 2\alpha_{i,j,k} + \gamma_{i,k,l} + 2\eta_{i,k,j} = 0.\end{cases} \tag{25}$$

If $j = n$ then

$$\begin{aligned}B_{i,n+1}^{-\gamma_{n,l,k}-2\eta_{n,l,i}} &= B_{n,n+1}^{2\alpha_{i,n,l}-2\alpha_{i,n,k}+\gamma_{i,k,l}+2\eta_{i,k,n}}\\ &= B_{1,n+1}^{-(2\alpha_{i,n,l}-2\alpha_{i,n,k}+\gamma_{i,k,l}+2\eta_{i,k,n})}\cdots B_{n-1,n+1}^{-(2\alpha_{i,n,l}-2\alpha_{i,n,k}+\gamma_{i,k,l}+2\eta_{i,k,n})}\\ &\quad\times\rho_{n+1,1}^{2(2\alpha_{i,n,l}-2\alpha_{i,n,k}+\gamma_{i,k,l}+2\eta_{i,k,n})}\cdots\rho_{n+1,g}^{2(2\alpha_{i,n,l}-2\alpha_{i,n,k}+\gamma_{i,k,l}+2\eta_{i,k,n})},\end{aligned}$$

and comparing coefficients of the elements of $\mathcal{B}$ in K/H, we observe that Eqs. (25) also hold if $j = n$. So for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant l < k \leqslant g$,

$$\gamma_{j,l,k} + 2\eta_{j,l,i} = 0 \tag{26}$$

$$2\alpha_{i,j,l} - 2\alpha_{i,j,k} + \gamma_{i,k,l} + 2\eta_{i,k,j} = 0. \tag{27}$$

(b) Let $1 \leqslant i \leqslant n$. Then $\prod_{l=1}^{g}\rho_{i,l}^2 = B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n}$ in $P_n(M_g)$ by relation (c) of Theorem 3. For $1 \leqslant l \leqslant g$, note that

$$\begin{aligned}\bar{s}\left(\rho_{i,l}^2\right) &= \rho_{i,l}\rho_{n+1,1}^{\gamma_{i,l,1}}\cdots\rho_{n+1,g}^{\gamma_{i,l,g}}B_{1,n+1}^{\eta_{i,l,1}}\cdots B_{n-1,n+1}^{\eta_{i,l,n-1}}\rho_{i,l}\rho_{n+1,1}^{\gamma_{i,l,1}}\cdots\rho_{n+1,g}^{\gamma_{i,l,g}}B_{1,n+1}^{\eta_{i,l,1}}\cdots B_{n-1,n+1}^{\eta_{i,l,n-1}}\\ &= \rho_{i,l}^2B_{i,n+1}^{-2\eta_{i,l,i}-\gamma_{i,l,l}}\rho_{n+1,1}^{2\gamma_{i,l,1}}\cdots\rho_{n+1,g}^{2\gamma_{i,l,g}}B_{1,n+1}^{2\eta_{i,l,1}}\cdots B_{n-1,n+1}^{2\eta_{i,l,n-1}}.\end{aligned}$$

As we saw in Eqs. (10) and (11), $\rho_{i,l}^2$ belongs to the centraliser of K/H in $P_{n+1}(M_g)/H$, so

$$\bar{s}\left(\prod_{l=1}^{g}\rho_{i,l}^2\right) = \left(\prod_{l=1}^{g}\rho_{i,l}^2\right)\left(\prod_{l=1}^{g}B_{i,n+1}^{-2\sum_{l=1}^{g}\eta_{i,l,i}-\sum_{l=1}^{g}\gamma_{i,l,l}}\rho_{n+1,1}^{2\sum_{l=1}^{g}\gamma_{i,l,1}}\cdots\rho_{n+1,g}^{2\sum_{l=1}^{g}\gamma_{i,l,g}}\cdot B_{1,n+1}^{2\sum_{l=1}^{g}\eta_{i,l,1}}\cdots B_{n-1,n+1}^{2\sum_{l=1}^{g}\eta_{i,l,n-1}}\right).$$

