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On the complexity of the approximation of nonplanarity parameters for cubic graphs☆

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Abstract

Let $G=(V,E)$ be a simple graph. The NON-PLANAR DELETION problem consists in finding a smallest subset $E' \subset E$ such that $H=(V,E \backslash E')$ is a planar graph. The SPLITTING NUMBER problem consists in finding the smallest integer $k \geqslant 0$, such that a planar graph H can be defined from G by k vertex splitting operations. We establish the Max SNP-hardness of SPLITTING NUMBER and NON-PLANAR DELETION problems for cubic graphs.

Keywords: Topological graph theory; Complexity classes; Computational difficulty of problems; Splitting number; Maximum planar subgraph

1. Introduction

Let $G=(V,E)$ be a simple graph. The NON-PLANAR DELETION problem consists in finding a smallest subset $E' \subset E$ such that $H=(V,E \setminus E')$ is a planar graph. The MAXIMUM PLANAR SUBGRAPH problem consists in finding a largest subset $E' \subset E$ such that $H=(V,E')$ is a planar graph. Given $u \in V(G)$, say that a graph H is obtained from G by *splitting* vertex u if $V(H)=(V(G) \setminus \{u\}) \cup \{u_1,u_2\}$ and $E(H)=(E(G) \setminus \{(u,x) : x \in N(u)\}) \cup \{(u_1,x) : x \in N_1\} \cup \{(u_2,x) : x \in N_2\}$, where $N(u)$, the neighborhood of u in G, is partitioned into non-empty sets N_1 and N_2. The SPLITTING NUMBER problem

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consists in finding the smallest integer $k \geqslant 0$, such that a planar graph H can be defined from G by k splitting operations. In this work we establish the Max SNP-hardness of SPLITTING NUMBER and NON-PLANAR DELETION problems for cubic graphs.

A natural question in the study of the complexity of a graph-theoretical decision problem is to determine the best possible bounds on the vertex degrees for which the problem remains NP-complete. Yannakakis [7] considered the complexity of edge-deletion decision problems and obtained corresponding best possible vertex-degree bounds for the NP-completeness of the edge-deletion bipartite problem and of the edge-deletion comparability graph problem.

The NON-PLANAR DELETION decision problem was shown to be NP-complete by Yannakakis in the fundamental paper [7]. More recently, Călinescu et al. [3] showed that NON-PLANAR DELETION is Max SNP-hard, which implies [1] that there is a constant $\varepsilon > 0$ such that the existence of a polynomial approximation algorithm with performance ratio at least $1 + \varepsilon$ implies that P = NP. Both the NP-completeness and Max SNP-hardness proofs left the corresponding best possible vertex-degree bounds unanswered.

We have established [4] the complexity of the SPLITTING NUMBER decision problem by constructing a reduction from 3-SAT. We proved that the SPLITTING NUMBER decision problem is NP-complete when restricted to cubic graphs.

In the present paper, we prove that, for graphs with maximum degree 3, we have $Opt_{\mathrm{SN}}(G) = Opt_{\mathrm{NPD}}(G)$, where $Opt_{\mathrm{SN}}(G)$ and $Opt_{\mathrm{NPD}}(G)$ denote, respectively, the optimum values for SPLITTING NUMBER and NON-PLANAR DELETION of G. Consequently, the NP-completeness of the SPLITTING NUMBER decision problem when restricted to cubic graphs implies the NP-completeness of the NON-PLANAR DELETION decision problem when restricted to cubic graphs.

In order to establish that SPLITTING NUMBER and consequently that NON-PLANAR DELETION are Max SNP-hard even for cubic graphs, we use the concept of *L-reductions* [5], a special kind of reduction that preserves approximability. To achieve the optimum vertex-degree bound with respect to Max SNP-hardness, we have strengthened our initial NP-completeness proof [4] by considering this time the Max SNP-complete problem MAX3SAT$_{\bar{3}}$ [5], a restricted version of MAX3-SAT, where each variable appears at most three times in the set of clauses.

The published results [7,3] on the complexity of NON-PLANAR DELETION did not use graphs with maximum vertex degree 3. Thus, our complexity results for non-planarity parameters SPLITTING NUMBER and NON-PLANAR DELETION are optimum with respect to the allowed maximum vertex degree, because a graph with maximum degree 2 is a collection of paths and circuits that define a planar graph.

2. The Max SNP-hardness of splitting number

In this section we prove that SPLITTING NUMBER is Max SNP-hard, by L-reducing the Max SNP-complete problem MAX3SAT$_{\bar{3}}$ [5] to SPLITTING NUMBER. These two optimization problems are defined as follows:

$\text{MAX3SAT}_{\bar{3}}$

Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$ literals, and each variable appears at most three times in the set of clauses.

Goal: Find a truth assignment for U which maximizes the number of clauses in C having at least one true literal.

SPLITTING NUMBER

Instance: Graph G.

Goal: Find the smallest integer $k \geqslant 0$, such that a planar graph H can be defined from G by k splitting operations.

We construct in polynomial time a special instance G for SPLITTING NUMBER from a general instance I for $\text{MAX3SAT}_{\bar{3}}$. We follow the steps of the construction published in [4] where we have established the NP-completeness of SPLITTING NUMBER decision problem by reduction of 3–SATISFIABILITY to it. We need to adapt this published construction in order to obtain the claimed L-reduction from $\text{MAX3SAT}_{\bar{3}}$ to SPLITTING NUMBER optimization problem. In particular, the two main properties we are going to establish are:

- We establish bounds for the size of the parameters and the optimum value for $\text{MAX3SAT}_{\bar{3}}$ by proving that: if $I=(U,C)$ is an instance of $\text{MAX3SAT}_{\bar{3}}$ with $|U|=n$ variables and $|C|=m$ clauses, then $\lceil n/3 \rceil \leqslant Opt_{\text{MAX3SAT}_{\bar{3}}}(I) \leqslant m \leqslant n$.
- The special instance G for SPLITTING NUMBER constructed from a general instance I for $\text{MAX3SAT}_{\bar{3}}$ satisfies: $Opt_{\text{SN}}(G) = 4n + 5Opt_{\text{MAX3SAT}_{\bar{3}}}(I) + 6(m - Opt_{\text{MAX3SAT}_{\bar{3}}}(I))$.

