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On unit groups of Lie centre-by-metabelian algebras

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Abstract

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We prove that the group of units of a Lie centre-by-metabelian algebra need not be centre-by-metabelian. This settles a question raised by Sharma and Srivastava.

In view of several known results, namely the result of Gupta and Levin who have proved that if an algebra is Lie nilpotent, then its group of units must be nilpotent, and the result of Sharma and Srivastava that Lie metabelian algebras have metabelian unit groups, it is natural to ask whether the unit groups of Lie centre-by-metabelian algebras must be centre-by-metabelian (the question was raised in [3]). Some results in the positive direction have been obtained by Smirnov [4] in the case of algebras of exponent 4 over certain fields of characteristic 2; and by Sharma and Srivastava [3] for certain algebras over fields of odd characteristic. (The reader is referred to these articles for the details.) However, we shall exhibit an example to demonstrate the following:

The main result. *The group of units of a Lie centre-by-metabelian Lie nilpotent algebra need not be centre-by-metabelian.*

The notation. Let R be an algebra; then for $x, y \in R$ we define Lie commutators as $(x, y) = xy - yx$; and left-normed Lie commutators by $(x, y, z) = ((x, y), z)$. Let $L_n(R)$ be the ideal of R generated by the left-normed Lie commutators of weight n, $(x_1, \ldots, x_n)$, where $x_i \in R$. (X, Y) denotes the additive subgroup of R generated by the set $\{(x, y) \mid x \in X, y \in Y\}$; in this notation the smallest ideal whose quotient is Lie centre-by-metabelian is the ideal generated by $((R, R), (R, R), R)$. Further, let $U = U(R)$ denote the group of units of R, and in the standard notation let $[x, y] = x^{-1}y^{-1}xy$, $[x, y, z] = [[x, y], z]$ and $\gamma_n(U)$ be the nth term of the lower central series of U. If A, B are subgroups of U we let $[A, B]$ be the subgroup generated by $\{[a, b] \mid a \in A, b \in B\}$. As usual $U' = [U, U]$, $U'' = [U', U']$. In this language, the result of Gupta and Levin reads $\gamma_n(U) \subseteq 1 + L_n(R)$. We shall describe an R such that

$$[U'', U] - 1 \not\subseteq R((R, R), (R, R), R)R .$$

The example. Let $K = Z_2[[x, y, z]]$ and let $\Delta = \langle x, y, z \rangle$, the fundamental ideal of K. Under the operation of Lie commutation K becomes a Lie algebra K^*; elements of the Lie subalgebra of K^* generated by $\{x, y, z\}$ are called *special* Lie elements of K. By a special Lie element of *weight* n we mean a special Lie element which is homogeneous of degree n in K. Let $\Delta_n(K)$ be the ideal of K generated by the special Lie elements of weight at least n.

The example is the algebra $R = K/J$, where J is the ideal of K generated by
(i) $\Delta^8 + \Delta_4(K) \cap \Delta^7 + \Delta\Delta_3(K)\Delta_3(K)$;
(ii) $((K, K), (K, K), K)$.

Clearly (because of the generators of type (ii)) R is Lie centre-by-metabelian; it is Lie nilpotent because Δ^8 is factored out; we shall prove that $U(R)$ is not centre-by-metabelian.

The proof. We want to show that $[[\bar{x}, \bar{y}], [\bar{x}, \bar{z}], \bar{x}] \neq 1$ in R. (Here $\bar{w} = 1 + w$ for $w \in \{x, y, z\}$. Note that $\bar{x}, \bar{y}, \bar{z}$ are in $U(R)$.) The central place in proving that statement is occupied by the following lemmas:

Lemma 1. $[[\bar{x}, \bar{y}], [\bar{x}, \bar{z}], \bar{x}] - 1 \in J$ *if and only if* $(y, x)^2(z, x, x)$ *and* $(z, x)^2(y, x, x)$ *are in* J.

Lemma 2. *Let θ be the endomorphism of K given by $x^\theta = x$, $y^\theta = y$, $z^\theta = y$. Then* $(y, x)^2(y, x, x) \notin J^\theta$.