Further,

$$\bar{s}\left(B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n}\right)=B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n}\cdot\prod_{l=1}^{i-1}\left(\rho_{n+1,1}^{\alpha_{l,i,1}}\cdots\rho_{n+1,g}^{\alpha_{l,i,g}}B_{1,n+1}^{\beta_{l,i,1}}\cdots B_{n-1,n+1}^{\beta_{l,i,n-1}}\right)$$
$$\times\prod_{l=i+1}^{n}\left(\rho_{n+1,1}^{\alpha_{i,l,1}}\cdots\rho_{n+1,g}^{\alpha_{i,l,g}}B_{1,n+1}^{\beta_{i,l,1}}\cdots B_{n-1,n+1}^{\beta_{i,l,n-1}}\right)$$
$$=B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n}\cdot\rho_{n+1,1}^{\sum_{l=1}^{i-1}\alpha_{l,i,1}+\sum_{l=i+1}^{n}\alpha_{i,l,1}}\cdots\rho_{n+1,g}^{\sum_{l=1}^{i-1}\alpha_{l,i,g}+\sum_{l=i+1}^{n}\alpha_{i,l,g}}\cdot B_{1,n+1}^{\sum_{l=1}^{i-1}\beta_{l,i,1}+\sum_{l=i+1}^{n}\beta_{i,l,1}}\cdots B_{n-1,n+1}^{\sum_{l=1}^{i-1}\beta_{l,i,n-1}+\sum_{l=i+1}^{n}\beta_{i,l,n-1}}.$$

Now in $P_{n+1}(M_g)/H$, $\prod_{l=1}^{g}\rho_{i,l}^{2}=B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n}B_{i,n+1}$, hence

$$B_{i,n+1}^{1-2\sum_{l=1}^{g}\eta_{i,l,i}-\sum_{l=1}^{g}\gamma_{i,l,l}}\rho_{n+1,1}^{2\sum_{l=1}^{g}\gamma_{i,l,1}}\cdots\rho_{n+1,g}^{2\sum_{l=1}^{g}\gamma_{i,l,g}}B_{1,n+1}^{2\sum_{l=1}^{g}\eta_{i,l,1}}\cdots B_{n-1,n+1}^{2\sum_{l=1}^{g}\eta_{i,l,n-1}}$$
$$=\rho_{n+1,1}^{\sum_{l=1}^{i-1}\alpha_{l,i,1}+\sum_{l=i+1}^{n}\alpha_{i,l,1}}\cdots\rho_{n+1,g}^{\sum_{l=1}^{i-1}\alpha_{l,i,g}+\sum_{l=i+1}^{n}\alpha_{i,l,g}}\cdot B_{1,n+1}^{\sum_{l=1}^{i-1}\beta_{l,i,1}+\sum_{l=i+1}^{n}\beta_{i,l,1}}\cdots B_{n-1,n+1}^{\sum_{l=1}^{i-1}\beta_{l,i,n-1}+\sum_{l=i+1}^{n}\beta_{i,l,n-1}}.$$

Thus for all $1\leqslant i\leqslant n$,

$$B_{i,n+1}^{1-2\sum_{l=1}^{g}\eta_{i,l,i}-\sum_{l=1}^{g}\gamma_{i,l,l}}=\rho_{n+1,1}^{\sum_{l=1}^{i-1}\alpha_{l,i,1}+\sum_{l=i+1}^{n}\alpha_{i,l,1}-2\sum_{l=1}^{g}\gamma_{i,l,1}}\cdots\rho_{n+1,g}^{\sum_{l=1}^{i-1}\alpha_{l,i,g}+\sum_{l=i+1}^{n}\alpha_{i,l,g}-2\sum_{l=1}^{g}\gamma_{i,l,g}}$$
$$\times B_{1,n+1}^{\sum_{l=1}^{i-1}\beta_{l,i,1}+\sum_{l=i+1}^{n}\beta_{i,l,1}-2\sum_{l=1}^{g}\eta_{i,l,1}}\cdots B_{n-1,n+1}^{\sum_{l=1}^{i-1}\beta_{l,i,n-1}+\sum_{l=i+1}^{n}\beta_{i,l,n-1}-2\sum_{l=1}^{g}\eta_{i,l,n-1}}. \tag{28}$$

4. Proofs of Theorems 1 and 2

In this section, we use the calculations of Section 3 to prove Theorem 1, from which we shall deduce Theorem 2.