2.1. The special instance G

The special instance G for SPLITTING NUMBER constructed from a general instance I for $\text{MAX3SAT}_{\bar{3}}$ contains two types of subgraphs: the *truth setting* (T_i) and the *satisfaction testing* (S_j) subgraphs defined, respectively, in Figs. 1(c) and (d). For each variable $u_i \in U$ there is a T_i. Note that each T_i is a modified $K_{3,3}$ (Figs. 1(a) and (b)), in the sense that the graph T_i can be obtained from the graph $K_{3,3}$ by replacing, as shown in Figs. 1(b) and (c), each one of the six vertices of $K_{3,3}$ by a supervertex, each one of six edges by a superedge, one edge by the graph *left side*, one edge by the graph *right side* and by the attachment to the bottom horizontal line of a square as defined in Figs. 1(b) and (c). For each clause $c_j \in C$ there is an S_j.

Each one of these two types of subgraphs has three types of vertices: white vertices that are supervertices, stripped vertices that are linking supervertices, and black vertices that are standard vertices. There are superedges linking supervertices (see Fig. 2). The construction of G is performed such that the subgraph of G induced by the vertices of the supervertices is a planar graph. We note in Fig. 2 that each supervertex has at the infinite face $3(4n+6m+1)$ standard vertices. This number of $3(4n+6m+1)$ standard vertices at the infinite face defines 3 sequences of $4n+6m+1$ consecutive standard vertices. Each sequence of $4n+6m+1$ standard vertices can be linked to a sequence of $4n+6m+1$ standard vertices in another supervertex in order to define a superedge. A supervertex adjacent to a standard vertex v has only one standard vertex adjacent to

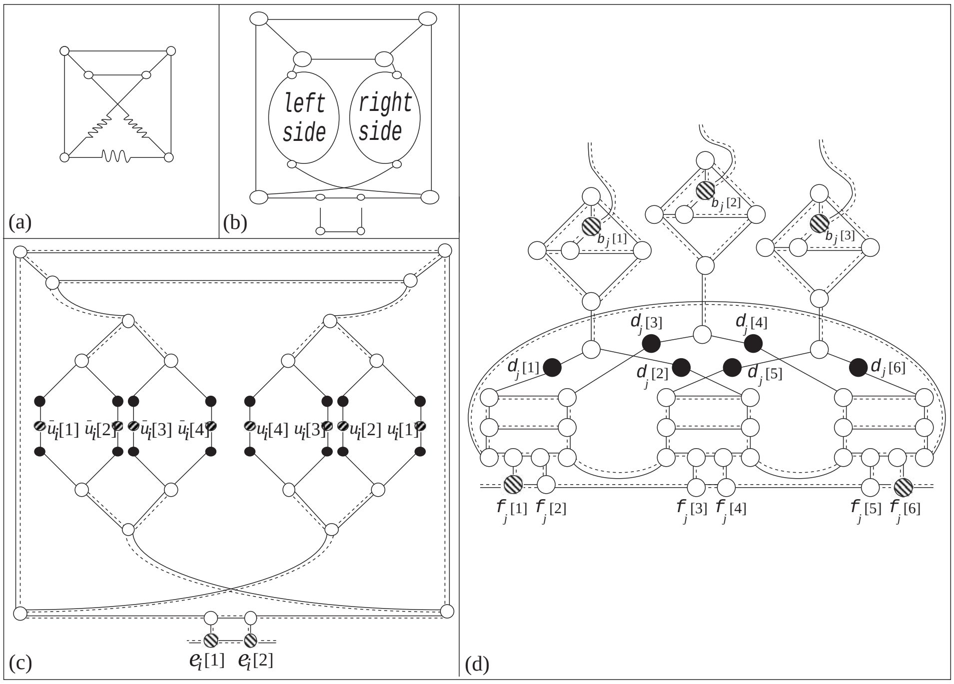


Fig. 1. (c) Truth-setting subgraph T_i and (d) satisfaction-testing subgraph S_j.

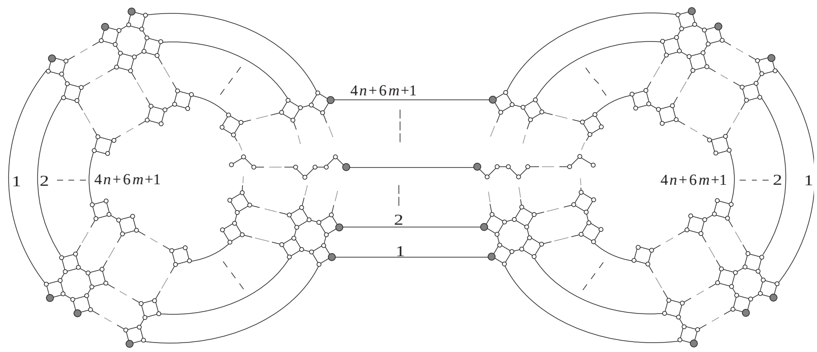


Fig. 2. Two supervertices and one superedge.

v, which is the first standard vertex in the clockwise direction of one of the sequences of $4n+6m+1$ standard vertices. As we will see in the sequel, these supervertices and superedges are big enough to ensure that the number of splittings needed to obtain a planar graph from G by splitting a supervertex is greater than $4n+6m$. This key property is used in our proof to forbid splittings in white or stripped vertices.

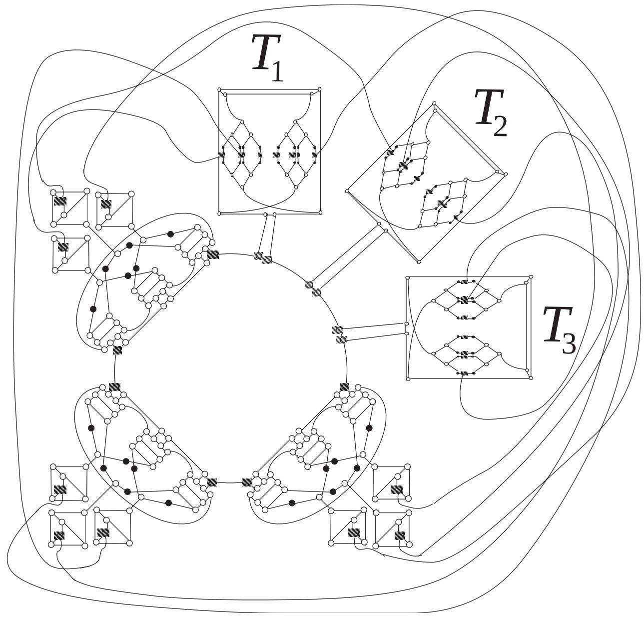


Fig. 3. Graph G obtained from the MAX3SAT$_{\bar{3}}$ instance $I=(U,C)=(\{u_1,u_2,u_3\},\{(u_1\vee\bar{u}_2\vee\bar{u}_3),(\bar{u}_1\vee u_2\vee\bar{u}_3),(\bar{u}_1\vee\bar{u}_2\vee u_3)\})$.

