Discrete Mathematics 42 (1982) 287–312
North-Holland Publishing Company

CHROMATIC INVARIANTS OF SIGNED GRAPHS

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Dedicated to Professor Fred Supnick of The City College of the City University of New York

Received 27 May 1980
Revised January 1982

We continue the study initiated in "Signed graph coloring" of the chromatic and Whitney polynomials of signed graphs. In this article we prove and apply to examples three types of general theorem which have no analogs for ordinary graph coloring. First is a balanced expansion theorem which reduces calculation of the chromatic and Whitney polynomials to that of the simpler balanced polynomials. Second is a group of formulas based on counting colorings by their magnitudes or their signs; among them are a combinatorial interpretation of signed coloring (which implies an equivalence between proper colorings of certain signed graphs and matchings in ordinary graphs) and a signed-graphic switching formula (which for instance gives the polynomials of a two-graph in terms of those of its associated ordinary graphs). Third are addition/deletion formulas obtained by constructing one signed graph from another through adding and removing arcs; one such formula expresses the chromatic polynomial as a combination of those of ordinary graphs, while another (in one example) yields a complementation formula for ordinary matchings. The examples treated are the sign-symmetric graphs (among them in effect the classical root systems), all-negative graphs (corresponding to the even-cycle graphic matroid), signed complete graphs (equivalent to two-graphs), and two varieties of signed graphs associated with matchings and colorings of ordinary graphs. Our results are interpreted as counting the acyclic orientations of a signed graph; geometrically this means counting the faces of the corresponding arrangement of hyperplanes or zonotope.

Introduction

A signed graph is a graph whose arcs are labelled with signs. Like an ordinary graph, a signed graph can be colored (by signed colors) and has a chromatic polynomial (the function which counts proper colorings). In [10] we explored this aspect of signed graph coloring, demonstrating the relationships among the chromatic (and the related Whitney) polynomial, the matroid, and the acyclic orientations of a signed graph. This for the most part parallels unsigned graph theory. Here we present a part of the signed theory which has no analog for ordinary graphs, a part concerned with means of computing the chromatic and Whitney polynomials of signed graphs.

We have four related goals. First, to show the remarkable range of reductions and interpretations possible for signed-graphic polynomials; foremost among them is the balanced expansion formula, which underlies nearly all computations. Second, to point out the simple and pretty forms taken by the polynomials in

0012-365X/82/0000–0000/$02.75

certain cases, such as those of sign-symmetric and all-negative graphs. Third, to treat in detail a wide variety of examples, some of which have special interest for their close connection to previously recognized graphical structures (see later in this introduction). And finally, but historically our original motivation, to calculate the numbers of regions (and lower-dimensional faces) of n-dimensional arrangements of hyperplanes in which every hyperplanar equation has one of the forms $x_i = x_j$, $x_i + x_j = 0$, and $x_i = 0$, thus generalizing the sign-symmetric results of [8]. As we noted in [10] this geometry problem is equivalent to finding the number of acyclic orientations of the signed graph associated to the arrangement (and of its contraction graphs), which in turn is done by computing the chromatic (and the Whitney) polynomial of the graph. Hence our effort has been to make the polynomial formulas as explicit as we possibly can.

The theoretical part of this paper contains three types of results. The balanced expansion theorem is the most fundamental, for it reduces computation of the chromatic and Whitney polynomials to computation of their balanced versions, and it is the latter which are treated by our other theorems. The second class of results is a diverse one based on the technique of counting colorings according to their magnitudes or signs. Treating magnitudes as primary leads to an important combinatorial interpretation of the balanced chromatic polynomial (more precisely, of its coefficients in falling semifactorial expansion) in terms of matchings in contraction graphs (Theorem 2.1) and also to expressions for the balanced polynomials in terms of chromatic polynomials of ordinary graphs (Theorem 2.3 et seq.) which seem to be interesting even for ordinary graphs (Corollary 2.5). Treating signs as primary leads to quite different formulas, giving the balanced signed polynomials as sums of the polynomials of positive (essentially, unsigned) graphs (Theorem 2.2). The last kind of general result is the addition/deletion theorem and related lemmas, all based on the idea of constructing a signed graph from a simpler one, say a sign-symmetric one, by adding and removing arcs. One gets another set of expressions for the (balanced) chromatic polynomial in terms of ordinary graphs.

We treat six kinds of examples. For full graphs, in which every node supports a negative loop or a half arc, the balanced expansions are particularly striking. The very simple sign-symmetric graphs, combinatorial generalizations of the classical root systems, are easily handled through balanced expansion or through counting by signs or magnitudes. The balanced polynomials of an all-negative graph are again found through counting by magnitudes; this gives us the characteristic polynomial of the even-circle matroid of the underlying graph (since $G(-\Gamma)$ is this matroid; see [9, Section 7D] for that fact and for citations to the literature). Signed complete graphs (which are equivalent to two-graphs; see [9, Section 7E] for this fact and for references) are susceptible to counting by signs and by magnitudes; among the results are a relationship between colorings of a two-graph and of the graphs in its switching class.

Especially interesting from the viewpoint of ordinary graph theory are two kinds of signed graph associated with an ordinary graph Γ. One consists of a positive complete graph and all negative links *not* in Γ. The general combinatorial interpretation reduces in this case to the fact that the balanced chromatic polynomial encodes the matching vector of Γ (the numbers of matchings of each size). Applying the addition lemma of the addition/deletion theorem shows that the matching vector of the complementary graph determines that of Γ (Section 6.1). The other associated signed graph contains a negative complete graph and all the positive links in Γ. Its balanced chromatic polynomial encodes the chromatic polynomial of Γ (Section 6.2). There are also interactions between the two kinds of associated graph (Section 6.3). These examples show that matchings and colorings of ordinary graphs can be embedded in a single theoretical framework; it is not yet known whether that is an important observation.

Evaluating the chromatic and Whitney polynomials at $\lambda = -1$ gives formulas for the numbers of acyclic orientations of a signed graph and of its contraction graphs; equivalently, for the numbers of faces of the corresponding arrangement of hyperplanes and zonotope (cf. [10]). We give results both generally and for particular examples.

The article concludes with mention of some extensions, e.g. to voltage graphs.

Notation and terminology

For a review of the basic concepts of signed graphs the reader should see Section 1.1 of [10], of which this paper is essentially the second half. Here are defined some additional notations and terms.

Let $\Gamma = (N, E)$ be a graph. If $S \subseteq E$ we write

$$c(S) = \text{the number of components of } S,$$

where S is regarded as the spanning subgraph (N, S). If π is a partition of N, we define the subset S *induced by* π to be

$$S:\pi = \bigcup\{S:B : B \in \pi\},$$

with node set $\bigcup \pi$ if regarded as a subgraph; similar are $\Gamma:\pi$ and $\Sigma:\pi$. A node set X or a partition π is *stable* in Γ if it supports no arcs, i.e. if $E:X = \emptyset$ or $E:\pi = \emptyset$ respectively.

Here are some important partition concepts. For any set X, Π_X denotes the set of partitions of X. In particular Π_n denotes the set of partitions of the standard node set N (whose size is n). For the graph Γ, $\Pi(\Gamma)$ is the set of partitions $\pi \in \Pi_n$ such that each block of π is connected by $\Gamma:\pi$; similarly $\Pi(S)$ is defined for any arc set S. The *kernel* of a set mapping $k: X \to Y$ is the partition of X given by

$$\operatorname{Ker} k = \{k^{-1}(y) : y \in k(X)\}.$$

A *bipartition* is a partition into two blocks of which one (or both) may be void.

If $\Sigma = (N, E, \sigma)$ is a signed graph, Σ^* denotes Σ with all half arcs removed. E_+ or $E_+(\Sigma)$ denotes the set $\sigma^{-1}(+)$ of positive arcs and Σ_+ denotes the positive subgraph of Σ, that is $(N, E_+(\Sigma), +)$. In general for any arc set S, S_+ denotes the positive part of S.

The *zero-free exact Whitney coefficients* $\psi^*_{\kappa r}(\Sigma)$ are defined by the falling semifactorial expansion

$$w^{\mathrm{b}}_{\Sigma}(x, \lambda) = \sum_{\kappa=0}^{n} {}_2(\lambda)_\kappa \sum_{r=0}^{n} \psi^*_{\kappa r}(\Sigma) x^r.$$

The *chromatic number* $\gamma(\Sigma)$ of the signed graph Σ is the smallest nonnegative integer μ for which $\chi_\Sigma(2\mu+1)>0$; thus it equals the smallest index for which $\psi_\kappa(\Sigma) \neq 0$. The *strict chromatic number* $\gamma^*(\Sigma)$ is the smallest nonnegative integer μ for which $\chi^{\mathrm{b}}_\Sigma(2\mu)>0$; it equals the smallest index for which $\psi^*_\kappa(\Sigma) \neq 0$.