Proof of Theorem 1. As we mentioned in the Introduction, the existence of an algebraic section for p_* is equivalent to that of a cross-section for p.

The case $n=1$ was treated in Theorem 1 of [20], using the fact that if $M=M_g$, where $g\geqslant 3$, then M is homeomorphic to the connected sum of one or two copies of $\mathbb{R}P^2$ with a compact, orientable surface without boundary of genus at least one.

Conversely, suppose that there exist $m\in\mathbb{N}$ and $n\geqslant 2$ for which the homomorphism $p_*\colon P_{n+m}(M)\longrightarrow P_n(M)$ admits a section. We shall argue for a contradiction. By [20, Proposition 3], it suffices to consider the case $m=1$. We first analyse the general structure of the coefficients $\alpha_{i,j,r}$, $\beta_{i,j,q}$, $\gamma_{k,l,r}$, $\eta_{k,l,q}$ defined by Eq. (7).

(a) Taking $j=n$ in Eq. (16) implies that $\gamma_{i,k,l}=0$ for all $1\leqslant i\leqslant n-1$ and $1\leqslant k<l\leqslant g$.

(b) By Eq. (27),

$$\gamma_{i,k,l}=-2\eta_{i,k,j}-2(\alpha_{i,j,l}-\alpha_{i,j,k})$$

for all $1\leqslant i<j\leqslant n$ and $1\leqslant l<k\leqslant g$. Taking $j=n$, we obtain

$$\gamma_{i,k,l}=-2\eta_{i,k,n}-2(\alpha_{i,n,l}-\alpha_{i,n,k})=0$$

since $\eta_{i,k,n}=0$ by definition and $\alpha_{i,n,r}=2\gamma_{i,1,1}$ for all $1\leqslant i\leqslant n-1$ and $1\leqslant r\leqslant g$ by Eq. (22).

It thus follows from (a) and (b) that

$$\gamma_{i,k,l}=0\quad\text{for all } 1\leqslant i\leqslant n-1 \text{ and } 1\leqslant k,l\leqslant g, k\neq l. \tag{29}$$

(c) By Eq. (22), $\gamma_{i,k,k}=\frac{1}{2}\alpha_{i,n,1}$ for all $1\leqslant i\leqslant n-1$ and $1\leqslant k\leqslant g$. So

$$\gamma_{i,k,k}=\gamma_{i,1,1}\quad\text{for all } 1\leqslant i\leqslant n-1 \text{ and } 1\leqslant k\leqslant g. \tag{30}$$

(d) By Eq. (16), for all $1\leqslant k<l\leqslant g$ and $1\leqslant i<j\leqslant n$, we have

$$\eta_{i,k,j}=-\frac{1}{2}\gamma_{i,k,l}=0,$$

using Eq. (29). So by taking $l=g$ we obtain

$$\eta_{i,k,j}=0\quad\text{for all } 1\leqslant i<j\leqslant n \text{ and } 1\leqslant k\leqslant g-1.$$

(e) By Eq. (27)

$$\eta_{i,k,j}=\frac{1}{2}\left(2\left(\alpha_{i,j,l}-\alpha_{i,j,k}\right)+\gamma_{i,k,l}\right)$$

for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant l < k \leqslant g$. But $\gamma_{i,k,l} = 0$ by Eq. (29), and $\alpha_{i,j,l} - \alpha_{i,j,k} = 0$ by Eq. (18) if $j \leqslant n-1$ and by Eq. (22) if $j = n$. Setting $l = 1$, it follows that

$$\eta_{i,k,j} = 0 \quad \text{for all } 1 \leqslant i < j \leqslant n \text{ and } 2 \leqslant k \leqslant g.$$