The only part in the construction of G that depends on which literals occur in which clauses is the following collection of edges produced sequentially when j grows from 1 until m. Let x_a, x_b and x_d, be the three literals in clause c_j. Note that a literal $x_a = u_a$, if x_a is a positive literal; and $x_a = \bar{u}_a$, if x_a is a negative literal with $a \in \{1,2,\ldots,n\}$. We associate literal x_a to a vertex in the set $\{u_a[l] : 1 \leqslant l \leqslant 4\}$ of T_a, if x_a is a positive literal; or to a vertex in the set $\{\bar{u}_a[l] : 1 \leqslant l \leqslant 4\}$ of T_a, if x_a is a negative literal. We denote this vertex associated to x_a by $x_a[l_a]$. Thus, have the following sets of edges emanating of the subgraphs T_a, T_b, T_d, and S_j: $\{(b_j[1],x_a[l_a]),(b_j[2],x_b[l_b]),(b_j[3],x_d[l_d])\}$, where, for $s \in \{a,b,d\}$, l_s is the minimum number in the set $\{1,2,3,4\}$ such that there is no vertex $b_{j'}[h]$, $h \in \{1,2,3\}$ linked to $x_s[l_s]$ with $j' \leqslant j$.

There is a set of edges, called the *ring* connecting the subgraphs T_i and S_j: $[\bigcup_{i=1}^{n-1}\{(e_i[2],e_{i+1}[1])\}] \cup [\bigcup_{j=1}^{m-1}\{(f_j[6],f_{j+1}[1])\}] \cup \{(e_n[2],f_1[1]),(f_m[6],e_1[1])\}$ (see Fig. 3).

We observe that $Opt_{\mathrm{SN}}(G) \leqslant 4n+6m$. This upperbound can be justified as follows. We can define a set Z with 4 splittings in a suitable set of black vertices in each one of the n subgraphs T_i's (either in left side or in the right side) totalizing $4n$ splittings, such that, we remove the crossings among the edges of each T_i; and with 6 splittings, one in each black vertex of each one of the m subgraphs S_j's, this subset with $6m$ splittings remove the crossings in each S_j allowing to define a plane drawing for each subgraph $K_{3,3}\setminus\{e\}$ of S_j that can be embedded in a suitable planar region of a resulting subgraph from the T_i's.

Let G be a simple graph and $e=(u,v)\in E(G)$. Say that a graph Q is obtained from G by *contracting* e if $V(Q)=(V(G)\setminus\{u,v\})\cup\{w\}$ and $E(Q)=(E(G)\setminus\{(u,x),(v,y): x\in N(u), y\in N(v)\})\cup\{(w,x): x\in((N(u)\cup N(v))\setminus\{u,v\})\}$, where $w\notin V(G)$; and $N(u)$ and $N(v)$ are, respectively, the neighborhood of u and v. Say that a graph G is *contractible* to a graph Q if there is a sequence of graphs $G=G_0, G_1, G_2, G_3, \ldots, G_k=Q$, where G_{i+1} is obtained from G_i by contracting $e\in G_i$. If a graph G is connected, then G is contractible to a graph with one vertex Q, since the resulting graph of a contraction has one less vertex and is still connected. Consider a connected subgraph of a graph G induced by a subset $S\subset V(G)$. Say that a graph Q is obtained from a graph G by *contracting the set S of vertices to a single vertex* if Q is obtained from G by a sequence of contractions defining a graph with one vertex from the subgraph of G induced by S.

Let G be the graph defined from the instance $I=(U,C)$ of MAX3SAT$_{\bar{3}}$. We say that two supervertices s_1 and s_2 are *adjacent* in G if there are standard vertices $x_1\in V(s_1)$ and $x_2\in V(s_2)$, such that $(x_1,x_2)\in E(G)$. Let Z be a set of splittings defining a graph H from G. Let Q be the graph obtained from the subgraph of G induced by the set of vertices of the supervertices of G, by contracting each set of vertices of each supervertex to a single vertex. We say that *no supervertex is split* in Z if H has a subgraph contractible to Q.

Lemma 1. *Let Z be a set of splittings defining a graph H from G. If $|Z|\leqslant 4n+6m$, then no supervertex is split in Z.*

Proof. Let Q be the graph obtained from the subgraph of G induced by the set of vertices of the supervertices of G, by contracting each set of vertices of each supervertex to a single vertex. Let s_1 and s_2 be two adjacent supervertices in G and let s_1+s_2 be the graph induced by the vertices of s_1 and s_2. Note that there are $4n+6m+1$ vertex disjoint cycles each in $3(4n+6m+1)$ vertices in each supervertex s_1 and s_2. Note that there are $4n+6m+1$ vertex disjoint paths in $6(4n+6m+1)$ vertices with vertices in each one of the $4n+6m+1$ cycles as shown in Fig. 2.

Since $|Z|\leq 4n+6m$, there are at least one cycle contained in s_1, one cycle contained in s_2, and one path contained in s_1+s_2 with no splitting in Z. For every pair of adjacent supervertices in G, let H' be the subgraph of H induced by the set of vertices of these cycles and paths. The resulting graph from H' by contracting each one of these cycles to a single vertex is isomorphic to Q. Hence, no supervertex is split in Z. □

Let Z be a set of splittings defining a graph H from G, with $|Z|\leqslant 4n+6m$, and let G' be a subgraph of G containing a set S of supervertices. Start with $C=\emptyset$ and $P=\emptyset$. For each supervertex s of S, add to C the set of $4n+6m+1$ vertex disjoint cycles of s each cycle in $3(4n+6m+1)$ vertices. For each pair of adjacent supervertices in G' add to P the set of $4n+6m+1$ vertex disjoint paths each path in $6(4n+6m+1)$ vertices. The *resulting graph from the supervertices* of G' in H is the subgraph of H induced by the vertices of the cycles of C and of the paths of P with no vertices in Z.

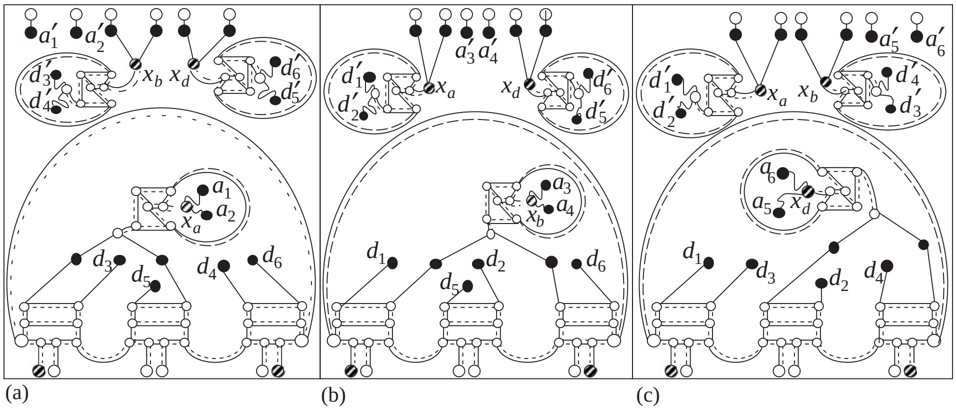


Fig. 4. A set Z' of splittings with $|Z'| = 5$.