Assuming we have proved the lemmas, the argument is as follows: Suppose $[[\bar{x}, \bar{y}], [\bar{x}, \bar{z}], \bar{x}] - 1 \in J$. Then, by Lemma 1, $(y, x)^2(z, x, x) \in J$ and therefore $(y, x)^2(y, x, x)$ is in J^θ. However, this contradicts Lemma 2. This establishes the result: $[[\bar{x}, \bar{y}], [\bar{x}, \bar{z}], \bar{x}] - 1 \notin J$.

We now proceed to prove the lemmas.

Proof of Lemma 1. Let $u = [[\bar{x}, \bar{y}], [\bar{x}, \bar{z}]]$; using $[u, \bar{x}] - 1 = u^{-1}\bar{x}^{-1}(u, \bar{x})$, we get

$$[u, \bar{x}] - 1 = (u^{-1} - 1)\bar{x}^{-1}(u, \bar{x}) + \bar{x}^{-1}(u, \bar{x}) \equiv \bar{x}^{-1}(u, \bar{x})$$

since the first summand is zero modulo Δ^8. Therefore, $[u, \bar{x}] - 1 \in J$ iff $(u, \bar{x}) \in J$.

Let $v=[\bar{x}, \bar{y}]$, $w=[\bar{x}, \bar{z}]$; then $u=[v, w]$ and using the identity

$$(*) \qquad (rs, t)=r(s, t)+(r, t)s$$

we obtain that $(u, \bar{x})=(v^{-1}w^{-1}(v, w), \bar{x})$ equals

$$(v^{-1}w^{-1}-1)(v, w, \bar{x})+(v, w, \bar{x})+(v^{-1}w^{-1}, \bar{x})(v, w)\,.$$

The third summand is in Δ^7 so that modulo Δ^8 it takes the form $b((x, y), (x, z))$, and hence is in J. Similarly, the first summand is in Δ^7, so that modulo Δ^8 it becomes a multiple of $((x, y), (x, z), x)$ and thus belongs to J. The second summand, upon further expansion of v, w, and using $\bar{a}^{-1}=1+a+a^2+\cdots$, is congruent modulo Δ^8 to

$$\begin{aligned}
&((x, y), (x, z), x)\\
&\quad+\sum_{i+j=1}^{3}(x^i(x, y), x^j(x, z), x)+\sum_{i+j=1}^{3}(x^i(x, y), z^j(x, z), x)\\
&\quad+\sum_{i+j=1}^{3}(y^i(x, y), x^j(x, z), x)+\sum_{i+j=1}^{3}(y^i(x, y), z^j(x, z), x)\\
&\quad+(xy(x, y), (x, z), x)+((x, y), xz(x, z), x)\,.
\end{aligned}$$

A typical element in the first three lines of this sum would be $(x^i(x, y), z^j(x, z), x)$; since $x^i(x, y)=(x^{i+1}, y)-(x^i, yx)$ by $(*)$ we see that these elements are actually in $((K, K), (K, K), K)$. Therefore,

$$(u, \bar{x})\equiv(xy(x, y), (x, z), x)+((x, y), xz(x, z), x) \quad \bmod J\,.$$

Finally, repeatedly applying $(*)$, we obtain

$$\begin{aligned}
&(xy(x, y), (x, z), x)\\
&\quad=(xy((x, y), (x, z)), x)+((x, z, xy)(x, y), x)\\
&\quad\equiv(x, z, xy)(x, y, x)+(x, z, xy, x)(x, y)\\
&\quad\equiv(x(x, z, y), x)(x, y)+((x, z, x)y, x)(x, y)\\
&\quad\equiv(y, x)^2(z, x, x)
\end{aligned}$$

modulo J. Similarly, $((x, y), xz(x, z), x)\equiv(z, x)^2(y, x, x)$. Thus, we have proved that $[[\bar{x}, \bar{y}], [\bar{x}, \bar{z}], \bar{x}]-1\in J$ if and only if $(y, x)^2(z, x, x)+(z, x)^2(y, x, x)\in J$.

