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A Laplace variational iteration strategy for the solution of differential equations

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ABSTRACT

The aim of this article is to introduce a novel Laplace variational numerical scheme, based on the variational iteration method (VIM) and Laplace transform, for the solution of certain classes of linear and nonlinear differential equations. The strategy is outlined and then illustrated through a number of test examples. The results assert that this alternative approach yields accurate results, converges rapidly and handles impulse functions and the ones with discontinuities.

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1. Introduction

In this article, a novel Laplace variational iteration strategy, based on the (VIM) and Laplace transform, is presented for the exact/numerical solution of linear and nonlinear differential equations [1]. Recently, there has been extensive research interest in the (VIM) for solving a wide spectrum of equations including algebraic, differential, partial-differential, functional-delay and integro-differential equations [2–6]. The main thrust of the method is to construct a correction functional using a general Lagrange multiplier chosen in a proper way that its correction solution is improved with respect to the initial trial function.

The published results on the (VIM) failed to point out that, in numerous cases, the integral of the correction functional is a convolution, thus manipulation of Laplace transform ought to come into place. In this paper, we address this deficiency by introducing an alternative Laplace correction functional and expressing the integral as a convolution. This approach can tackle functions with discontinuities as well as impulse functions effectively. The method is implemented for a number of examples and the results confirm the simplicity, suitability and effectiveness of this technique using only few terms of the iterative scheme.

The balance in this paper is as follows. In Section 2, the Laplace transform variational iteration method will be presented. In Section 3, the approach is implemented for few test examples. Section 4 includes a conclusion that briefly summarizes the results.

2. Laplace transform variational iteration method

Consider the following general nonlinear differential equation:

$$Ly(x) + Ny(x) = f(x), \quad (2.1)$$

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where L is a linear operator, N is a nonlinear operator and $f(x)$ is a known analytical function. Before we begin the implementation, we shall present the variational iteration method scheme in constructing the correction functional. The (VIM) admits the use of the correction functional for Eq. (2.1) given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi) [Ly_n(\xi) + N\tilde{y}_n(\xi) - f(\xi)] d\xi, \quad n = 0, 1, 2, \dots, \quad (2.2)$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory. The subscript n denotes the n th approximation and \tilde{y}_n is a restricted variation ($\delta\tilde{y}_n = 0$). In a wide range of problems that appear in the literature, the general form of Lagrange multiplier is found to be of the form

$$\lambda = \bar{\lambda}(x - \xi).$$

In this article, we will make the assumption that λ is expressed in this latter way. In such a case, the integration is basically the convolution; hence Laplace transform is appropriate to use. Operating with Laplace transform on both sides of (2.2) the correction functional will be constructed in the following manner:

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] + \mathcal{L}\left[\int_0^x \bar{\lambda}(x - \xi) [Ly_n(\xi) + N\tilde{y}_n(\xi) - f(\xi)] d\xi\right], \quad n = 0, 1, 2, \dots \quad (2.3)$$

Therefore

$$\begin{aligned} \mathcal{L}[y_{n+1}(x)] &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x) * (Ly_n(x) + N\tilde{y}_n(x) - f(x))] \\ &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x)] \mathcal{L}[Ly_n(x) + N\tilde{y}_n(x) - f(x)]. \end{aligned} \quad (2.4)$$

To find the optimal value of $\bar{\lambda}(x - \xi)$ we first take the variation with respect to $y_n(x)$. Thus

$$\frac{\delta}{\delta y_n} \mathcal{L}[y_{n+1}(x)] = \frac{\delta}{\delta y_n} \mathcal{L}[y_n(x)] + \frac{\delta}{\delta y_n} \mathcal{L}[\bar{\lambda}(x)] \mathcal{L}[Ly_n(x) + N\tilde{y}_n(x) - f(x)], \quad (2.5)$$

and hence upon applying the variation this simplifies to

$$\mathcal{L}[\delta y_{n+1}] = \mathcal{L}[\delta y_n] + \delta \mathcal{L}[\bar{\lambda}] \mathcal{L}[y_n]. \quad (2.6)$$

We assume that L is a linear differential operator with constant coefficients given by

$$L(y) \equiv a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a_2 y'' + a_1 y' + a_0 y, \quad (2.7)$$

where a_i 's are constants. It is important to note that if the coefficients contain only non-constant terms of the form x^k , then the Laplace variational approach is still valid.