1. Balanced expansion formulas

In examples it is usually much easier to compute the balanced than the ordinary chromatic and Whitney polynomials. Fortunately there are simple expressions for the unbalanced polynomials in terms of their balanced relatives. These expressions I regard as a fundamental theorem, for they show balance to be central to the matroid and coloring theory of signed graphs. Another, rather technical reason for thinking one of them, equation (1.3), interesting is that it shows the balanced chromatic polynomial to provide an interpretation of the Tutte polynomial $t(z, 0)$ of $G(\Sigma)$.

Theorem 1.1. *The chromatic and balanced chromatic polynomials of a signed graph Σ are related by the equation*

$$\chi_\Sigma(\lambda) = \sum_{W} \chi^{\mathrm{b}}_{\Sigma:W}(\lambda - 1), \tag{1.1}$$

the range of summation being all $W \subseteq N(\Sigma)$ whose complements are stable; except that $\chi_\Sigma(\lambda) \equiv 0$ if Σ has any free loops. The Whitney and balanced Whitney polynomials are related by the equation

$$w_\Sigma(x, \lambda) = \sum_{W \subseteq N} w^{\mathrm{b}}_{\Sigma:W}(x, \lambda - 1) x^{\mathrm{rk}(\Sigma:W^c)} \tag{1.2}$$

in which W ranges over all subsets of $N = N(\Sigma)$.

In particular for a full graph $\Sigma^\cdot$,

$$\chi_{\Sigma^\cdot}(\lambda) = \chi^{\mathrm{b}}_{\Sigma^\cdot}(\lambda - 1) \tag{1.3}$$

(if there are no free loops) and

$$w_{\Sigma^\cdot}(x, \lambda) = \sum_{W \subseteq N} w^{\mathrm{b}}_{\Sigma:W}(x, \lambda - 1) x^{n-\#(W)}. \tag{1.4}$$

Proof. There are at least three ways to prove Theorem 1. We shall take the combinatorial approach of counting colorings. Thus let $\lambda = 2\mu + 1$ where $\mu \geq 0$ is an integer.

Proof of (1.1) *and* (1.3). The left-hand side of (1.1) counts the number of proper colorings of Σ by the colors $\{0, \pm 1, \ldots, \pm\mu\}$.

The right-hand side counts the number of ways to choose a stable set W^c, color it 0, and color $\Sigma : W$ properly with the colors $\{\pm 1, \ldots, \pm\mu\}$. Clearly all these colorings are distinct. They are also proper, because W^c, being stable, supports no half arcs or loops (wherefore 0 is a proper color for it) and no links (whence all nodes in W^c may take the same color). So the *right-hand side counts a certain* class of proper colorings of Σ.

We must show this class includes all proper colorings. Let k be any proper coloring of Σ and $X = k^{-1}(0)$. This set must be stable or k would not be proper. And $W = X^c$ is colored in a zero-free way. Therefore k is one of the proper colorings of Σ enumerated by the term $\chi^{b}_{\Sigma:W}(2\mu)$ in the right-hand summation in (1.1). So we have proved that every proper coloring is counted by the right-hand side of equation (1.1). Hence the equation is true. Equation (1.3) is immediate.

Proof of (1.2) *and* (1.4). Because of how coefficients of x^{n-c} are defined, the equations are entailed by the assertion that, for each c, the class of colorings of Σ whose $I(k)$ have c balanced components is in one-to-one correspondence with the disjoint union over all $W \subseteq N$ of the zero-free colorings of $\Sigma : W$ which have $c - b(\Sigma : W^c)$ balanced components. It is easy to check that the mapping $k \to (W, k \mid W)$, where $W = \{v : k(v) \neq 0\}$, is just such a correspondence. The inverse mapping is $(W, k_W) \to k$ defined by $k \mid W = k_W$, $k \mid W^c \equiv 0$. □

It follows from (1.3) that *the full chromatic number* $\gamma(\Sigma^{\cdot})$ *equals the strict chromatic number* $\gamma^*(\Sigma)$.

Formula (1.3) (combined with [10, Theorem 2.3]) shows that

$$(-1)^n \chi^{b}_{\Sigma^{\cdot}}(-z) = t(z, 0), \tag{1.5}$$

where $t(z, x)$ is the Tutte polynomial of $G(\Sigma^{\cdot})$ (cf. [3] and [1]). This is a new interpretation of the Tutte polynomial. It can be extended to $x \neq 0$ by considering the 'balanced dichromatic polynomial' (cf. [7]); that will appear elsewhere.

2. Counting by color magnitudes and signs

The balanced expansion theorem still leaves us the task of computing the balanced polynomials of Σ. One technique for doing so is to group colorings by their sign or magnitude parts. We call this process 'counting by magnitudes', or by signs, depending on the order of analysis. The first results are fundamental combinatorial interpretations of the zero-free exact chromatic coefficients based on counting by magnitudes.

Theorem 2.1. *Let Σ be a signed graph and $E=E(\Sigma)$.*

(A) *Suppose Σ has no free loops. Then the coefficient $\psi^*_\kappa(\Sigma)$ is equal to the number of pairs (π, φ) of partitions of N such that $\pi \leq \varphi$, $\#\varphi=\kappa$, π is stable in Σ_+, and φ is a (partial) matching of the blocks of π with $E_-:\varphi=E_-:\pi$. Equivalently*

$$\psi^*_\kappa(\Sigma)=\sum_{j=0}^{n-k} \sum_{\substack{\pi \in \Pi_n:\#\pi=j+\kappa,\\ \pi \text{ stable in } \Sigma_+}} m_j([E_-/\pi]^c). \tag{2.1}$$

(B) *The coefficient $\psi^*_{\kappa r}(\Sigma)$ is equal to the number of pairs (π, φ) of partitions of N such that $\pi \leq \varphi$, $\#\varphi=\kappa$, φ is a (partial) matching of the blocks of π, and*

$$\text{rk}([E_+:\pi]\cup[(E_-:\varphi)\setminus(E_-:\pi)])=r.$$

Proof. (B) Given a ze...-free signed coloring k^*, we set $l=|k^*|$, $\varphi=\text{Ker}\, l$, and $\pi=\text{Ker}\, k^*$. An alternate definition of $\psi^*_{\kappa r}(\Sigma)$ is the number of symmetry classes of colorings k^* such that l uses exactly κ magnitudes and $\text{rk}\, I(k^*)=r$ (cf. [10, Section 2.2], where the case $r=0$ is discussed): in other words, such that $\#\varphi=\kappa$ and $\text{rk}\, I(k^*)=r$. Notice that

$$I(k^*)=[E_+:\pi]\cup[(E_-:\varphi)\setminus(E_-:\pi)]. \tag{*}$$

Because $k^*=l\cdot\text{sign}(k^*)$, φ is a matching of the blocks of π.

The symmetry class of k^* is completely described by π and φ. Conversely any π and φ as above determine a symmetry class of colorings counted by $\psi^*_{\kappa r}(\Sigma)$. Thus part (B) is proved.

(A) Setting $r=0$ in the above argument yields the first part of (A). Equation (2.1) is an easy variant. □

Equation (2.1) is the expression which leads to most of the interesting interpretations of ψ^*_κ in the important special cases we treat later. There seems to be no similarly simple interpretation of $\psi^*_{\kappa r}$, although a complicated analog is obtained by combining (2.1) above with equation (2.2) of [10].

Another kind of formula results from considering as primary the bipartition of N due to the sign part of a coloring. We call this 'counting by signs'.

Theorem 2.2. *Let Σ be a signed graph with no half arcs. Then*

$$w^b_\Sigma(x,\lambda)=\sum_\nu w(E_+(\Sigma^\nu);x,\lambda), \tag{2.2}$$

summed over all switching functions ν. If Σ has no free loops,

$$\chi^b_\Sigma(x,\lambda)=\sum_\nu \chi(E_+(\Sigma^\nu);\lambda). \tag{2.3}$$

Proof. Set $\lambda=2\mu$, where μ is a positive integer. Any zero-free signed coloring of Σ in μ colors is uniquely expressible as a product νl, where ν is a sign function on

N and l is a coloring using the color set $\{1, 2, \ldots, \mu\}$. We have

$$w^{\mathrm{b}}_{\Sigma}(x, 2\mu) = \sum_{\nu} \sum_{l} x^{\operatorname{rk} I(\nu l)}.$$

We can view ν as a switching function: then $l = (\nu l)^{\nu}$ so $I(\Sigma^{\nu}; l) = I_{\Sigma}(\nu l)$. Now consider an arc e. If it is improper for l in Σ^{ν}, it must be positive in Σ^{ν}. So $I(\Sigma^{\nu}; l) = I(\Sigma^{\nu}_{+}; l)$ and

$$w^{\mathrm{b}}_{\Sigma}(x, 2\mu) = \sum_{\nu} \sum_{l} x^{\operatorname{rk} I(\Sigma^{\nu}_{+}; l)},$$

from which (2.2) follows. Equation (2.3) is obtained by setting $x = 0$. □

Reversing the order of summation leads to the formulas we call 'counting by magnitudes'.