By (d) and (e) we thus have

$$\eta_{i,k,j} = 0 \quad \text{for all } 1 \leqslant i < j \leqslant n \text{ and } 1 \leqslant k \leqslant g. \tag{31}$$

(f) Suppose that $1 \leqslant j < i \leqslant n-1$. Then

$$\begin{aligned} \eta_{i,k,j} &= -\frac{1}{2}\gamma_{i,k,l} \quad \text{for all } 1 \leqslant k < l \leqslant g, \text{ by Eq. (26)} \\ &= 0 \quad \text{by Eq. (29)}. \end{aligned}$$

So taking $l = g$, we have $\eta_{i,k,j} = 0$ for all $1 \leqslant k \leqslant g-1$. Further, for all $1 \leqslant l < k \leqslant g$,

$$\begin{aligned} \eta_{i,k,j} &= -\frac{1}{2}\gamma_{i,k,l} \quad \text{by Eq. (15)} \\ &= 0 \quad \text{by Eq. (29)}. \end{aligned}$$

Hence it follows from Eq. (31) and (f) that

$$\eta_{i,k,j} = 0 \quad \text{for all } 1 \leqslant i, j \leqslant n-1, i \neq j, \text{ and } 1 \leqslant k \leqslant g. \tag{32}$$

(g) From Eq. (23), we obtain

$$\beta_{i,n,s} = -\gamma_{i,1,1} \quad \text{for all } 1 \leqslant s \leqslant n-1, s \neq i. \tag{33}$$

(h) By Eqs. (21) and (32), we see that

$$\gamma_{i,1,1} = \beta_{i,i+1,i+1} = \cdots = \beta_{i,n-1,n-1} \quad \text{for all } 1 \leqslant i \leqslant n-2. \tag{34}$$

(i) By Eqs. (20) and (32), we obtain

$$\gamma_{i,1,1} = \beta_{1,i,1} = \cdots = \beta_{i-1,i,i-1} \quad \text{for all } 2 \leqslant i \leqslant n-1. \tag{35}$$

Analysing Eq. (28), we are now able to complete the proof of Theorem 1 as follows. Let $i \in \{1, \ldots, n-1\}$. Then the coefficient of $B_{i,n+1}$ yields:

$$1 - 2\sum_{l=1}^{g} \eta_{i,l,i} - \sum_{l=1}^{g} \gamma_{i,l,l} = \sum_{l=1}^{i-1} \beta_{l,i,i} + \sum_{l=i+1}^{n} \beta_{i,l,i} - 2\sum_{l=1}^{g} \eta_{i,l,i}. \tag{36}$$

Now

$$\sum_{l=1}^{i-1} \beta_{l,i,i} = \sum_{l=1}^{i-1} \gamma_{l,1,1} \quad \text{by Eq. (34)},$$

and

$$\sum_{l=i+1}^{n} \beta_{i,l,i} = \sum_{l=i+1}^{n} \gamma_{l,1,1} \quad \text{by Eq. (35)}.$$

So using Eq. (30), Eq. (36) becomes

$$1 - g\gamma_{i,1,1} = \beta_{i,n,i} + \sum_{l=1}^{n-1} \gamma_{l,1,1} - \gamma_{i,1,1}.$$