The Laplace transform of the first term of the operator L is given by

$$\mathcal{L}[a_n y^{(n)}] = a_n s^n \mathcal{L}[y] - a_n \sum_{k=1}^n s^{k-1} y^{(n-k)}(0), \quad (2.8)$$

so the variation with respect to y is

$$\delta \mathcal{L}[a_n y^{(n)}] = a_n s^n \mathcal{L}[\delta y]. \quad (2.9)$$

The other term in the operator L , namely $a_{n-1} y^{(n-1)}, \dots, a_1 y' + a_0 y$ yields similar results. Hence using (2.9), Eq. (2.6) reduces to

$$\mathcal{L}[\delta y_{n+1}] = \mathcal{L}[\delta y_n] + \mathcal{L}[\bar{\lambda}] \left(\sum_{k=0}^n a_k s^k \right) \mathcal{L}[\delta y_n] = \left[1 + \mathcal{L}[\bar{\lambda}] \left(\sum_{k=0}^n a_k s^k \right) \right] \mathcal{L}[\delta y_n]. \quad (2.10)$$

The extremum condition of y_{n+1} requires that $\delta y_{n+1} = 0$. This means that the right-hand side of Eq. (2.10) should be set to zero. Hence, we have the stationary condition

$$\mathcal{L}[\bar{\lambda}] = -\frac{1}{\sum_{k=0}^n a_k s^k}. \quad (2.11)$$

Taking the Laplace inverse of the last equation gives the optimal value of $\bar{\lambda}$. For this value of $\bar{\lambda}$, we have the following iteration formulation:

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] + \mathcal{L}\left[\int_0^x \bar{\lambda}(x - \xi) [Ly_n(\xi) + N\tilde{y}_n(\xi) - f(\xi)] d\xi\right], \quad \text{where } n \geq 0. \quad (2.12)$$

3. Examples

To illustrate this alternative novel approach, the Laplace variational scheme is applied on specific initial- and boundary-value problems.

Example 1. Consider the boundary-value problem

$$\begin{aligned} y'' + y + x &= 0, \\ y(0) &= 0, \quad y(1) = 0. \end{aligned} \quad (3.13)$$

The exact solution for this problem is $y = \csc(1) \sin x - x$. For this case, the Laplace variational iteration correction functional will be constructed in the following manner:

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] + \mathcal{L}\left[\int_0^x \bar{\lambda}(x - \xi) (y_n''(\xi) + y_n(\xi) + \xi) d\xi\right], \quad (3.14)$$

or equivalently, upon applying the properties of Laplace transform, we have

$$\begin{aligned} \mathcal{L}[y_{n+1}(x)] &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x) * (y_n''(x) + y_n(x) + x)] \\ &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x)] \mathcal{L}[y_n''(x) + y_n(x) + x] \\ &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x)] \left((s^2 + 1)\mathcal{L}[y_n(x)] - sy_n(0) - y_n'(0) + \frac{1}{s^2} \right). \end{aligned} \quad (3.15)$$

Taking the variation with respect to $y_n(x)$ of both sides of the latter equation, leads to

$$\frac{\delta}{\delta y_n} \mathcal{L}[y_{n+1}(x)] = \frac{\delta}{\delta y_n} \mathcal{L}[y_n(x)] + \frac{\delta}{\delta y_n} \mathcal{L}[\bar{\lambda}(x)] \left((s^2 + 1)\mathcal{L}[y_n(x)] - sy_n(0) - y_n'(0) + \frac{1}{s^2} \right), \quad (3.16)$$

and upon simplification we get

$$\mathcal{L}[\delta y_{n+1}] = \mathcal{L}[\delta y_n] + \mathcal{L}[\bar{\lambda}] (s^2 + 1)\mathcal{L}[\delta y_n] = \mathcal{L}[\delta y_n] (1 + \mathcal{L}[\bar{\lambda}] (s^2 + 1)). \quad (3.17)$$