Theorem 2.3. *Let Σ be a signed graph, $E = E(\Sigma)$, and $\Delta = |\Sigma^*|$. Then*

$$w^{\mathrm{b}}_{\Sigma}(x, \lambda) = \sum_{F \in \operatorname{Lat} \Delta} \chi_{\Delta/F}(\tfrac{1}{2}\lambda) \sum_{\nu : N \to \{\pm\}} x^{\operatorname{rk} F^{\nu}_{+}} \tag{2.4a}$$

$$= \sum_{F \in \operatorname{Lat} \Delta} 2^{c(F)} \chi_{\Delta/F}(\tfrac{1}{2}\lambda) \sum_{A} x^{\operatorname{rk} A}, \tag{2.4b}$$

where the range of A is all balanced subsets of F for which $-(\Sigma \mid F/A)$ is balanced. (All such A are balanced flats.) If Σ has no free loops,

$$\chi^{\mathrm{b}}_{\Sigma}(\lambda) = \sum_{\substack{F \in \operatorname{Lat} \Delta; \\ -\Sigma \mid F \text{ blanced}}} 2^{c(F)} \chi_{\Delta/F}(\tfrac{1}{2}\lambda). \tag{2.5}$$

Proof. For (2.5) we set $x = 0$ in (2.4b). To prove (2.4) we begin as in the proof of Theorem 2.2. Then we write $F = I_{\Delta}(l)$; we observe that F is a flat of Δ, $I(\nu l) = I(\Sigma \mid F; \nu)$, and the number of μ-colorings l for which $F = I_{\Delta}(l)$ is $\chi_{\Delta/F}(\mu)$. Thus

$$w^{\mathrm{b}}_{\Sigma}(x, 2\mu) = \sum_{F \in \operatorname{Lat} \Delta} \chi_{\Delta/F}(\mu) \sum_{\nu} x^{\operatorname{rk} I(\Sigma \mid F;\, \nu)}. \tag{$*$}$$

To prove (2.4a), observe that $I_{\Sigma|F}(\nu) = I(\Sigma^{\nu} \mid F; +) = F^{\nu}_{+}$.

To prove (2.4b) we view the inner sum in ($*$) as $w^{\mathrm{b}}_{\Sigma|F}(x, 2)$. By [10, (2.2)],

$$w^{\mathrm{b}}_{\Sigma}(x, 2\mu) = \sum_{F} \chi_{\Delta/F}(\mu) \sum_{A \in \operatorname{Lat}^{\mathrm{b}} \Sigma|F} x^{\operatorname{rk} A} \chi^{\mathrm{b}}_{\Sigma|F/A}(2).$$

As we see from Lemma 2.4 below, many of the terms $\chi^{\mathrm{b}}_{\Sigma|F/A}(2) = 0$. Thus we get (b).

Suppose $-(\Sigma \mid F/A)$ is balanced. Then $\Sigma \mid F/A$ can have no positive loops. So A is a flat. □

Lemma 2.4. *Let Φ be a signed graph with no half arcs or free loops. Then*

$$\chi^{b}_{\Phi}(2) = \begin{cases} 2^{c(\Phi)} & \text{if } \Phi \text{ is balanced,} \\ 0 & \text{if not.} \end{cases}$$

Proof. Say $\nu : N \to \{\pm 1\}$ is a proper coloring of Φ. For each arc, its propriety for ν in Φ implies it is improper in $-\Phi$. Thus $I_{-\Phi}(\nu) = E(\Phi)$. But the improper arcs of a zero-free coloring form a balanced set, by [10, Lemma 1.4]. So $-\Phi$ must have been balanced if ν exists.

If $-\Phi$ is indeed balanced, each component can be properly colored in just two ways. That shows what the number of colorings is. □

Theorem 2.3 is interesting even in its specialization to ordinary graphs, regarded as all-positive signed graphs. I do not know a reference for these formulas.

Corollary 2.5. *If Δ is an ordinary graph, then*

$$\chi_{\Delta}(\lambda) = \sum_{\substack{F \in \operatorname{Lat} \Delta \\ \text{bipartite}}} 2^{c(F)} \chi_{\Delta/F}(\tfrac{1}{2}\lambda),$$

$$w_{\Delta}(x, \lambda) = \sum_{F \in \operatorname{Lat} \Delta} 2^{c(F)} \chi_{\Delta/F}(\tfrac{1}{2}\lambda) \sum_{\substack{A \leq F: \\ F/A \text{ bipartite}}} x^{\operatorname{rk} A}.$$

Proof. An all-negative arc set is balanced if and only if it is bipartite. Use that fact in (2.5) and (2.4b) to get the polynomials of $+\Delta$, which are the same as those of Δ. □

Finally we present formulas intermediate between those of Theorems 2.1 and 2.3.

Corollary 2.6. *Let Σ be a signed graph. Then*

$$\psi^{*}_{\kappa r}(\Sigma) = \sum_{\substack{\varphi \in \Pi_n: \\ \#\varphi = \kappa}} 2^{c(\Sigma:\varphi) - \kappa} a_r(\Sigma : \varphi), \tag{2.6}$$

where $a_r(\Sigma : \varphi)$ = the number of rank r balanced arc sets A (equivalently, balanced flats) in $\Sigma : \varphi$ such that $-[(\Sigma^* : \varphi)/A]$ is balanced. If Σ has no free loops,

$$\psi^{*}_{\kappa}(\Sigma) = \sum_{\substack{\varphi \in \Pi_n;\, \#\varphi = \kappa, \\ -\Sigma^* : \varphi \text{ balanced}}} 2^{c(\Sigma:\varphi) - \kappa}. \tag{2.7}$$

Proof. Equation (2.7) is immediate from (2.6). For the latter, suppose we are coloring in κ colors, respecting a fixed partition φ of N into κ parts. Let l be the magnitude part of such a coloring. There are 2^n ways to choose a sign ν, but these are grouped into symmetry classes of 2^κ signed colorings. So

$$\sum_r \psi^*_{\kappa r}(\Sigma)x^r = \sum_{\#\varphi=\kappa} 2^{-\kappa} \sum_\nu x^{\mathrm{rk}\, I(\Sigma:\varphi;\nu)}.$$

As in the proof of (2.4b), $I_{\Sigma:\varphi}(\nu) = E^\nu_+ : \varphi$ and by Lemma 2.4 the number of ways a given balanced flat $A \leq E:\varphi$ appears as $\Sigma^\nu_+ : \varphi$ equals $2^{c(\Sigma:\varphi)}$ or 0. □

3. Sign-symmetric graphs

Probably the simplest forms assumed by the chromatic and Whitney polynomials are those for signed expansions of ordinary graphs. If Γ is an ordinary graph, the *signed expansion* $\pm\Gamma$ has a positive and a negative arc for each arc in Γ. A *sign-symmetric* graph is a signed expansion with perhaps added half arcs and negative loops.

Theorem 3.1. *The balanced chromatic and Whitney polynomials of the signed expansion $\pm\Gamma$ of an ordinary graph Γ are given by*

$$\chi^b_{\pm\Gamma}(\lambda) = 2^n \chi_\Gamma(\tfrac{1}{2}\lambda) \tag{3.1}$$

and

$$w^b_{\pm\Gamma}(x, \lambda) = 2^n w_\Gamma(x, \tfrac{1}{2}\lambda). \tag{3.2}$$

Proof. Theorem 3.1 is a corollary of equations (2.3) and (2.2): one merely observes that $E_+(\pm\Gamma^\nu) = +\Gamma$. Almost any other formulas in Section 2 also give proofs. And the direct proof by counting colorings is very easy. □

It follows that *the strict chromatic number $\gamma^*(\pm\Gamma)$ equals the chromatic number of Γ.*

In view of the balanced expansion formulas, Theorem 1 implies formulas for the unblanced polynomials χ_Σ and w_Σ when $\Sigma = \pm\Gamma$, $\pm\Gamma^\bullet$, and $\pm\Gamma \cup U^\bullet$ for any $U \subseteq N$. They, as well as a proof by deletion and contraction and the results for $\Gamma = K_n$, can be found in [8].

4. Addition and deletion formulas

Earlier we calculated the balanced chromatic polynomial of Σ by splitting its proper colorings. Now we shall break down Φ itself. We regard it as obtained from a different signed graph Ψ by removing some arcs from Ψ and adjoining others not in it. To get some good from this addition/deletion method we have to choose our initial graph Ψ carefully. If Ψ is sign-symmetric (Theorem 4.4), things work out rather well.

We treat deletion first. Addition is best regarded as the inverse operation.