Theorem 2 (Fundamental Property of the construction of G from I). *Let $I = (U, C)$ be an instance of* MAX3SAT$_{\bar{3}}$, *with $|U| = n$ and $|C| = m$. A truth assignment for U with c satisfied clauses defines a feasible set of splittings Z' for G of size $|Z'| = 4n + 5c + 6(m - c)$. Conversely, if Z' is a feasible set of splittings for G of size $|Z'| \leqslant 4n + 6m$, then there exists a subset $Z'' \subseteq Z'$ such that Z'' is a feasible set of splittings for G whose size satisfies $|Z'| \geqslant |Z''| = 4n + 5c + 6(m - c)$. Moreover, Z'' defines a truth assignment for U with c satisfied clauses, where each graph T_i requires* 4 *splittings in Z'', each graph S_j corresponding to a satisfied clause requires* 5 *splittings in Z'', and each graph S_j corresponding to a non-satisfied clause requires* 6 *splittings in Z''.*

Proof. Suppose a truth assignment for U with c satisfied clauses is given. We shall define a suitable set Z' of splittings proving the first part of the Fundamental Property. First of all, for each $i \in \{1, 2, 3, \ldots, n\}$ we define a planar graph from T_i by adding to Z' either 4 black vertices on the right side of T_i if u_i is true, or 4 black vertices on the left side of T_i if u_i is false. We remark that this set of $4n$ splittings defines a planar resulting graph from each T_i. For each clause $c_j = (x_a \vee x_b \vee x_d)$, Fig. 4 shows that if c_j is a satisfied clause, then it is enough to add 5 splittings to Z' in order to define a planar graph from S_j, in this case each S_j can borrow one splitting in T_a (Fig. 4(a)), T_b (Fig. 4(b)) or T_d (Fig. 4(c)), according to if the corresponding literal with value true in c_j is x_a, x_b or x_d. If c_j is a non-satisfied clause, then it is enough to add to Z' the 6 splittings of one of the sets of splittings in Figs. 4(a), (b) or (c) in order to define a planar graph from S_j. This completes the definition of Z'. Since, all the resulting graphs from the T_i's and S_j's are planar and disjoint in vertices we have that Z' defines a planar resulting graph from G. Hence, the set Z' of splittings is a feasible solution of size $|Z'| = 4n + 5c + 6(m - c)$, as required.

Now we prove the second part of the Fundamental Property. Let F be the planar graph that Z' defines from G. Since $|Z'| \leqslant 4n + 6m$, Lemma 1 says that no supervertex is split. Since F is planar and no supervertex is split, for each $i \in \{1, 2, 3, \ldots, n\}$ the resulting graph of the left side of T_i in F has white vertices in different connected components, or the resulting graph of the right side of T_i has white vertices in different connected components in F.

Let L_i be the subset of Z' on the side of T_i having white vertices in different connected components in F. Consider the truth assignment that sets $u_i = T$ if and only if the subset L_i is on the right side of T_i. Let c be the number of clauses of C satisfied by this truth assignment. Since the first part of the Fundamental Property ensures that there is a feasible solution Z'' of size $4n + 5c + 6(m - c) = |Z''|$, it is enough to prove that $|Z''| \leqslant |Z'|$.

By definition $4n \leqslant |\bigcup_{i=1}^{n} L_i|$. Let G' be the graph that $\bigcup_{i=1}^{n} L_i$ defines from G. By definition, the set $\bigcup_{i=1}^{n} L_i$ partitions the set of subgraphs S_j's into two sets, the set of the S_j's that correspond to satisfied clauses and the set of the S_j's that correspond to non-satisfied clauses.

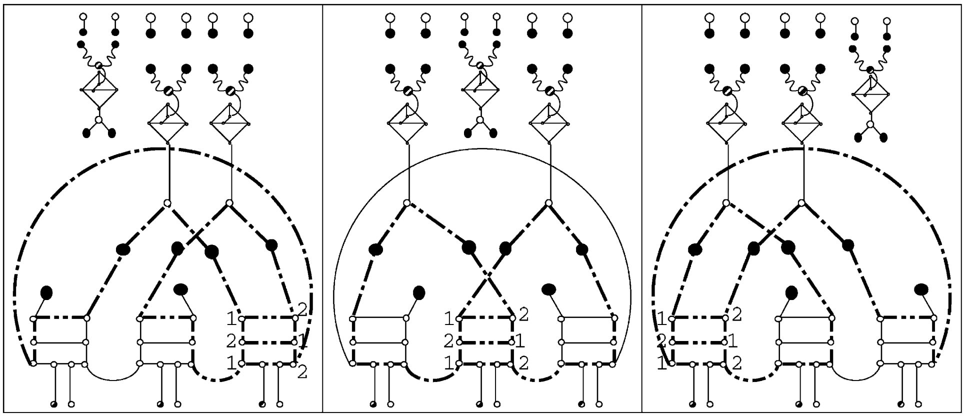
Let H be the subgraph of G' induced by the vertices of S_j and the vertices of the resulting graphs from T_a, T_b and T_d in G', where $c_j = (x_a \vee x_b \vee x_d)$. We consider two cases:

(1) We prove that if S_j corresponds to a non-satisfied clause c_j, then S_j requires 6 additional splittings in Z'. Note that, there is a graph $K_{3,3} - e$ in supervertices of S_j with a supervertex adjacent to a stripped vertex of T_a. For simplicity, we say that there is a graph $K_{3,3} - e$ of S_j adjacent to T_a. In addition, this stripped vertex of T_a is adjacent to 2 standard black vertices of T_a. Note that, if in the resulting graph of H in F, the resulting graph from the supervertices of this $K_{3,3} - e$ is in the same connected component as the resulting graph from the white vertices of S_j incident to the ring and the white vertices of T_a, then there is a subdivision of $K_{3,3}$ in the planar graph F, a contradiction. The same argument is valid for the 2 $K_{3,3} - e$'s of S_j adjacent to T_b or of T_d.