The extremum condition of y_{n+1} requires that $\delta y_{n+1} = 0$. This means that the right-hand side of Eq. (3.17) should be set to zero. Hence, we have $1 + \mathcal{L}[\bar{\lambda}] (s^2 + 1) = 0$, that is, $\mathcal{L}[\bar{\lambda}] = -\frac{1}{s^2 + 1}$. Therefore

$$\bar{\lambda}(x) = -\sin x. \quad (3.18)$$

Substituting Eq. (3.18) into (3.14) results in the following iterative scheme

$$\begin{aligned} \mathcal{L}[y_{n+1}(x)] &= \mathcal{L}[y_n(x)] - \mathcal{L}\left[\int_0^x \sin(x - \xi) (y_n''(\xi) + y_n(\xi) + \xi) d\xi\right] \\ &= \mathcal{L}[y_n(x)] - \mathcal{L}[\sin x] \mathcal{L}[y_n''(x) + y_n(x) + x]. \end{aligned} \quad (3.19)$$

Let $y_0 = y(0) + y'(0)x = Ax$. Then

$$\begin{aligned} \mathcal{L}[y_1] &= \mathcal{L}[y_0] - \mathcal{L}[\sin x] \mathcal{L}[y_0'' + y_0 + x] = \mathcal{L}[Ax] - \mathcal{L}[\sin x] \mathcal{L}[Ax + x] \\ &= \frac{A}{s^2} - \frac{1}{s^2 + 1} \left(\frac{A + 1}{s^2} \right). \end{aligned} \quad (3.20)$$

Inverse Laplace transform yields

$$y_1 = (A + 1) \sin x - x. \quad (3.21)$$

Using the boundary condition $y(1) = 0$ in (3.21), we get $A = \csc 1 - 1$. Consequently $y_1 = \csc(1) \sin x - x$, which is the exact solution of our problem.

Example 2. Consider the following boundary-value problem that arises from the Euler–Lagrange equation upon solving the Brachistochrone problem.

$$\begin{aligned} y'' - yy'' - \frac{1}{2} - \frac{1}{2}y^2 &= 0, \\ y(0) &= 0, \quad y(1) = -0.5. \end{aligned} \quad (3.22)$$

The exact solution for this boundary-value problem, written in implicit form, is given by

$$F(x, y) = -\sqrt{-y^2 + 0.381510869y + 0.618489131} - x + 0.5938731505 - 0.8092445655 \tan^{-1} \left(\frac{y - 0.1907554345}{\sqrt{-y^2 + 0.381510869y + 0.618489131}} \right) = 0. \quad (3.23)$$

The Laplace variational iteration correction functional is constructed as follows:

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] + \mathcal{L} \left[\int_0^x \bar{\lambda}(x - \xi) \left(y_n''(\xi) - \tilde{y}_n(\xi) \tilde{y}_n''(\xi) - \frac{1}{2} - \frac{1}{2} \tilde{y}_n'^2(\xi) \right) d\xi \right]; \quad (3.24)$$

hence we have

$$\begin{aligned} \mathcal{L}[y_{n+1}(x)] &= \mathcal{L}[y_n(x)] + \mathcal{L} \left[\bar{\lambda}(x) * \left(y_n''(x) - \tilde{y}_n(x) \tilde{y}_n''(x) - \frac{1}{2} - \frac{1}{2} \tilde{y}_n'^2(x) \right) \right] \\ &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x)] \left(s^2 \mathcal{L}[y_n(x)] - sy_n(0) - y_n'(0) - \mathcal{L}[\tilde{y}_n(x) \tilde{y}_n''(x)] - \frac{1}{2s} - \frac{1}{2} \mathcal{L}[\tilde{y}_n'^2(x)] \right). \end{aligned} \quad (3.25)$$

Taking the variation with respect to $y_n(x)$ results in

$$\begin{aligned} \frac{\delta}{\delta y_n} \mathcal{L}[y_{n+1}(x)] &= \frac{\delta}{\delta y_n} \mathcal{L}[y_n(x)] + \frac{\delta}{\delta y_n} \mathcal{L}[\bar{\lambda}(x)] \left(s^2 \mathcal{L}[y_n(x)] - sy_n(0) - y_n'(0) \right. \\ &\quad \left. - \mathcal{L}[\tilde{y}_n(x) \tilde{y}_n''(x)] - \frac{1}{2s} - \frac{1}{2} \mathcal{L}[\tilde{y}_n'^2(x)] \right). \end{aligned} \quad (3.26)$$