Each monomial has its frequency pattern (the triple (i, j, k) telling that x occurs i times, y occurs j times and z occurs k times in that monomial); and to each triple there is an additive map of K which leaves monomials of that frequency pattern invariant and annihilates all other monomials. Since J is invariant under these frequency maps, and the two summands of $(y, x)^2(z, x, x) + (z, x)^2(y, x, x)$ have different frequency patterns, it follows that $(y, x)^2(z, x, x) + (z, x)^2(y, x, x) \in J$ iff $(y, x)^2(z, x, x)$ and $(z, x)^2(y, x, x)$ are in J. This completes the proof of Lemma 1. $\square$

For the proof of Lemma 2 we shall need the following:

Lemma 3.

$$\begin{aligned} &K((K, K), (K, K), K)K \cap \Delta^7 \\ &\quad \subseteq ((\Delta^3, \Delta), \Delta_2, \Delta) + ((\Delta^2, \Delta^2), \Delta_2, \Delta) \\ &\qquad + ((\Delta^2, \Delta), (\Delta^2, \Delta), \Delta) + \Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3 + \Delta^8. \end{aligned}$$

Furthermore, each of

$$((\Delta\Delta_2, \Delta), \Delta_2, \Delta)\,, \quad ((\Delta_2, \Delta^2), \Delta_2, \Delta)\,, \quad ((\Delta_2, \Delta), (\Delta^2, \Delta), \Delta)$$

is contained in the ideal $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$.

Proof. Clearly, modulo $\Delta_4 \cap \Delta^7 + \Delta^8$,

$$\begin{aligned} &K((K, K), (K, K), K)K \cap \Delta^7 \\ &\quad \equiv ((\Delta^2, \Delta), \Delta_2, \Delta, \Delta) + \Delta((\Delta^2, \Delta), \Delta_2, \Delta) \\ &\qquad + ((\Delta^2, \Delta), \Delta_3, \Delta) + ((\Delta^2, \Delta), (\Delta^2, \Delta), \Delta) \\ &\qquad + ((\Delta^2, \Delta^2), \Delta_2, \Delta) + ((\Delta^3, \Delta), \Delta_2, \Delta)\,. \end{aligned}$$

Consider $((\Delta^2, \Delta), \Delta_2, \Delta, \Delta)$; we have, using $(*)$,

$$\begin{aligned} &((\Delta^2, \Delta), \Delta_2, \Delta, \Delta) \\ &\quad \subseteq (\Delta\Delta_2, \Delta_2, \Delta, \Delta) + (\Delta_2\Delta, \Delta_2, \Delta, \Delta) \\ &\quad \subseteq (\Delta\Delta_4, \Delta, \Delta) + (\Delta_3, \Delta_2, \Delta, \Delta) \\ &\qquad + (\Delta_4\Delta, \Delta, \Delta) + (\Delta_2\Delta_3, \Delta, \Delta)\,. \end{aligned}$$

Another application of $(*)$ shows that every summand is in $\Delta_4 \cap \Delta^7$.

In a similar way we have by $(*)$:

$$\begin{aligned}
&\Delta((\Delta^2, \Delta), \Delta_2, \Delta)\\
&\quad\subseteq \Delta(\Delta\Delta_2, \Delta_2, \Delta) + \Delta(\Delta_2\Delta, \Delta_2, \Delta)\\
&\quad\subseteq \Delta(\Delta\Delta_4, \Delta) + \Delta(\Delta_3\Delta_2, \Delta)\\
&\qquad + \Delta(\Delta_4\,\Delta, \Delta) + \Delta(\Delta_2\Delta_3, \Delta)\,.
\end{aligned}$$

Expanding one more time we see that each of the summands is contained in the ideal $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$. Futhermore,

$$((\Delta^2, \Delta), \Delta_3, \Delta) \subseteq ((\Delta^2, \Delta), (\Delta^2, \Delta), \Delta)\,.$$

Therefore modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3 + \Delta^8$,

$$K((K, K), (K, K), K)K \cap \Delta^7$$

is contained in

$$((\Delta^3, \Delta), \Delta_2, \Delta) + ((\Delta^2, \Delta^2), \Delta_2, \Delta) + ((\Delta^2, \Delta), (\Delta^2, \Delta), \Delta)\,.$$