Hence, each $K_{3,3} - e$ requires at least 2 splittings in S_j or at least 2 splittings in T_a, T_b or T_d. Since there are $(m - c)$ non-satisfied clauses, there are $6(m - c)$ additional splittings in Z'.

(2) We prove that if S_j corresponds to a satisfied clause c_j, then S_j requires 5 additional splittings in Z' besides the $4n + 6(m - c)$ splittings required by the set $\bigcup_{i=1}^{n} L_i$ and by the subgraphs S_j corresponding to non-satisfied clauses. In Fig. 5 we define three non-planar graphs H_i, $i = 1, 2, 3$. In Fig. 5, we depict in H_1, H_2 and H_3 a subdivision for $K_{3,3}$, as a subgraph. For the convenience of the reader, we label the two color classes with 1 and 2, respectively. Each graph H_1, H_2 and H_3 corresponds to the resulting subgraph from a subgraph of H defined by a set of splittings with 2 splittings in each T_a, T_b and T_d, and 2 splittings in S_j. We use H_1, H_2 and H_3 in order to show that a subset of the set of splittings which defines H_1, H_2 or H_3 from H still defines a non-planar graph from H.

We remark that, if the resulting graphs in F from the 3 $K_{3,3} - e$'s in S_j adjacent to T_a, T_b and T_d are, respectively, in the same component as T_a, T_b and T_d, in the resulting graph of H in F, then there are at least 6 splittings of Z' in the vertices of S_j, since each one of the 3 $K_{3,3} - e$'s requires 2 splittings at the vertices of S_j in Z'.



(a) (b) (c)

Fig. 5. Subdivision of $K_{3,3}$ as subgraph of H_i, $i = 1, 2, 3$.

We observe also that at least 2 splittings are required in Z' at the vertices of S_j because no supervertex is split and because of the depicted subdivisions of $K_{3,3}$ in Figs. 5(a)–(c).

Note that in H there are no splittings in the vertices of S_j. Hence for H, there are four possibilities according to the number of connected components containing the white vertices of S_j, T_a, T_b or T_d in H being 1, 2, 3 or 4. We consider these four possibilities next.

(a) The first case is when the vertices of S_j are in a different connected component of H with respect to the resulting graphs from the white vertices of T_a, T_b and T_d. In this case, for each graph T_a, T_b and T_d there is 1 additional splitting in $\bigcup_{i=1}^{n} L_i$. These 3 splittings, plus the 2 additional splittings in Z' at the vertices of S_j, yield $3 + 2 = 5$ additional splittings in Z'.

(b) The second case is when the vertices of S_j are in a different connected component of H with respect to the vertices of two of the resulting graphs from the white vertices of T_a, T_b and T_d, say T_a and T_b. In this case, there are 2 additional splittings in $\bigcup_{i=1}^{n} L_i$, one in T_a and one in T_b. We consider two different subcases. If the resulting graph in F of the white vertices of the $K_{3,3} - e$ in S_j adjacent to T_d is not in the same component as the resulting graph of the white vertices of T_d, then there is 1 additional splitting in Z' in the vertices of T_d and case (a) above shows that 2 additional splittings are required in Z' at the vertices of S_j, yielding $2 + 1 + 2 = 5$ additional splittings in Z'. If the resulting graph in F of the white vertices of the $K_{3,3} - e$ in S_j adjacent to T_d is in the same component as the resulting graph of the white vertices of T_d, then the 2 additional splittings in S_j defined in Fig. 5(c) are required in Z', and the $K_{3,3}$ depicted in this figure shows that 1 additional splitting is required in Z' in the vertices of S_j, yielding $2 + 2 + 1 = 5$ additional splittings in Z'. Figs. 5(a) and (b) can be used, analogously, in the analysis when the vertices of S_j are in a different connected component of H with respect to the vertices of a different pair of T_a and T_b in the set $\{T_a, T_b, T_d\}$.

(c) The third case is when the vertices of S_j are in a different connected component of H with respect to the vertices of exactly one of the resulting graphs from the white vertices of T_a, T_b and T_d in H, say T_a. In this case, there is 1 additional splitting in $\bigcup_{i=1}^{n} L_i$ in the vertices of T_a. If the resulting graph in F of the white vertices of one of the $K_{3,3} - e$'s of S_j adjacent to T_b or T_d, say in T_b, is not in the same component as the resulting graph of the white vertices of T_b, then we have 1 additional splitting in T_b and cases (a) and (b) above show that S_j requires at least 3 additional splittings in Z', yielding $1+1+3=5$ additional splittings in Z'. If the resulting graph in F of the white vertices of the 2 $K_{3,3}-e$'s of S_j adjacent to T_b and T_d are, respectively, in the same connected component as the white vertices of T_b and T_d in the resulting graph of H in F, then 4 additional splittings are required at the vertices of S_j in Z': 2 required by the $K_{3,3}-e$ adjacent to T_b and 2 required by the $K_{3,3}-e$ adjacent to T_d, which yields $1+4=5$ additional splittings in Z'.

(d) The fourth case is when the vertices of S_j are in the same connected component as the resulting graphs from the white vertices of T_a, T_b and T_d in H. If the resulting graph in F of the white vertices of one of the $K_{3,3} - e$'s of S_j adjacent to T_a, T_b or T_d, say T_a, is not in the same connected component as the white vertices of T_a in the resulting graph of H in F, then there is 1 additional splitting in T_a and cases (a)–(c) above show that S_j requires at least 4 additional splittings in Z'. If the resulting graphs in F of the white vertices of the $K_{3,3} - e$'s of S_j adjacent to T_a, T_b and T_d are respectively, in the same connected component as the white vertices of T_a, T_b, and T_d in the resulting graph of H in F, then 6 additional splittings are required in S_j in Z'.

Hence, for each one of the c satisfied clauses at least 5 additional splittings are required besides the $4n + 6(m-c)$ splittings required in $\bigcup_{i=1}^{n} L_i$ and in the set of the non-satisfied clauses, this means that $4n + 5c + 6(m-c) = |Z''| \leqslant |Z'|$. □

Figs. 6(a), (b) and 7 give an example where a set Z' of splittings defines a planar graph F from G which is the graph obtained in turn from the instance of MAX3SAT$_{\bar{3}}$: $I=(U,C)=(\{u_1,u_2,u_3\},\{(u_1 \vee \bar{u}_2 \vee \bar{u}_3),(\bar{u}_1 \vee u_2 \vee \bar{u}_3),(\bar{u}_1 \vee \bar{u}_2 \vee u_3)\})$. Fig. 6(a) shows the graph G. Fig. 6(b) shows the graph G' obtained from G by a set of 4×3 splittings defined by the truth assignment $u_1=u_2=u_3=T$. Fig. 7 shows a plane drawing for the graph F obtained from G by a set Z' of splittings. Note that in this example we have a satisfying truth assignment, which defines the size $|Z'| = 4\times 3 + 5\times 3 + 6(3-3)$.