Making the above correction functional stationary we have

$$\mathcal{L}[\delta y_{n+1}] = \mathcal{L}[\delta y_n] + \mathcal{L}[\bar{\lambda}] s^2 \mathcal{L}[\delta y_n] = \mathcal{L}[\delta y_n] (1 + s^2 \mathcal{L}[\bar{\lambda}]) = 0. \quad (3.27)$$

This implies that $1 + s^2 \mathcal{L}[\bar{\lambda}] = 0$, or $\mathcal{L}[\bar{\lambda}] = -\frac{1}{s^2}$. Therefore

$$\bar{\lambda}(x) = -x. \quad (3.28)$$

Substituting Eq. (3.28) into (3.24) yields the iteration scheme:

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] + \mathcal{L} \left[\int_0^x (\xi - x) \left(y_n''(\xi) - y_n(\xi) y_n''(\xi) - \frac{1}{2} - \frac{1}{2} y_n'^2(\xi) \right) d\xi \right], \quad (3.29)$$

or

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] + \mathcal{L}[-x] \mathcal{L} \left[y_n''(x) - y_n(x) y_n''(x) - \frac{1}{2} - \frac{1}{2} y_n'^2(x) \right]. \quad (3.30)$$

Let $y_0 = y(0) + y'(0)x = Ax$. Then

$$\mathcal{L}[y_1] = \mathcal{L}[Ax] + \mathcal{L}[-x] \mathcal{L} \left[-\frac{1}{2} - \frac{1}{2} A^2 \right] = \frac{A}{s^2} + \frac{1}{2s^2} \left(\frac{A^2 + 1}{s} \right). \quad (3.31)$$

Inverse Laplace gives the first iterate

$$y_1 = Ax + \frac{1}{4} (A^2 + 1)x^2. \quad (3.32)$$

Matching y_1 with the boundary conditions $y(1) = -0.5$, $A = -1$ and hence $y_1 = -x + x^2/2$. Substituting (3.32) into (3.30) yields

$$y_2 = Ax + \frac{1}{4} (A^2 + 1)x^2 + \frac{1}{6} (A^3 + A)x^3 + \frac{1}{48} (A^4 + 2A^2 + 1)x^4. \quad (3.33)$$

If we stop at this second iterate, the boundary condition $y(1) = -0.5$ gives the value of $A = -0.7460779052$. Proceeding similarly and using the scheme (3.29) the third iterate is found to be

$$\begin{aligned} y_3 &= 0.001156656714x^8 - 0.009046734096x^7 + 0.03457969268x^6 - 0.1011538798x^5 \\ &\quad + 0.1917675862x^4 - 0.2225556398x^3 + 0.4131913831x^2 - 0.8079390650x. \end{aligned} \quad (3.34)$$

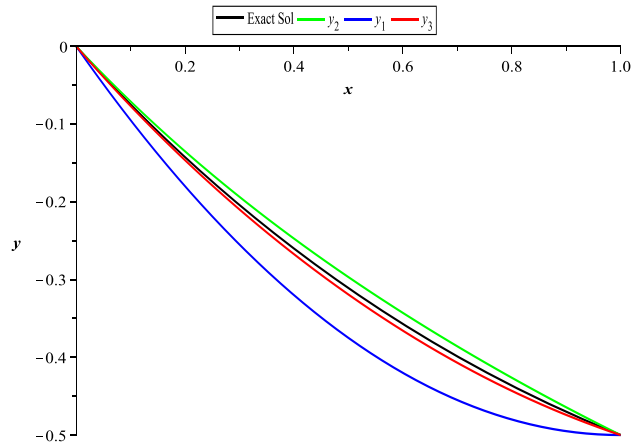


Fig. 1. Numerical solutions of Example 2.