Deletion Lemma 4.1. *Let $\Sigma \subseteq \Phi$ be signed graphs on the same node set. Let $D = E(\Phi) \setminus E(\Sigma)$. Then*

$$\chi_\Sigma(\lambda) = \sum_{\substack{S \in \mathrm{Lat}\,\Phi;\\ S \subseteq D}} \chi_{\Phi/S}(\lambda) \quad \text{and} \quad \chi^{\mathrm{b}}_\Sigma(\lambda) = \sum_{\substack{S \in \mathrm{Lat}^{\mathrm{b}}\,\Phi;\\ S \subseteq D}} \chi^{\mathrm{b}}_{\Phi/S}(\lambda).$$

Proof. Although a strictly algebraic proof can be carried through (for instance the first formula is a special case of Crapo [2, p. 606, Corollary 5]), a proof by counting colorings is simpler and more interesting. Let us classify the proper colorings of Σ according to their sets of impropriety in Φ. Any such set $S = I(k)$ is closed [10, Lemma 1.4] and does not intersect $E(\Sigma)$. The number of k such that $I(k) = S$ is $\chi_{\Phi/S}(2\mu+1)$ by [10, Lemma 1.5]. That proves the first formula. When we consider zero-free colorings, $I(k) = S$ entails S is balanced; and since it is balanced the number of zero-free colorings k with $I(k) = S$ equals $\chi^{\mathrm{b}}_{\Phi/S}(2\mu)$. That proves the other formula. □

Addition Lemma 4.2. *Let $\Psi \subseteq \Phi$ be signed graphs on the same node set. Let $D = E(\Phi) \setminus E(\Psi)$. Then*

$$\chi_\Phi(\lambda) = \sum_{\substack{T \in \mathrm{Lat}\,\Phi;\\ T \subseteq D}} \mu_\Phi(\emptyset, T)\chi_{(\Psi \cup T)/T}(\lambda)$$

and

$$\chi^{\mathrm{b}}_\Phi(\lambda) = \sum_{\substack{T \in \mathrm{Lat}^{\mathrm{b}}\,\Phi;\\ T \subseteq D}} \mu_\Phi(\emptyset, T)\chi^{\mathrm{b}}_{(\Psi \cup T)/T}(\lambda),$$

where μ_Φ is the Möbius function of Lat Φ.

Proof. Lemma 4.2 is trivial if $\emptyset$ is not closed, so let us assume it is closed. Then $\emptyset$ is the 0 element of Lat Φ. We shall apply Möbius inversion to the deletion lemma. Put $f(Q) = \chi_{(\Psi \cup Q)/Q}(\lambda)$ and $g(Q) = \chi_{\Phi/Q}(\lambda)$ if $Q \subseteq D$ and $f(Q) = g(Q) = 0$ otherwise. Substituting $(\Psi \cup T)/T$ for Σ and Φ/T for Φ in Lemma 4.1, we have

$$f(T) = \sum_{\substack{S \in \mathrm{Lat}(\Phi/T)\\ S \subseteq D \setminus T}} \chi_{(\Phi/T)/S}(\lambda) = \sum_{\substack{R \in \mathrm{Lat}\,\Phi\\ T \subseteq R \subseteq D}} \chi_{\Phi/R}(\lambda) = \sum_{\substack{R \in \mathrm{Lat}\,\Phi\\ T \subseteq R}} g(R).$$

In the language of incidence algebra, $f = \zeta_\Phi * g$. Inverting, $g = \mu_\Phi * f$. Evaluation at $\emptyset$ gives the first formula. The proof of the second is similar. □

Addition/Deletion Lemma 4.3. *Let Σ and Ψ be signed graphs on the same nodes and $\Phi = \Sigma \cup \Psi$. Let $A = E(\Sigma) \setminus E(\Psi)$ and $D = E(\Psi) \setminus E(\Sigma)$. Then*

$$\chi_\Sigma(\lambda) = \sum_{\substack{R \subseteq A \cup D,\\ R,\, R \cap D \in \mathrm{Lat}\,\Phi}} \mu_\Phi(R \cap D, R)\chi_{(\Psi \cup R)/R}(\lambda) \tag{4.1}$$

and

$$\chi^{\mathrm{b}}_{\Sigma}(\lambda)=\sum_{\substack{R\subseteq A\cup D;\\ R,R\cap D\in \mathrm{Lat}^{\mathrm{b}}\Phi}}\mu_{\Phi}(R\cap D,R)\chi^{\mathrm{b}}_{(\Psi\cup R)/R}(\lambda). \tag{4.2}$$

Proof. We apply the deletion lemma to $\Sigma=\Phi\setminus D$, then the addition lemma to $\Phi/S=(\Psi/S)\cup A$.

$$\begin{aligned}\chi_{\Sigma}(\lambda)&=\sum_{\substack{S\in\mathrm{Lat}\,\Phi;\\ S\subseteq D}}\chi_{\Phi/S}(\lambda)\\ &=\sum_{\substack{S\in\mathrm{Lat}\,\Phi;\\ S\subseteq D}}\sum_{\substack{T\in\mathrm{Lat}\,\Phi/S;\\ T\subseteq A}}\mu_{\Phi/S}(\emptyset,T)\chi_{(\Psi/S)/T}(\lambda)\\ &=\sum_{\substack{R\in\mathrm{Lat}\,\Phi;\\ R\subseteq A\cup D,\\ R\cap D\in\mathrm{Lat}\,\Phi}}\mu_{\Phi}(R\cap D,R)\chi_{\Psi/R}(\lambda);\end{aligned}$$

where $R=S\cup T$, thus $S=R\cap D$; and $\mu_{\Phi/S}(\emptyset,T)=\mu_{\Phi}(S,R)$ because the interval $[\emptyset,T]$ in $\mathrm{Lat}\,\Phi/S$ is isomorphic to $[S,R]$ in $\mathrm{Lat}\,\Phi$. The proof of the balanced formula is similar; we use the fact that, since S is balanced in Φ, T is balanced in Φ/S if and only if $T\cup S$ is balanced in Φ (cf. [10, Lemma 2.5]). Thus the result is established. □

Let us describe how we apply this lemma. Suppose, to illustrate, that Σ has set W of full nodes and no balanced loops, and Γ is the graph whose arcs are the negative links of Σ with their signs suppressed. Then Σ is obtained from the sign-symmetric graph $\Psi=\pm\Gamma\cup W^{\bullet}$ by removing a set of positive links, $+D$, and adding another, $+A$. We regard A and D as lying in the graph $\Delta=\Gamma\cup A$; then $D\subseteq E(\Gamma)$. We also identify A, D, Γ, Δ on the one hand with $+A$, $+D$, $+\Gamma$, $+\Delta$ on the other (which is possible because $G(+\Delta)=G(\Delta)$); then $\Delta=\Psi_{+}\cup A$. Thus $\mathrm{Lat}\,\Phi$, $\mathrm{Lat}^{\mathrm{b}}\,\Phi$, and μ_{Φ} in Lemma 4.3 can be replaced by $\mathrm{Lat}\,\Delta$, $\mathrm{Lat}\,\Delta$, and μ_{Δ}, respectively.

In general we can always represent Σ, if it has no balanced loops, as a sign-symmetric graph to which some links have been added and from which others have been taken away. The Addition/Deletion Lemma when applied to this representation yields Theorem 4.4, which reduces χ_{Σ} to a combination of ordinary chromatic polynomials. In the context of Theorem 4.4, a *lifting of* $A\cup D$ *to signed arcs* means a mapping $l: A\cup D\to\pm E(\Delta)$ such that $l(e)$ has the same endpoints as e. Recall that Γ/F is an abbreviation for $(\Gamma\cup F)/F$.

Addition/Deletion Theorem 4.4. *Let* $\Gamma\subseteq\Delta$ *be ordinary simple graphs on the node set* N, $D\subseteq E(\Gamma)$, *and* $A=E(\Delta)\setminus E(\Gamma)$. *Let* $l: A\cup D\to\pm E(\Delta)$ *be a lifting of* $A\cup D$ *to signed arcs, and let* $\Sigma=(\pm\Gamma\cup l(A))\setminus l(D)$. *Then*

$$\chi^{\mathrm{b}}_{\Sigma}(\lambda)=\sum_{F}\mu_{\Delta|F}(F\cap D,F)2^{c(F)}\chi_{\Gamma/F}(\tfrac{1}{2}\lambda),$$

summed over all $F \subseteq A \cup D$ *such that* $l(F) \in \mathrm{Lat}^{\mathrm{b}}(\pm\Gamma \cup l(A))$ *and* $F \cap D \in \mathrm{Lat}\, \Delta \mid F$.