2.2. The L-reduction

Let A and B be two optimization problems. We say that A *L-reduces* to B if there are two polynomial-time algorithms f and g, and positive constants α and β, such that for each instance I of A,

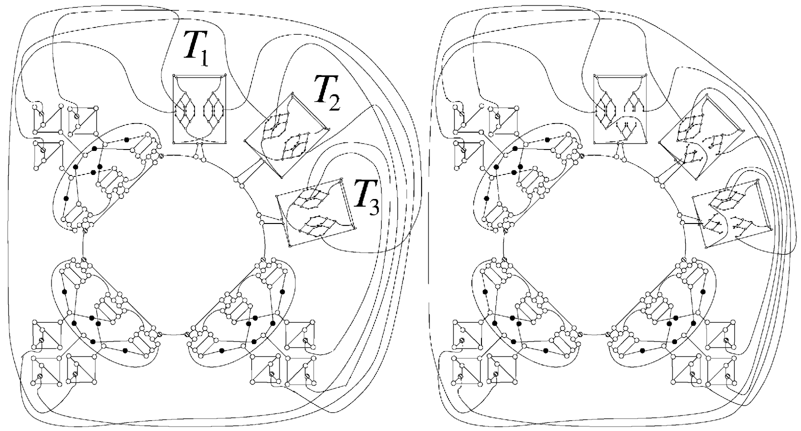


Fig. 6. Graph G (a) obtained from the MAX3SAT$_{\bar{3}}$ instance $I = (U, C) = (\{u_1, u_2, u_3\}, \{(u_1 \vee \bar{u}_2 \vee \bar{u}_3), (\bar{u}_1 \vee u_2 \vee \bar{u}_3), (\bar{u}_1 \vee \bar{u}_2 \vee u_3)\})$. Set of 4×3 splitting. (b) defined with satisfying truth assignment $u_1 = u_2 = u_3 = T$.

(1) algorithm f produces an instance $I' = f(I)$ of B such that the optima of I and I', satisfy $Opt_B(I') \leqslant \alpha.Opt_A(I)$;
(2) given any feasible solution of I' with cost c', algorithm g produces a solution of I with cost c such that $|c - Opt_A(I)| \leqslant \beta.|c' - Opt_B(I')|$.

Given an instance $I = (U, C)$ for MAX3SAT$_{\bar{3}}$, we first establish in Lemma 3 bounds for the size of an instance I and for the size of its optimum value.

Lemma 3. *If $I = (U, C)$ is an instance of* MAX3SAT$_{\bar{3}}$ *with $|U| = n$ and $|C| = m$, then* $\lceil n/3 \rceil \leqslant Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) \leqslant m \leqslant n$.

Proof. Consider $I = (U, C)$ an instance of MAX3SAT$_{\bar{3}}$ with $|U| = n$ and $|C| = m$. Since each variable occurs at most 3 times in the set of clauses, the number m of clauses satisfies $3m \leqslant 3n$. Therefore we have the inequality $m \leqslant n$, as required.

Now in order to establish the claimed bounds for $Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I)$, note first that $Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) \leqslant m$. Now to establish the claimed lower bound, it is enough to exhibit a truth assignment for I with $\lceil n/3 \rceil$ satisfied clauses. For each variable $u_i \in U$, $i \in \{1, 2, \ldots, n\}$, set $u_i = T$, if and only its positive literal occurs in C. Note that this truth assignment for U can be defined in time polynomial in the size of I. Now to each variable u_i we have a corresponding literal x_i with value true. Let k be the minimum number of clauses that fit those n literals with value true. Since each clause has size 3, integer k is the least integer satisfying $3k \geqslant n$, i.e., $k = \lceil n/3 \rceil$ is the least integer greater than or equal to $n/3$. Hence, we have at least $\lceil n/3 \rceil$ satisfied clauses, and we have the inequalities $\lceil n/3 \rceil \leqslant Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) \leqslant m$, as required. □

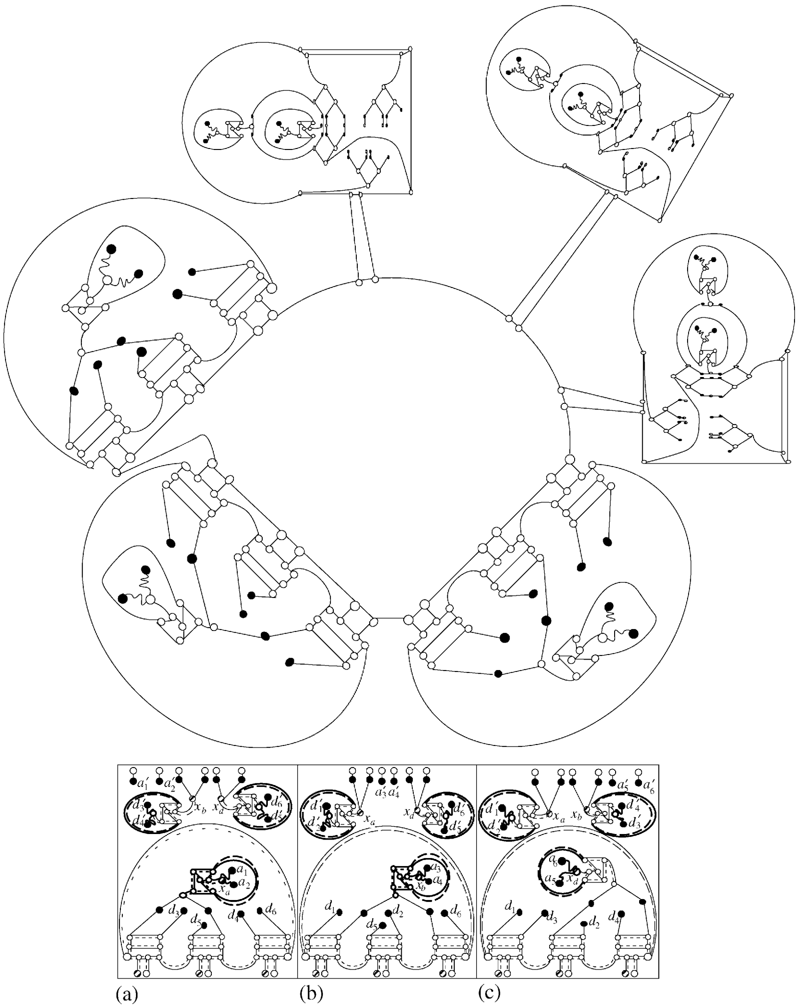


Fig. 7. Set Z' of $4 \times 3 + 5 \times 3$ splittings defining a planar graph from G.