In Fig. 1 we plot the exact solution together with the numerical solution obtained by the Laplace variational method using one, two and three iterates. Fig. 1 shows the improvement of the approximate solutions y_n as n increases.

Example 3. Consider the nonlinear boundary-value problem

$$\begin{aligned} y''(x) + 2(y'(x))^2 - 8y(x) &= 4, \\ y(0) = 0, \quad y(1) &= 2. \end{aligned} \tag{3.35}$$

The exact solution for this problem is $y = x + x^2$. The Laplace variational iteration correction functional is expressed as:

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] + \mathcal{L}\left[\int_0^x \bar{\lambda}(x - \xi) (y_n''(\xi) - 8y_n(\xi) - 4 + 2\tilde{y}_n'^2(\xi)) d\xi\right]. \tag{3.36}$$

Therefore we have

$$\begin{aligned} \mathcal{L}[y_{n+1}(x)] &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x) * (y_n''(x) - 8y_n(x) - 4 + 2\tilde{y}_n'^2(x))] \\ &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x)] \left((s^2 - 8)\mathcal{L}[y_n(x)] - sy_n(0) - y_n'(0) - \frac{4}{s} + 2\mathcal{L}[\tilde{y}_n'^2] \right). \end{aligned} \tag{3.37}$$

Taking the variation with respect to $y_n(x)$ and making the above correction functional stationary we obtain

$$\mathcal{L}[\delta y_{n+1}] = \mathcal{L}[\delta y_n] (1 + (s^2 - 8)\mathcal{L}[\bar{\lambda}]) = 0. \tag{3.38}$$

This implies that $\mathcal{L}[\bar{\lambda}] = -\frac{1}{s^2 - 8}$. Therefore

$$\bar{\lambda}(x) = -\frac{1}{\sqrt{8}} \sinh(\sqrt{8}x). \tag{3.39}$$

Substituting Eq. (3.39) into (3.36), we get the appropriate iterative scheme:

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] - \frac{1}{\sqrt{8}} \mathcal{L}\left[\int_0^x \sinh(\sqrt{8}(x - \xi)) (y_n''(\xi) - 8y_n(\xi) - 4 + 2y_n'^2(\xi)) d\xi\right], \tag{3.40}$$

or

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] - \frac{1}{\sqrt{8}} \mathcal{L}[\sinh \sqrt{8}x] \mathcal{L}[y_n''(x) - 8y_n(x) - 4 + 2y_n'^2(x)]. \tag{3.41}$$

Let $y_0 = y(0) + y'(0)x = Ax$. Then

$$\begin{aligned} \mathcal{L}[y_1] &= \mathcal{L}[Ax] - \frac{1}{\sqrt{8}} \mathcal{L}[\sinh \sqrt{8}x] \mathcal{L}[-8Ax - 4 + 2A^2] \\ &= \frac{A}{s^2} - \frac{1}{s^2 - 8} \left(-\frac{8A}{s^2} + \frac{2A^2 - 4}{s} \right). \end{aligned} \tag{3.42}$$

Thus

$$y_1 = \frac{1}{4}(A^2 - 2) - \frac{1}{4}(A^2 - 2) \cosh(2\sqrt{2}x) + \frac{1}{4}\sqrt{2}A \sinh(2\sqrt{2}x). \tag{3.43}$$

The higher iterates are obtained in a similar fashion. To find the value of A , we use the boundary condition $y(1) = 2$.

Example 4. Consider the following fifth order boundary-value problem

$$\begin{aligned} y^{(5)}(x) &= e^{-x}y^2(x), \\ y(0) &= 1, \quad y'(0) = 1, \quad y''(0) = 1, \quad y(1) = e, \quad y'(1) = e. \end{aligned} \tag{3.44}$$

The exact solution for (3.44) is $y = e^x$. The Laplace variational iteration correction functional is given by:

$$\mathcal{L} [y_{n+1}(x)] = \mathcal{L} [y_n(x)] + \mathcal{L} \left[\int_0^x \bar{\lambda}(x - \xi) (y_n^{(5)}(\xi) - e^{-\xi}y_n^2(\xi)) d\xi \right]. \tag{3.45}$$