The second statement of the lemma follows in much the same way. □

Proof of Lemma 2. Suppose $(y, x)^2(y, x, x) \in J^\theta$. By Lemma 3, J^θ is contained in

$$\begin{aligned}
&\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3 + \Delta^8 + ((\Delta^3, \Delta), \Delta_2, \Delta)^\theta\\
&\quad + ((\Delta^2, \Delta^2), \Delta_2, \Delta)^\theta + ((\Delta^2, \Delta), (\Delta^2, \Delta), \Delta)^\theta\,.
\end{aligned}$$

Hence, modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3 + \Delta^8$, $(y, x)^2(y, x, x)$ is a linear combination of elements from

$$((\Delta^3, \Delta), \Delta_2, \Delta)^\theta\,, \quad ((\Delta^2, \Delta^2), \Delta_2, \Delta)^\theta\,, \quad \text{and} \quad ((\Delta^2, \Delta), (\Delta^2, \Delta), \Delta)^\theta\,.$$

Furthermore, we may assume that the summands in this linear combination are all of the frequency pattern $(4, 3, 0)$. We shall show that such elements are actually in $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$. Let us look at $((abc, d), (e, f), g) \in ((\Delta^3, \Delta), \Delta_2, \Delta)^\theta$ and assume that its frequency pattern is $(4, 3, 0)$. Modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$,

$$((abc, d), (e, f), g) \equiv ((a^\pi b^\pi c^\pi, d), (e, f), g)$$

for all permutations π of the set $\{a, b, c\}$—by Lemma 3; hence we only have four possibilities:

$$((y^2x, x), (x, y), x) \quad ((x^2y, x), (x, y), y),$$

$$((x^2y, y), (x, y), x) \quad \text{and} \quad ((x^3, y), (x, y), y).$$

Modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$,

$$\begin{aligned}
&((x^2y, x), (x, y), y) \\
&\quad = ((x^2(y, x), (x, y)), y) \\
&\quad \equiv ((x, y, x^2)(x, y), y) \\
&\quad \equiv (x(x, y, x)(x, y), y) + ((x, y, x)x(x, y), y) \\
&\quad \equiv 2(x, y)(x, y)(x, y, x) = 0 .
\end{aligned}$$

Similar calculations will prove that the other elements are in $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$, too. Consider $((ab, cd), (e, f), g) \in ((\Delta^2, \Delta^2), \Delta_2, \Delta)^\theta$, having frequency pattern $(4, 3, 0)$; modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$, and using Lemma 3, the nontrivial choices are $((x^2, y^2), (x, y), x)$ and $((x^2, xy), (x, y), y)$. Repeated application of (*) will show that each of these is in $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$. Finally, let us look at a generator $((ab, c), (de, f), g)$ of $((\Delta^2, \Delta), (\Delta^2, \Delta), \Delta)^\theta$. Modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$, we may assume that x precedes y in the expressions ab and de (by Lemma 3); hence, the generator is congruent to one of:

$$((x^2, y), (xy, y), x), \quad ((x^2\ y), (xy, x), y), \quad ((x^2, y), (xy, x), x),$$

$$((xy, y), (xy, x), x), \quad ((xy, x), (y^2, x), x).$$

(Using, of course, that the frequency pattern is $(4, 3, 0)$.) Expanding these by (*) will show that each of them is zero modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$. For example,

$$\begin{aligned}
&((xy, y), (xy, x), x) \\
&\quad = ((y, x)y, x(y, x), x) \\
&\quad = ((y, x)(y, x(y, x)), x) + (((y, x), x(y, x))y, x) \\
&\quad \equiv ((y, x)^3, x) + ((y, x, x)(y, x)y, x) \\
&\quad \equiv 3(y, x)^2(y, x, x) + (y, x)^2(y, x, x) = 0 .
\end{aligned}$$

Therefore, we are forced to conclude that modulo Δ^8, $(y, x)^2(y, x, x)$ is a linear combination of elements from $\Delta_4 \cap \Delta^7$ and $\Delta\Delta_3\Delta_3$. But this yields a nontrivial relation between the basic products of degree 7—which is impossible (see [2, Theorem 5.8]). Lemma 2 is thus proved. □

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