Given an instance $I = (U, C)$ for MAX3SAT$_{\bar{3}}$, the polynomial-time algorithm f produces from I a graph G. We relate in Lemma 4 the optimum value for I to the optimum value for G.

Lemma 4. *If* $I = (U, C)$ *is an instance for* MAX3SAT$_{\bar{3}}$ *with* $|U| = n$, $|C| = m$, *and* $f(I) = G$, *then*

$$Opt_{\mathrm{SN}}(G) = 4n + 5Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + 6(m - Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I)).$$

Proof. Consider first a truth assignment for I with $Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I)$ satisfied clauses. By Theorem 2, there exists a feasible solution Z' for G, i.e., a set Z' of splittings with size: $|Z'| = 4n + 5Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + 6(m - Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I))$, which defines a planar graph from G. This establishes the inequality: $Opt_{\mathrm{SN}}(G) \leqslant 4n + 5Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + 6(m - Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I))$.

On the other hand, let Z' be any feasible solution for G with size $|Z'| \leqslant 4n + 5Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + 6(m - Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I))$. Since $|Z'| \leqslant 4n + 6m$, by Theorem 2, there exists a truth assignment with c satisfied clauses such that $|Z'| \geqslant 4n + 5c + 6(m - c) = 4n + 6m - c \geqslant 4n + 6m - Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) = 4n + 5Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + 6(m - Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I))$, which establishes the claimed equality. □

We are now ready to define parameters α and β for the L-reduction and prove:

Theorem 5. SPLITTING NUMBER *is Max SNP-hard.*

Proof. Theorem 2 says that a truth assignment for U with c satisfied clauses defines a feasible solution Z' for $f(I) = G$ with size $|Z'| = 4n + 6m - c \leqslant 4n + 6m$. Hence, $Opt_{\mathrm{SN}}(G) \leqslant 4n + 6m$. Now, by applying Lemma 3 we get $Opt_{\mathrm{SN}}(G) \leqslant 4n + 6m \leqslant 4n + 6n = 10n = 30n/3 \leqslant 30\lceil n/3 \rceil \leqslant 30.Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I)$, which shows that $\alpha = 30$ suffices.

On the other hand, let us define algorithm g and constant β. For let Z' be a feasible solution for G with cost c', i.e., $c' = |Z'|$ is the size of this set of splittings Z' which defines a planar graph from G. We distinguish two cases for c': If $c' > 4n + 6m$, then choose as image of Z' under g any feasible solution for I, and let c be the number of clauses satisfied by this truth assignment. If $c' \leqslant 4n + 6m$, then choose by Theorem 2 as image of Z' under g a truth assignment for U with c satisfied clauses such that $|Z'| = c' \geqslant 4n + 5c + 6(m - c)$. Thus, by Lemma 4 we obtain $|Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) - c| = |-Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + c| = |(-6+5)Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + (-5+6)c| = |(-6+5)Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + (-5+6)c + (4-4)n + (6-6)m| = |4n + 5Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) + 6(m - Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I)) - 4n - 5c - 6(m - c)| = |Opt_{\mathrm{SN}}(G) - (4n + 5c + 6(m - c))|$. Now, since: $Opt_{\mathrm{SN}}(G) \leqslant 4n + 5c + 6(m - c) \leqslant c'$, we have that: $|Opt_{\mathrm{SN}}(G) - (4n + 5c + 6(m - c))| \leqslant |Opt_{\mathrm{SN}}(G) - c'|$. Therefore, $|Opt_{\mathrm{MAX3SAT}_{\bar{3}}}(I) - c| \leqslant |Opt_{\mathrm{SN}}(G) - c'|$, which shows that $\beta = 1$ suffices. This ends the L-reduction. □

3. Splitting number, non-planar deletion and cubic graphs

In Section 2 we have established the Max SNP-hardness of SPLITTING NUMBER. The special instance of SPLITTING NUMBER, the graph G constructed as image of a general instance I of $\mathrm{MAX3SAT}_{\bar{3}}$, is a graph of maximum degree 3.

For graphs of maximum degree 3, we have the following relationship between the problems SPLITTING NUMBER and NON-PLANAR DELETION:

Lemma 6. *Let G be a graph of maximum degree 3. Then, we have $Opt_{\mathrm{SN}}(G) = Opt_{\mathrm{NPD}}(G)$, where $Opt_{\mathrm{SN}}(G)$ and $Opt_{\mathrm{NPD}}(G)$ denote, respectively, the optimum values for* SPLITTING NUMBER *and* NON-PLANAR DELETION *of G.*

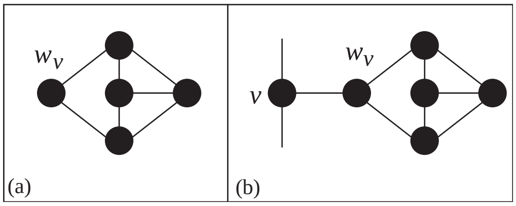


Fig. 8. Auxiliary graph for the proof of Corollary 9.

Proof. A leaf is a vertex of degree 1. Any splitting in a graph of maximum degree 3 yields one or two leaves. In addition, a crossing in the edge incident to a leaf can always be removed by considering a different drawing in the plane. Thus, if L is the set of leaves of G, then $Opt_{SN}(G) = Opt_{SN}(G \setminus L)$.

Let Z be a feasible solution of SPLITTING NUMBER for G, i.e., Z is a set of splittings which defines a planar graph H from G. Define a subset L of $V(H)$, $|L| = |Z|$, such that L is obtained from Z by adding to L one leaf obtained in each splitting of Z. By construction, the planar graph $H \setminus L$ is isomorphic to a subgraph of G with $|E(H \setminus L)| = |E(G)| - |Z|$, i.e., we have that $|Z| \geqslant Opt_{NPD}(G)$ and hence $Opt_{SN}(G) \geqslant Opt_{NPD}(G)$.