Thus

$$\begin{aligned} \mathcal{L} [y_{n+1}(x)] &= \mathcal{L} [y_n(x)] + \mathcal{L} [\bar{\lambda}(x) * (y_n^{(5)}(x) - e^{-x}y_n^2(x))] \\ &= \mathcal{L} [y_n(x)] + \mathcal{L} [\bar{\lambda}(x)] (s^5 \mathcal{L} [y_n(x)] - s^4 y_n(0) - s^3 y_n'(0) \\ &\quad - s^2 y_n''(0) - s y_n'''(0) - y_n^{(4)}(0) - \mathcal{L} [e^{-x}y_n^2(x)]). \end{aligned} \tag{3.46}$$

Taking the variation with respect to $y_n(x)$ and making the above correction functional stationary we have $\mathcal{L} [\delta y_{n+1}] = \mathcal{L} [\delta y_n] (1 + s^5 \mathcal{L} [\bar{\lambda}]) = 0$, that is

$$\mathcal{L} [\bar{\lambda}] = -\frac{1}{s^5}. \tag{3.47}$$

Therefore

$$\bar{\lambda}(x) = -\frac{1}{24}x^4. \tag{3.48}$$

Substituting Eq. (3.48) into (3.45) results in the following iterative scheme:

$$\mathcal{L} [y_{n+1}(x)] = \mathcal{L} [y_n(x)] - \frac{1}{24} \mathcal{L} \left[\int_0^x (x - \xi)^4 (y_n^{(5)}(\xi) - e^{-\xi}y_n^2(\xi)) d\xi \right], \tag{3.49}$$

or

$$\mathcal{L} [y_{n+1}(x)] = \mathcal{L} [y_n(x)] - \frac{1}{24} \mathcal{L}[x^4] \mathcal{L} [y_n^{(5)}(x) - e^{-x}y_n^2(x)]. \tag{3.50}$$

Letting

$$y_0 = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 = 1 + x + \frac{1}{2!}x^2 + Ax^3 + Bx^4,$$

we get

$$\begin{aligned} \mathcal{L}[y_1] &= \mathcal{L}[1 + x + \frac{1}{2!}x^2 + Ax^3 + Bx^4] - \frac{1}{24} \mathcal{L}[x^4] \mathcal{L} \left[-e^{-x} \left(1 + x + \frac{1}{2!}x^2 + Ax^3 + Bx^4 \right)^2 \right] \\ &= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{6A}{s^4} + \frac{24B}{s^5} + \frac{1}{s^5} \left(\frac{1}{1+s} + \frac{2}{(1+s)^2} + \frac{4}{(1+s)^3} + \frac{12A+6}{(1+s)^4} \right. \\ &\quad \left. + \frac{48A+48B+6}{(1+s)^5} + \frac{120A+240B}{(1+s)^6} + \frac{720(B+A^2)}{(1+s)^7} + \frac{10080AB}{(1+s)^8} + \frac{40320B^2}{(1+s)^9} \right). \end{aligned} \tag{3.51}$$

Upon taking the inverse Laplace, the resulting expression for y_1 is too long so we choose not to include it. Upon using the two conditions $y(1) = e$ and $y'(1) = e$, the values of A, B are found to be $A = 0.1666728160$ and $B = 0.04166338259$. The following Taylor series expansion of y_1 was obtained:

$$y_1 = 1.00000 + 1.000006x + \frac{1}{2}x^2 + 0.1666728160x^3 + 0.04166338259x^4 + \frac{1}{120}x^5 + \dots, \tag{3.52}$$

which matches highly accurately with Taylor's expansion of the exact solution $y = e^x$.