Proof. To begin the proof let us translate the balanced addition/deletion formula (4.2) into the language of Theorem 4.4. We must replace A and D of (4.2) by $l(A)$ and $l(D)$ here; then R becomes $l(F)$. Φ and Ψ become $\pm\Gamma \cup l(A)$ and $\pm\Gamma$. Thus the summation of (4.2) is over $F \subseteq A \cup D$ such that $l(F)$ and $l(F \cap D)$ are both closed and balanced in $\pm\Gamma \cup l(A)$. By balance of the signed graph $l(F)$, the interval $[\emptyset, l(F)]$ in $\mathrm{Lat}(\pm\Gamma \cup l(A))$ is isomorphic to $[\emptyset, F]$ (see [9, Corollary 5.5]). Thus μ_Φ may be replaced by $\mu_{\Delta|F}$ and the condition on $l(F \cap D)$ is equivalent to requiring that $F \cap D$ be closed in F. We have now shown that the conditions of summation for $\chi^{\mathrm{b}}_{\Sigma}(\lambda)$ in (4.2) may be replaced by those of Theorem 4.4.

Let us write $T = F \setminus D$. Thus $(\Psi \cup R)/R = (\pm\Gamma \cup l(T))/l(F)$. Since we are concerned with the balanced polynomial of $(\Psi \cup R)/R$, we may change over to $\pm(\Gamma \cup T)/l(F)$; this only differs in having more unbalanced loops (those derived from $-l(T)$ in $\pm(\Gamma \cup T)$). But in $\pm(\Gamma \cup T)/l(F)$, the arcs of $-l(F)$ become unbalanced loops, so they are immaterial; thus we obtain the same balanced sets by turning our attention to $\pm[(\Gamma \cup T)/F]$, or briefly $\pm(\Gamma/F)$. Now by equation (3.1)

$$\chi^{\mathrm{b}}_{(\Psi \cup R)/R}(\lambda) = \chi^{\mathrm{b}}_{\pm(\Gamma/F)}(\lambda) = 2^{c(F)} \chi_{\Gamma;F}(\tfrac{1}{2}\lambda).$$

That concludes the proof of the theorem. □

Unfortunately it does not seem possible to find similar expansions of the Whitney polynomials. That is because addition/deletion results are related to the deletion/contraction formulas [10, Theorem 2.3]. Our proofs have not used them but they are the algebraic technique for proving the deletion lemma. Conversely the lemma with $D = \{e\}$ implies [10, Theorem 2.3]. So since the latter does not hold for Whitney polynomials one cannot expect an exact analog of our deletion lemma. Whether there is yet an appropriate variant—which would be a very nice thing to have in work with signed-graphic arrangements of hyperplanes—is not known.

5. All-negative graphs; the even-circle chromatic polynomial

The formulas for the chromatic polynomial of an all-negative graph $-\Gamma$ are especially elegant. Recall that $G(-\Gamma)$ is the even-circle matroid of Γ (cf. [9, Section 7D]) so what we are computing here is the 'even-circle chromatic polynomial'.

Certain bipartite subgraphs play an important role in work with $-\Gamma$. Let $E = E(\Gamma)$ and $N = N(\Gamma)$. If X and Y are disjoint node sets, by $E:\langle X, Y\rangle$ we mean the set of arcs of Γ with one end in X and the other in Y (thus half arcs as well as loops are excluded). If $X_1, Y_1, \ldots, X_k, Y_k$ are disjoint node sets, we call

$$(E:\langle X_1, Y_1\rangle) \cup \cdots \cup (E:\langle X_k, Y_k\rangle) \tag{5.1}$$

a *bipartitionally induced arc set.*

Lemma 5.1 [9, Corollary 7D.2]. *The balanced flats of $-\Gamma$, i.e. of the even-circle matroid $G(-\Gamma)$, are precisely the bipartitionally induced arc sets.* □

From this and equation (2.4) of [10] we know the balanced chromatic and Whitney polynomials (therefore by Theorem 1.1 the ordinary ones):

$$\chi^{b}_{-\Gamma}(\lambda)=\sum_{A}\mu(A)\lambda^{c(A)} \tag{5.2}$$

and

$$\chi_{-\Gamma}(\lambda)=\sum_{A}\mu(A)(\lambda-1)^{c(A)}\sum_{\substack{X\subseteq N(A)^c\\ \text{stable in }\Gamma}}(\lambda-1)^{-\#X}, \tag{5.3}$$

where A ranges over all bipartitionally induced arc sets and $\mu(A)$ denotes the Möbius invariant of the bipartite graph A.

The combinatorial interpretation of $\chi^{b}_{-\Gamma}$, derived from equation (2.1), is

$$\psi^{*}_{\kappa}(-\Gamma)=\sum_{j}\sum_{\substack{\pi\in\Pi_n;\\ \#\pi=j+\kappa}} m_j([\Gamma/\pi]^c). \tag{5.4}$$

Therefore *the strict chromatic number of $-\Gamma$ equals the largest size of a matching in the complement of any contraction of Γ.*

These facts do not offer much insight into the numerical behavior of $-\Gamma$. An interesting result is the next theorem, which shows a remarkable relationship between $-\Gamma$ and Γ. A *cut set* of Γ is the set of links between two complementary node sets; thus $\emptyset$ is one cut set.

Theorem 5.2. *Let Γ be an ordinary graph. Then*

$$\chi^{b}_{-\Gamma}(\lambda)=\sum_{F\in\operatorname{Lat}\Gamma}2^{c(F)}\chi_{\Gamma/F}(\tfrac{1}{2}\lambda)=2^{n}w_{\Gamma}(\tfrac{1}{2},\tfrac{1}{2}\lambda) \tag{5.5}$$

and

$$w^{b}_{-\Gamma}(x,\lambda)=\sum_{F\in\operatorname{Lat}\Gamma}2^{c(F)}\chi_{\Gamma/F}(\tfrac{1}{2}\lambda)\sum_{A}x^{n-c(A)}, \tag{5.6}$$

where A is summed over all cut sets of $\Gamma\,|\,F$.

Proof. We get (5.5) from (2.5) by noticing that $-\Sigma\,|\,F=+\Gamma\,|\,F$, which is always balanced.

Equation (5.6) is almost equally immediate from (2.4b); but it is necessary to prove that A is a cut set and any cut set is an A. Suppose A is a cut set of F. Then it is a balanced flat (by Lemma 5.1). Furthermore $\Gamma\,|\,F$ can be switched to make A positive without changing any other signs.

Conversely suppose A is the bipartitionally induced arc set (5.1) and $-(\Sigma\,|\,F/A)$ is balanced. That means it is possible to switch Σ to Σ^{ν} in which A is positive and $F\setminus A$ is negative. Then A is the cut set $E:\langle X,X^c\rangle$, where $X=\nu^{-1}(+)$. □

Remark. Variant Proof of (5.5). Direct application of counting by magnitudes is quite simple.

One might ask what would happen if we applied Theorem 4.4. If we chose Γ in $(4.4)=\Gamma$ here, $A=\emptyset$, $D=E$, and $l(D)=+E$, equation (5.5) would result. If we took $\Gamma(4.4)=\emptyset$, $A=E$, $D=\emptyset$, $l(A)=-E$, we would get (5.2). As far as I can see there are no other interesting cases.

From (5.5) we have for the all-negative complete graphs:

$$\chi^{b}_{-K_n}(\lambda)=\sum_{\kappa=0}^{n} S(n,\kappa)\cdot{}_2(\lambda)_\kappa, \tag{5.7}$$

$$\chi_{-K_n}(\lambda)=\sum_{\kappa=0}^{n} [S(n,\kappa)+nS(n-1,\kappa)]\cdot{}_2(\lambda-1)_\kappa. \tag{5.8}$$

And from equation (5.7) we can immediately write down expressions for the numbers $w^{b}_{k}(-K_n)$ (the coefficient in $\chi^{b}_{-K_n}(\lambda)$ of λ^{n-k}), called the *balanced Whitney numbers of the first kind* of $-K_n$. We have also a different formula:

$$w^{b}_{k}(-K_n)=(-1)^k \sum_{\substack{\pi\in\Pi_n;\\ \#\pi=n-k}} \prod_{B\in\pi} g(\#B), \tag{5.9}$$

where

$$g(n)=|w^{b}_{n-1}(-K_n)|=\sum_{j=0}^{n-1} j!\,S(n-1,j). \tag{5.10}$$

The latter sum is interesting because there is no cancellation among its terms. To prove (5.9) we define $g(n)$ by the first equality in (5.10), then apply Lemma 5.1 and the matroid formula for $\chi^{b}_{\Sigma}(\lambda)$ (in [10, Theorem 2.4]), noticing that $\operatorname{rk}(A)=\operatorname{rk}\pi(A)$ for a bipartitionally induced arc set A in K_n. To establish the second equality in (5.10) we observe that A, if of rank $n-1$, is equal to $K_{p,n-p}$ for some $p=1,2,\ldots,n-1$. Thus

$$g(n)=\tfrac{1}{2}\sum_{p=1}^{n-1}\binom{n}{p}|\mu(K_{p,n-p})|,$$

μ being the Möbius invariant of $G(K_{p,n-p})$, that is the coefficient of λ in $\chi_{K_{p,n-p}}(\lambda)$. Knowing the latter we can write $g(n)=(-1)^{n-1}G_n'(0)$, where

$$G_n(\lambda)=\tfrac{1}{2}\sum_{p=0}^{n}\binom{n}{p}\sum_{j=0}^{n} S(n,j)(\lambda)_j(\lambda-j)^{n-p}.$$

Then for instance expanding the Stirling numbers or calculating the exponential generating function one derives the second expression in (5.10).