On the other hand, let L be a feasible solution of NON-PLANAR DELETION for G, i.e., L is a set of edges whose removal leaves a planar subgraph of G. Hence, a planar graph is also obtained from G by splitting, for each edge (u, v) of L one of its endpoints, say v, with degree greater than 1, into v_1 and v_2, such that $\{u\}$ is the neighborhood of v_1. Thus, we have that $|L| \geqslant Opt_{SN}(G)$, and hence $Opt_{NPD}(G) \geqslant Opt_{SN}(G)$. □

Corollary 7. NON-PLANAR DELETION *for graphs of maximum degree* 3 *is Max SNP-hard.*

Corollary 8. SPLITTING NUMBER and NON-PLANAR DELETION *are Max SNP-hard when restricted to graphs not containing a subdivision of* K_5 *as a subgraph.*

Proof. It follows from Theorem 5 and Corollary 7 because a graph of maximum degree 3 does not have a subdivision of K_5 as a subgraph. □

Corollary 9. SPLITTING NUMBER *and* NON-PLANAR DELETION *are Max SNP-hard for cubic graphs.*

Proof. By Lemma 6, it suffices to show that SPLITTING NUMBER is Max SNP-hard for cubic graphs. For, we use the strategy of Theorem 5 by modifying locally the graph G in Theorem 5 as follows. Consider the auxiliary graph G_v depicted in Fig. 8(a). For each vertex v of degree 2 in G, we add to G a copy of G_v, such that w_v is the vertex of G_v adjacent to v as shown in Fig. 8(b). □

4. Conclusion and further work

We have established that for cubic graphs there is a constant threshold $c > 1$ such that if SPLITTING NUMBER or NON-PLANAR DELETION can be approximated in polynomial time with ratio better than c, then $P = NP$.

Since MAXIMUM PLANAR SUBGRAPH and NON-PLANAR DELETION are complementary problems with respect to the number of edges of the graph, for the decision versions of these two problems, every result for NON-PLANAR DELETION is also a result for MAXIMUM PLANAR SUBGRAPH. In particular, Lemma 6 says that the NP-completeness of SPLITTING NUMBER for cubic graphs [4] implies both the NP-completeness of MAXIMUM PLANAR SUBGRAPH and of NON-PLANAR DELETION for cubic graphs.

The trivial polynomial-time approximation algorithm for MAXIMUM PLANAR SUBGRAPH produces a spanning tree and achieves a performance ratio of $\frac{1}{3}$: every spanning tree of a connected graph on n vertices has $n-1$ edges, and every planar graph on n vertices has at most $3n-3=3(n-1)$ edges.

Recently, Călinescu et al. [3] published the first non-trivial polynomial-time approximation algorithm for MAXIMUM PLANAR SUBGRAPH achieving a higher performance of $\frac{4}{9}$.

Note that a cubic graph on n vertices has $3n/2$ edges, hence the trivial polynomial-time approximation algorithm for MAXIMUM PLANAR SUBGRAPH achieves for cubic graphs a performance ratio of $\frac{2}{3}$, the best known. We are currently trying to obtain a non-trivial polynomial-time approximation algorithm for MAXIMUM PLANAR SUBGRAPH restricted to cubic graphs.

Note that if a graph G is dense (i.e., $|E(G)|=\Theta(n^2)$), then $Opt_{\mathrm{NPD}}(G)=\Theta(n^2)$. On the other hand, if a graph G is sparse (i.e., $|E(G)|=\mathrm{O}(n)$), then $Opt_{\mathrm{NPD}}(G)$ can be O(1). This stands in contrast with the fact that for a general connected graph G, we have $Opt_{\mathrm{MPS}}(G)=\Theta(n)$, given that a spanning tree has $n-1$ edges and that every planar graph has at most $3n-6$ edges. The fact that, for a given graph, the optima of NON-PLANAR DELETION and MAXIMUM PLANAR SUBGRAPH do not necessarily have the same order, implies that the identity map cannot be used as f in an L-reduction from NON-PLANAR DELETION and MAXIMUM PLANAR SUBGRAPH, and explains the difficulty in defining an L-reduction from NON-PLANAR DELETION to MAXIMUM PLANAR SUBGRAPH. Călinescu et al. [3] established both the Max SNP-hardness of MAXIMUM PLANAR SUBGRAPH and NON-PLANAR DELETION by presenting two distinct L-reductions from the same variant of the traveling salesman problem.

We are also trying to exhibit an L-reduction in order to establish the Max SNP-hardness, or to construct a better polynomial time approximation algorithm, for MAXIMUM PLANAR SUBGRAPH restricted to cubic graphs. We have two conjectures concerning the Max SNP-hardness of MAXIMUM PLANAR SUBGRAPH:

Conjecture 10. MAXIMUM PLANAR SUBGRAPH *is Max SNP-hard even when restricted to cubic graphs.*

The *girth* of a graph is the size of its smallest cycle.

Conjecture 11. SPLITTING NUMBER *is Max SNP-hard for cubic graphs with girth k, for some $k \geqslant 7$.*

Lemma 12. *The validity of Conjecture* 11 *implies the validity of Conjecture* 10.

Proof. Let H be a connected planar subgraph of G, with $V(H)=V(G)$. Its number of edges $|E(H)|$ satisfies $\sum_{f\in F} d(f)=2|E(H)|$, where F is the set of faces in a plane drawing of H, and $d(f)$ is the degree of a face f. Recall that the degree of a face f is defined to be the number of edges incident to its boundary with cut edges counted twice [2]. If H has girth at least 7, then $7|F| \leqslant \sum_{f\in F} d(f)=2|E(H)|$. By Euler's formula: $7|F|=7|E(H)|-7|V(G)|+14$, which implies $|E(H)| \leqslant (7|V(G)|-14)/5$.

Note that a cubic graph G has $3|V(G)|/2$ edges. Hence, $Opt_{\mathrm{NPD}}(G) \geqslant 3|V(G)|/2 - (7|V(G)| - 14)/5 = |V(G)|/10 + \frac{14}{5}$. Thus, $Opt_{\mathrm{NPD}}(G) > |V(G)|/10$. Therefore, $30.Opt_{\mathrm{NPD}}(G) > 30|V(G)|/10 = 3|V(G)| > 3|V(G)|/2 \geqslant Opt_{\mathrm{MPS}}(G)$.

Therefore, in order to define an L-reduction from NON-PLANAR DELETION to MAXIMUM PLANAR SUBGRAPH, we may take f as the identity map and $\alpha = 30$ in the L-reduction. To finish the L-reduction, it remains to define g and β. For, given a feasible solution for instance G of MAXIMUM PLANAR SUBGRAPH of cost c', take as its image by g the set of edges that are not in this planar subgraph. The cost of this feasible solution for NON-PLANAR DELETION is $c=|E(G)|-c'$. Since $Opt_{\mathrm{MPS}}(G)=|E(G)|-Opt_{\mathrm{NPD}}(G)$, then $|Opt_{\mathrm{NPD}}(G)-c| = |Opt_{\mathrm{MPS}}(G)-c'|$, and $\beta = 1$ suffices. □

A positive evidence for the validity of Conjecture 11 is the existence of an infinite number of cubic graphs with a fixed girth $k, k \geqslant 7$ [6].

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