Example 5. Consider the Riccati equation with the impulse function complimented with initial condition.

$$\begin{aligned} y'(x) &= 2y(x) - y^2(x) + 1 + \delta(x - 2), \\ y(0) &= 0. \end{aligned} \quad (3.53)$$

For $x \neq 2$, the exact solution is given by $y(x) = 1 + \sqrt{2} \tanh(\sqrt{2}x - \tanh^{-1}(\sqrt{2}/2))$. The Laplace variational iteration correction functional is expressed as

$$\mathcal{L}[y_{n+1}(x)] = \mathcal{L}[y_n(x)] + \mathcal{L}\left[\int_0^x \bar{\lambda}(x-\xi) (y_n'(\xi) - 2y_n(\xi) - 1 - \delta(\xi - 2) + \tilde{y}_n^2(\xi)) d\xi\right]. \quad (3.54)$$

Applying the Laplace transform, Eq. (3.54) becomes

$$\begin{aligned} \mathcal{L}[y_{n+1}(x)] &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x) * (y_n'(x) - 2y_n(x) - 1 - \delta(x - 2) + \tilde{y}_n^2(x))] \\ &= \mathcal{L}[y_n(x)] + \mathcal{L}[\bar{\lambda}(x)] \left((s-2)\mathcal{L}[y_n(x)] - y_n(0) - \frac{1}{s} - e^{-2s} + \mathcal{L}[\tilde{y}_n^2(x)] \right). \end{aligned} \quad (3.55)$$

Taking the variation with respect to $y_n(x)$ and making the above correction functional stationary, noting that $\delta\tilde{y}_n = 0$, we have

$$\mathcal{L}[\delta y_{n+1}] = \mathcal{L}[\delta y_n] (1 + (s-2)\mathcal{L}[\bar{\lambda}]) = 0. \quad (3.56)$$

This implies that $\mathcal{L}[\bar{\lambda}] = -\frac{1}{(s-2)}$; hence

$$\bar{\lambda}(x) = -e^{2x}. \quad (3.57)$$

Substituting Eq. (3.57) into (3.54), we get the following iterative algorithm

$$\begin{aligned} \mathcal{L}[y_{n+1}(x)] &= \mathcal{L}[y_n(x)] - \mathcal{L}\left[\int_0^x e^{2(x-\xi)} (y_n'(\xi) - 2y_n(\xi) - 1 - \delta(\xi - 2) + y_n^2(\xi)) d\xi\right] \\ \mathcal{L}[y_{n+1}(x)] &= \mathcal{L}[y_n(x)] - \mathcal{L}[e^{2x}] \mathcal{L}[y_n'(x) - 2y_n(x) - 1 - \delta(x - 2) + y_n^2(x)]. \end{aligned} \quad (3.58)$$

Let $y_0 = y(0) = 0$. Then from the scheme (3.58), the first iterate is

$$\mathcal{L}[y_1] = \mathcal{L}[0] - \mathcal{L}[e^{2x}] \mathcal{L}[-1 - \delta(x - 2)] = \frac{1}{s-2} \left(\frac{1}{s} + e^{-2s} \right), \quad (3.59)$$

and upon taking the inverse Laplace gives $y_1 = \frac{1}{2}e^{2x} - \frac{1}{2} + e^{2x-4} (1 - H(2-x))$, where H is the Heaviside unit step function. In a similar fashion we obtain the higher iterates, for instance the second iterate is

$$\begin{aligned} y_2 &= -\frac{3}{8} - \frac{1}{8}e^{4x} + \frac{1}{2}e^{2x-8} + e^{3x-8} \sinh x - e^{3x-2} \sinh(x-2)H(x-2) + \frac{1}{2}(1+x)e^{2x} \\ &\quad + \frac{1}{2}(1-H(2-x))(2x-1)e^{2x-4} + \left(\frac{1}{2}H(2-x) - 1\right)e^{4x-8}. \end{aligned} \quad (3.60)$$

The first few terms of Taylor's series expansion of y_2 , for $x \neq 2$, is

$$y_2 = x + x^2 + \frac{1}{3}x^3 - \frac{1}{3}x^4 - \dots$$

which matches with the series expansion of the exact solution of the problem.

4. Conclusion

A Laplace variational iteration method is proposed for the solution of linear and nonlinear differential equations. The test examples that are examined confirm that this approach is an effective alternative and in certain cases an enhancement of the (VIM) strategy. A Laplace correction functional is introduced that enables us to express the integral in many instances in the form of a convolution. Laplace transforms will make the variational problems easier to tackle, particularly in finding the general Lagrange multiplier. When solving non-homogeneous equations involving discontinuous functions, Heaviside functions and forcing functions, the Laplace variational technique is an appropriate tool.

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