As a consequence of (5.9) we have values for the Whitney numbers of $G(-K_n^{\bullet})$:

$$w_k(-K_n^{\bullet})=(-1)^k \sum_{\pi\in\Pi_n}\binom{n-\#\pi}{n-k}\prod_{B\in\pi} g(\#B). \tag{5.11}$$

6. Partial matching numbers and ordinary chromatic coefficients

The chromatic polynomials of signed graphs (without loops or half arcs) which contain a positive or negative complete graph are related to ordinary graphs in noteworthy ways.

6.1. *Graphs with every positive link: partial matching numbers*

A signed graph which contains $+K_n$ can be regarded as the result of deleting a negative subgraph $-\Gamma$ from the signed expansion $\pm K_n$. The interpretation of the coefficients of χ^b_Σ from equation (2.1), in which $\pi=0$ because $E_+=+K_n$, is:

Theorem 6.1. *If $\Gamma=(N,E)\subseteq K_n$ and $\Sigma=(\pm K_n)\setminus(-E)$, then*

$$\chi^b_\Sigma(\lambda)=\sum_{i=0}^{n} m_i(\Gamma)\cdot {}_2(\lambda)_{n-i}. \quad \square \tag{6.1}$$

Thus *the matching number of Γ equals $n-\gamma^*(\Sigma)$.*

Theorem 6.1 implies that by using the semifactorial matching polynomia (the right-hand side of (6.1)) one has a version of the matching problem to which the machinery of chromatic theory applies. For instance the deletion/contraction law for such polynomials is easily reduced to the well-known formula

$$m_i(\Gamma)=m_i(\Gamma\setminus e)+m_{i-1}(\Gamma:N(e)^c).$$

(We omit the details.)

From Theorem 2.1(B) we can calculate the balanced Whitney polynomial. For $B\subseteq N$, let $\alpha(\Gamma:B)$ be the proportion of bipartitions $\{X,Y\}$ of B in which $\Gamma:\langle X,Y\rangle$ is complete bipartite (and $X,Y\neq\emptyset$).

Corollary 6.2. *If $\Sigma=(\pm K_n)\setminus(-\Gamma)$, we have*

$$w^b_\Sigma(x,\lambda)=\sum_{\varphi\in\Pi_n} {}_2(\lambda)_{\#\varphi}\cdot(2x)^{n-\#\varphi}\prod_{B\in\varphi}[1-\alpha(\Gamma:B)(1-x^{-1})].$$

Proof. For the proof first note that Theorem 2.1(B) can be restated in the form

$$w^b_\Sigma(x,\lambda)=\sum_{\varphi\in\Pi_n} {}_2(\lambda)_{\#\varphi}\prod_{B\in\varphi}\sum_{\substack{\beta\in\Pi_B;\\ \#\beta\leq 2}} x^r,$$

where r depends on B and β. Let $\beta=\{X,Y\}$ if $\#\beta=2$. Since $E_+(\Sigma)=+K_n$, we have $r=\#B-1$ except that $r=\#B-2$ if the two halves of β are not connected by any negative edge: in other words if $\Gamma:\langle X,Y\rangle$ is complete bipartite. From this the corollary follows. $\square$

We get a striking result if we apply the addition lemma to $\Phi=+K_n\cup -E$ with $D=-E$. We must sum over those $T\subseteq -E$ which are balanced and closed in Φ. It follows that T is a partial matching in E, so the formula of the addition lemma becomes

$$\chi_{\Sigma}^{b}(\lambda) = \sum_{k \geq 0} m_k(E)(-1)^k \chi_{\Sigma_k}^{b}(\lambda), \tag{*}$$

Σ_k denoting a contraction $(+K_n \cup -M_k)/(-M_k)$ where M_k is a k-arc partial matching. Σ_k has the same polynomial as $(\pm K_{n-k}) \setminus (-K_{n-2k})$, as we now prove.

Let L consist of the left endpoints of the links in M_k and let R be the rest of N. If we switch L, the switched graph Σ_k^{ν} consists of a $+K_k$ on L and a $+K_{n-k}$ on R with every $v \in L$ and $w \in R$ linked by a negative arc, and a matching $+M_k$ of L onto $L' \subseteq R$. After we contract by M_k, we have merged L into R. The arcs (neglecting multiplicities) are a $+K_{n-k}$ on R, negative arcs between each $v \in L'$ and $w \in R$, and a negative loop at each $v \in L'$. For balanced polynomials the last named may be ignored, so in effect we have to deal with $(\pm K_{n-k}) \setminus (-K_{n-2k})$, as claimed.

Substituting into (*) from (6.1) and comparing coefficients of ${}_2(\lambda)_{n-l}$ yields

$$m_l(E^c) = \sum_{k=0}^{l} m_k(E)(-1)^k m_{l-k}(K_{n-2k}). \tag{6.2}$$

Thus the partial matching numbers of a simple graph (N, E) determine those of its complement. This relationship, which is apparently new, was discovered exactly as we have proved it here. It is, however, easy to prove directly. For more on this subject see [11].

Another explicit formula for w_{Σ}^{b} besides that in Corollary 2 is implied by

$$\psi_{\kappa r}^{*}(\Sigma) = (-1)^{n-r-\kappa} \sum_{q=0}^{L} \binom{n-q-\kappa}{r-q} \sum_{\substack{\pi \in \Pi_n; \\ \mathrm{rk}\,\pi = q}} m_{n-q-\kappa}([\Gamma^c/\pi]^c). \tag{6.3}$$

We omit the proof because of its complexity; it seems to require a generalization of (6.2) which appears in [11].

6.2. *Graphs with every negative link: chromatic coefficients of an ordinary graph*

A signed graph which contains $-K_n$ is the union of $-K_n$ with some all-positive graph, say $+\Gamma$. The balanced chromatic polynomial of Σ is a linear transform of $\chi_{\Gamma}(\lambda) = \sum_{\kappa} \psi_{\kappa}(\Gamma) \cdot (\lambda)_{\kappa}$, as we can see from (2.1) (in which $\varphi = \pi$ because $E_- = -K_n$).

Corollary 6.3. *Let $\Gamma = (N, E)$ and $\Sigma = -K_n \cup +E$. Then*

$$\psi_{\kappa}^{*}(\Sigma) = \psi_{\kappa}(\Gamma). \qquad \square$$

Thus *the chromatic number of Γ equals $\gamma^{*}(\Sigma)$.*

There is also a formula for w_{Σ}^{b} similar to that of Corollary 6.2, which we omit.

6.3. *Matchings vs. colorings*

Let $\Gamma=(N,E)$ again. The signed graph $\Sigma=+K_n\cup -E$ is the result of adding the arcs $+E^c$ to $\pm\Gamma$. Theorem 4.4 can be applied with $D=\emptyset$, $A=E^c$, $\Delta=K_n$, and $l(S)=+S$. The range of summation is $S=K_n:\pi$ where $\pi\in\Pi_n$ is stable in Γ. If we write μ_n for the Möbius function of the partition lattice Π_n, Theorem 4.4 in this case reads

$$\chi^b_\Sigma(\lambda)=\sum_{\substack{\pi \text{ stable}\\ \text{in } \Gamma}}\mu_n(0,\pi)2^{\#\pi}\chi_{\Gamma/\pi}(\tfrac12\lambda).$$

Here Γ/π means Γ with the blocks of π identified to points. Now let us expand $\chi_{\Gamma/\pi}(\frac12\lambda)$ in falling factorials, collect terms, and compare to Theorem 6.1. We have

$$m_{n-k}(\Gamma^c)=\sum_{\substack{\pi \text{ stable}\\ \text{in } \Gamma}}\mu_n(0,\pi)2^{\#\pi-k}\psi_k(\Gamma/\pi),\tag{6.4}$$

which shows a relationship between the exact coloring numbers of contractions by stable partitions and the partial matching numbers of the complementary graph. (Equation 6.4 can be proved directly by employing the identity

$$\sum_{\pi\in[\sigma,\tau]}\mu(\sigma,\pi)2^{\#\pi}=2^{\#\tau}\text{ or }0,$$

the former if τ is a matching of the blocks of σ, the latter otherwise. This identity is the special case $\Phi=+K_n$ of Lemma 2.4.) The inverse of (6.4) is

$$\psi_k(\Gamma)=\sum_{\pi \text{ stable}}m_{\#\pi-k}([\Gamma/\pi]^c)/2^{\#\pi-k}.\tag{6.5}$$

Example 6.4. In case $\Gamma=(N,\emptyset)$, equation (6.4) becomes

$$\frac{(n)_{2n-2k}}{(2n-2k)!!}=\sum_{i=0}^{n}s(n,i)2^iS(i,k),$$

where $s(n,i)$ is the Stirling number of the first kind.