Summing over all $i = 1, \ldots, n-1$, and setting $\varDelta = \sum_{l=1}^{n-1} \gamma_{l,1,1}$ and $L = \sum_{i=1}^{n-1} \beta_{i,n,i}$, we obtain

$$(n + g - 2)\varDelta = (n-1) - L. \tag{37}$$

Now let $i = n$, and let $k \in \{1, \ldots, n-1\}$. Since $\eta_{n,l,n} = 0$, the coefficient of $B_{k,n+1}$ in Eq. (28) yields:

$$\begin{aligned} \sum_{l=1}^{g} \gamma_{n,l,l} - 1 &= \sum_{l=1}^{n-1} \beta_{l,n,k} - 2\sum_{l=1}^{g} \eta_{n,l,k} = \beta_{k,n,k} + \sum_{\substack{l=1 \\ l\neq k}}^{n-1} \beta_{l,n,k} - 2\sum_{l=1}^{g} \eta_{n,l,k} \\ &= \beta_{k,n,k} - \sum_{\substack{l=1 \\ l\neq k}}^{n-1} \gamma_{l,1,1} - 2\sum_{l=1}^{g} \eta_{n,l,k} \quad \text{by Eq. (33)} \end{aligned}$$

$$= \beta_{k,n,k} - (\Delta - \gamma_{k,1,1}) + \sum_{l=1}^{g} \left(-\beta_{k,n,k} + \gamma_{n,l,l} - \gamma_{k,l,l}\right) \quad \text{by Eq. (24)}$$

$$= (1-g)\beta_{k,n,k} + \gamma_{k,1,1} - \Delta + \sum_{l=1}^{g} \gamma_{n,l,l} - \sum_{l=1}^{g} \gamma_{k,1,1} \quad \text{by Eq. (30)}$$

$$= (1-g)\beta_{k,n,k} + (1-g)\gamma_{k,1,1} - \Delta + \sum_{l=1}^{g} \gamma_{n,l,l}.$$

Hence $-1 = (1-g)\beta_{k,n,k} + (1-g)\gamma_{k,1,1} - \Delta$. Summing over all $k = 1, \ldots, n-1$, we obtain

$$(n+g-2)\Delta = (1-g)L + (n-1). \tag{38}$$

Equating Eqs. (37) and (38), we see that $(n-1) - L = (1-g)L + (n-1)$. Since $g \geqslant 3$, it follows that $L = 0$, and therefore

$$\Delta = \frac{n-1}{(n-1)+(g-1)}$$

by Eq. (37). This yields a contradiction to the fact that Δ is an integer, and thus completes the proof of Theorem 1. □

Remark. Although some of the relations derived in (a)–(i) do not exist if $n = 2$, one may check that the above analysis from Eq. (36) onwards is also valid in this case (with $\Delta = \gamma_{1,1,1}$ and $L = \beta_{1,2,1}$).

Proof of Theorem 2. (a) If $r > 0$ then the result follows applying the methods of the proofs of Proposition 27 and Theorem 6 of [20]. If $r = 0$ and M has non-empty boundary, let C be a boundary component of M. Then $M' = M \setminus C$ is homeomorphic to a compact surface with a single point deleted (which is the case $r = 1$), so (**PBS**) splits for M'. The inclusion of M' in M not only induces a homotopy equivalence between M and M', but also a homotopy equivalence between their nth configuration spaces. Therefore their nth pure braid groups are isomorphic, and the sequence (**PBS**) for M splits if and only it splits for M'.

(b) Suppose that $r = 0$ and that M is without boundary. If $M = \mathbb{S}^2$, $m = 1$ and $n \geqslant 3$ then the statement follows from [10]. The geometric construction of Fadell may be easily generalised to all $m \in \mathbb{N}$. If $n \in \{1, 2\}$, the result is obvious since $P_n(\mathbb{S}^2)$ is trivial. If $M = \mathbb{T}^2$ or $\mathbb{K}^2$, the fact that p_* has a section is a consequence of [9] and the fact that $\mathbb{T}^2$ and $\mathbb{K}^2$ admit a non-vanishing vector field. If $M = \mathbb{R}P^2$ then p_* admits a section if and only if $n = 2$ and $m = 1$ by [22]. Finally, if $M \neq \mathbb{R}P^2, \mathbb{S}^2, \mathbb{T}^2, \mathbb{K}^2$ then p_* admits a section if and only if $n = 1$ by Theorem 1 for the non-orientable case, and by [20] for the orientable case. □

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