The inverse formula, from (6.5), is

$$S(n,k)=\sum_{i=k}^{n}2^{-(i-k)}\frac{(i)_{2i-2k}}{(2i-2k)!!}.$$

7. Signed complete graphs

A signed complete graph is K_n with signs. For such graphs our general formulas are relatively simple; from them further relationships among partitions and matchings emerge. We shall treat principally the balanced polynomials; Theorem 1.1 shows how to compute the others.

A few results carry over to signed simple graphs; but in general the latter are more complex.

Let $\Sigma = (K_n, \sigma)$ be a signed complete graph. We write $\Sigma = K_Q$, where $Q = \sigma^{-1}(-)$ is the set of negative arcs, to emphasize which signed complete graph we have in mind. Note that $K_Q = -K_P$, where $P = \sigma^{-1}(+)$ is the set of positive arcs. We write $E = E(K_n) = P \cup Q$. The *two-graph* of K_Q (also called the *two-graph of* Q) is the class $\mathcal{T}(K_Q)$ of unbalanced triangles. (See [9, Section 7Ξ].) Since $\mathcal{T}(K_Q)$ determines the switching class of K_Q, we can think of $\chi_{K_Q}(\lambda)$ as the chromatic polynomial of the two-graph; and the same for the other polynomials.

Harary's criterion for balance (cf. [9, Proposition 2.1(ii)]) is in the case of a signed complete graph:

Lemma 7.1. *The following are equivalent:*
(i) K_Q *is balanced.*
(ii) N *partitions into* X *and* Y *(possibly void) so that* $P = E:X \cup E:Y$.
(iii) Q *is a complete bipartite graph on* N *or is void.* □

A criterion for $S \subseteq E$ to be a balanced flat (from [9, Proposition 5.6(ii)]) is that $S \cap Q$ be bipartitionally induced as in (5.1) and $S \cap P = P:\{X_1, Y_1, \ldots, X_k, Y_k\}$. Then by the algebraic connection expressed in [10, (2.4)], we have

$$\chi^b_{K_Q}(\lambda) = \sum_S \mu_{K_Q}(\emptyset, S)\lambda^{c(S)}, \tag{7.1}$$

S ranging over all balanced flats. (This reasoning applies also to signed simple graphs.)

Let us begin with two switching formulas derived from counting by magnitudes. *Graph switching* (or *Seidel switching*) means reversing the adjacencies between a node set and its complement. Let P^W stand for the graph (N, P) switched by $W \subseteq N$. From Theorem 2.2 we obtain

$$\chi^b_{K_Q}(\lambda) = \sum_{W \subseteq N} \chi(P^W; \tfrac{1}{2}\lambda), \tag{7.2}$$

$$w^b_{K_Q}(x, \lambda) = \sum_{W \subseteq N} w(P^W; x, \tfrac{1}{2}\lambda). \tag{7.3}$$

The polynomial of a signed complete graph is thus in a sense the average of those of the graphs in the graph-switching class of its positive part. (One can say also: the polynomial of a two-graph is the average of those of all complements of graphs in its switching class.) From (7.2) we deduce

$$\psi^*_k(K_Q) = 2^{-k} \sum_{W \subseteq N} \psi_k(P^W) \tag{7.4}$$

for the exact chromatic coefficients. (Formulas (7.2–4) hold for a signed simple graph (Δ, σ) with $P = \sigma^{-1}(+)$ if graph switching is carried out within Δ.)

Next we present the combinatorial interpretation of the balanced chromatic polynomial and a related formula for the Whitney polynomial. We shall regard a partition $\varphi \geq \alpha$ in Π_n, by abuse of notation, as a partition of the set α, writing

$\varphi \in \Pi_\alpha$. Let

$\beta_k(\Gamma)$ = the number of partitions of $N(\Gamma)$ into k blocks, each of which is either stable in Γ or induces a complete bipartite graph.

(These are also the partitions φ such that $-K_\Gamma : \varphi$ is balanced, i.e. every triangle contained in a block is in $\mathcal{T}(K_\Gamma)$.)

Corollary 7.2. *The zero-free exact chromatic coefficients of K_Q are given by*

$$\psi_\kappa^*(K_Q) = \beta_\kappa(P) = \sum_j \sum_{\substack{\pi \in \Pi_n;\ \#\pi = j+k, \\ \text{stable in } P}} m_j([Q/\pi]^c). \tag{7.5}$$

The balanced Whitney polynomial is given by

$$w^b_{K_Q}(x, \lambda) = \sum_{\alpha \in \Pi(P)} x^{n-\#\alpha} \sum_{\varphi \in \Pi_\alpha} \sum_R x^{\operatorname{rk} R} \cdot {}_2(\lambda)_{\#\varphi}, \tag{7.6}$$

where R ranges over subsets of $(Q/\alpha):\varphi$ such that, in each block of φ, $R \cup (P/\alpha)$ is either $\emptyset$ or a complete bipartite graph.

Proof. Equations (7.5) are immediate from the two parts of Theorem 2.1(A).

We deduce (7.6) from Theorem 2.1(B). Evidently by writing the Whitney polynomial expressions in the form $\sum_\varphi(\cdot)$ we can examine each block of φ separately. Let us therefore consider the case $\varphi = 1$. We have to prove that

$$\sum_{\alpha \in \Pi(P)} x^{n-\#\alpha} \sum_R x^{\operatorname{rk} R}, \tag{a}$$

the range of R being $R \subseteq Q/\alpha$ such that $R \cup (P/\alpha)$ is either $\emptyset$ or complete bipartite, is equal to

$$\sum_{\substack{\pi \in \Pi_n; \\ \#\pi \leqslant 2}} x^{\operatorname{rk}([P:\pi] \cup [Q \setminus (Q:\pi)])}. \tag{b}$$

If we set $\alpha = \pi(P:\pi)$ and $R = (Q/\alpha) \setminus (Q/\alpha : \pi)$, then (b) is the same as

$$\sum_{\alpha \in \Pi(P)} x^{n-\#\alpha} \sum_\pi x^{\operatorname{rk} R}, \tag{b$'$}$$

summed over bipartitions $\pi = \{X, Y\}$ of α for which $(P/\alpha):\pi = \emptyset$. Now clearly $R \cup (P/\alpha) = (E/\alpha):\langle X, Y \rangle$, which is void or complete bipartite. So α and R are as required by the sum (a).

Conversely given α and R as in (a), we let $\pi = \{X, Y\}$ = the bipartition of α determined by $R \cup (P/\alpha)$. Then $(P/\alpha):\pi = \emptyset$ and $R = (Q/\alpha):\langle X, Y \rangle = (Q/\alpha) \setminus (Q/\alpha : \pi)$. So α and R are as required in the summation (b$'$). □

Notice that φ in (7.6) is restricted to partitions such that $(P/\alpha):\varphi$ is bipartite. The range and rank of R too are partially predictable. For instance if $(P/\alpha):B$ is nonvoid bipartite and has $c>1$ components, there are 2^c possible R, all of rank $\#B-1$. But such observations are not enough to simplify (7.6).

A bevy of formulas results from application of Theorem 4.4 with different choices of the deletion edge set D. One can choose any $D \subseteq E$ (then $A = D^c$). The form taken by Theorem 4.4 is found by noting that $\Gamma = (N, D)$. $\Delta = K_n$, and the lifting is $l \mid D = -K_Q \mid D$, $l \mid A = K_Q \mid A$. Thus we have

$$\chi^{b}_{K_Q}(\lambda) = \sum_{F} \mu_{K_n|F}(F \cap D, F) 2^{c(F)} \chi_{(D \cup F)/F}(\tfrac{1}{2}\lambda), \tag{7.7}$$

summed over edge sets F for which $F \cap D$ is a closed subset of F (in the ordinary polygon closure) and $l(F)$ is a balanced flat in the signed graph $\Phi = \pm D \cup l(A) = K_Q \cup \pm D$.

Unfortunately most choices of D lead to hopelessly complicated versions of (7.7). The most trivial choices of D, on the other hand, yield results we already know: choosing $D = E$ we obtain (7.5), while from $D = \emptyset$ we get the matroidal form of the balanced chromatic polynomial (see [10, (2.4)]). The most interesting selections are $D = P$ and $D = Q$.

Case $D = P$. Here we are regarding K_Q as obtained from $\pm(N, P)$ by deleting $-P$ and adding $+Q$. The lifting is to negative arcs in $\Phi = -K_n \cup +D$. The conditions on $l(F)$ and $F \cap D$, if rephrased in terms of $\varphi = \pi(F)$ and $\pi = \pi(F \cap D)$, give the following result:

$$\chi^{b}_{K_Q}(\lambda) = \sum_{\varphi \in \Pi_n} 2^{\#\varphi} \chi_{P/\varphi}(\tfrac{1}{2}\lambda) \sum_{\pi} \mu([E:\varphi/\pi]/[P:\varphi]), \tag{7.8}$$

where the range of π is $\pi \leq \varphi$ such that π is stable in P, π partitions in exactly two parts each nonsingleton block of φ, and each component of $P:\varphi$ is complete bipartite or a single node; and where $E:\varphi/\pi$ is shorthand for $(E:\varphi)\setminus(E:\pi)$ (this is a bipartitionally induced subgraph of K_n). The strong restriction on π means that few φ—only those for which all the components of $P:\varphi$ are complete bipartite graphs or single nodes—will contribute to the sum. For each sum φ the sum over π is computable: it equals the product over all $B \in \varphi$ of a quantity $(-1)^{l+m-1} f(l, m)$ which depends only on the numbers m of bipartite and l of singleton components of $P:B$. The function f is

$$f(l, m) = 2^{m-1} \sum_{k=0}^{l} \binom{l}{k} m^{l-k} \sum_{j=0}^{k} S(k, j)(m+j-1)!; \tag{7.9}$$

in case $m = 0$ this equals the $g(l)$ defined in (5.10).

Case $D = Q$. Here we view K_Q as the result of deleting $+Q$ and adding $-P$ to $\pm(N, Q)$. The lifting is to positive arcs in $\Phi = +K_n \cup -Q$. The sum is over sets F of the form $E:\varphi$ for some partition φ such that $F \cap Q$ is closed in F; this reduces to

$$\chi^{b}_{K_Q}(\lambda) = \sum_{\varphi} \mu_n(\pi(Q:\varphi), \varphi) 2^{\#\varphi} \chi_{Q/\varphi}(\tfrac{1}{2}\lambda), \tag{7.10}$$

summed over all $\varphi \in \Pi_n$ for which $Q:\varphi$ is polygon-closed in K_n. Here μ_n denotes the Möbius function of the partition lattice. Note that (7.10) is very much like (2.5) but involves fewer summands.

8. Orientations

In this section we apply our results to the calculation of

$$o(\Sigma) = \text{the number of acyclic orientations of } \Sigma$$

and

$$o_k(\Sigma) = \text{the number of acyclic orientations of all } k\text{-node contractions,}$$

in general and for examples. The tool is [10, Corollary 4.1], which says that $o(\Sigma) = |\chi_\Sigma(-1)|$ and $o_k(\Sigma)$ is the coefficient of x^{n-k} in $(-1)^n w_\Sigma(-x, -1)$. One should keep in mind the geometric interpretations of $o_k(\Sigma)$ as the number of k-dimensional faces of the arrangement of hyperplanes $H[\Sigma]$ and as the number of $(n-k)$-dimensional faces of the acyclotope $Z[\Sigma]$ (see [10, Section 4]). One purpose of this section is to solve the geometric problem of counting faces.

We denote by $o_k^*(\Sigma)$ the coefficient of x^{n-k} in the polynomial $(-1)^n w_\Sigma^{\mathrm{b}}(-x, -2)$ and we write $o^*(\Sigma) = |\chi_\Sigma^{\mathrm{b}}(-2)|$.

8.1. *Balanced expansion formulas*

From (1.4) we have for a full graph $\Sigma^\bullet$ the expression

$$o_k(\Sigma^\bullet) = \sum_{W \subseteq N} o_k^*(\Sigma : W), \tag{8.1}$$

whose inverse is

$$o_k^*(\Sigma) = \sum_{W \subseteq N} (-1)^{\# W^c} o_k(\Sigma^\bullet : W). \tag{8.1'}$$

And note that the number of acyclic orientations of $\Sigma^\bullet$ satisfies

$$o(\Sigma^\bullet) = o^*(\Sigma), \tag{8.2}$$

which gives one combinatorial interpretation of o^*.

From (1.2) we obtain for any Σ the expression

$$o_k(\Sigma) = \sum_{l=0}^{k} (-1)^{k-l} \sum_{\substack{W \subseteq N \\ b(\Sigma : W^c) = k-l}} o_l^*(\Sigma : W) \tag{8.3}$$

and from (1.1) and (8.2) we get

$$o(\Sigma) = \sum_{W^c \text{ stable}} (-1)^{\# W^c} o(\Sigma^\bullet : W). \tag{8.4}$$

We can generalize (8.4). For any m-vertex signed graph Φ, let $\beta_i^{(j)}(\Phi)$ denote the number of subsets $V \subseteq N(\Phi)$ of $m-j$ vertices whose induced graph $\Phi : V$ has exactly i balanced components. Substituting (8.1′) into (8.3) yields the expression

$$o_k(\Sigma) = \sum_{l=0}^{k} (-1)^{k-l} \sum_{X \subseteq N} o_l(\Sigma^{\cdot} : X) \sum_{j \geq 0} (-1)^j \beta_{k-l}^{(j)}(\Sigma : X^c). \tag{8.5}$$

These formulas reduce the enumeration of acyclic orientations of general signed graphs to that for the simpler full graphs.

8.2. *All-positive graphs*

For an all-positive signed graph $+\Gamma$ (where Γ is ordinary; that is, without half arcs or free loops) we can interpret the numbers $o_k^*(+\Gamma)$ in two ways. By counting colorings (for example) it is clear that

$$\chi_{+\Gamma}^{\mathrm{b}}(\lambda) = \chi_\Gamma(\lambda) \quad \text{and} \quad w_{+\Gamma}^{\mathrm{b}}(x, \lambda) = w_\Gamma(x, \lambda).$$

Thus by the contraction formula (2.4) in [10] and Stanley's interpretation [5] of $|\chi_\Gamma(-2)|$, we see that $o_k^*(+\Gamma)$ equals the number of compatible pairs of acyclic orientations and 2-colorings of all k-node contractions of Γ.

On the other hand $G(+\Gamma^{\cdot}) \cong G(\Gamma + v_0)$, where $\Gamma + v_0$ means Γ plus one extra node v_0 adjacent to every node of Γ. (We observed this in [9, Section 7A].) Thus

$$\lambda^{-1} w_{\Gamma + v_0}(x, \lambda) = \sum_{W \subseteq N} w_{+\Gamma:W}^{\mathrm{b}}(x, \lambda - 1) \tag{8.6}$$

and similarly for the chromatic polynomials. We conclude by inverting (8.6), or from (8.1′), that

$$o_k^*(+\Gamma) = \sum_{W \subseteq N} (-1)^{n-k} o_k([\Gamma : W] + v_0). \tag{8.7}$$

In particular,

$$o^*(+\Gamma) = o(\Gamma + v_0). \tag{8.8}$$

These interpretations of $o_k^*(+\Gamma)$ lend more meaning to the switching formulas that follow.

8.3. *Switching formulas*

From Theorem 2.2 we have

$$o_k^*(\Sigma) = \sum_{\nu} o_k^*(\Sigma_+^{\nu}), \tag{8.9}$$

summed over all switching functions $\nu : N \to \{\pm\}$. Since Σ_+^{ν} is an all-positive graph, the right-hand side of (8.9) has the combinatorial interpretations in terms of ordinary graphs described just above. In particular

$$o_k(\Sigma^{\cdot}) = \sum_{W \subseteq N} \sum_{\nu} o_k(|(\Sigma : W)_+^{\nu}| + v_0), \tag{8.10}$$

the range of ν being switching functions $W \to \{\pm\}$.

8.4. *Other general formulas*

From Theorem 2.3 we can express the $o_k^*(\Sigma)$ in terms of contractions of the underlying ordinary graph $\Delta = |\Sigma^*|$. From (2.4a) we have

$$o_k^*(\Sigma) = \sum_{F,\nu} \sum_{c(F_\nu^*)=k} (-1)^{c(F_\nu^*)-c(F)} o(\Delta/F) \tag{8.11}$$

and from (2.5) we get

$$o(\Sigma^\cdot) = o^*(\Sigma) = \sum_{\substack{F \in \mathrm{Lat}\,\Delta \\ -\Sigma | F \text{ balanced}}} (-2)^{c(F)} o(\Delta/F). \tag{8.12}$$

Making the appropriate substitutions in Theorem 5.4 yields an ordinary-graphic expression via addition/deletion:

$$o(\Sigma^\cdot) = \sum_{\substack{F_1 \in \mathrm{Lat}\,\Delta; \\ F_1 \subseteq D, \\ l(F_1) \text{ balanced}}} (-1)^{\mathrm{rk}\, F_1} \sum_{\substack{F \in \mathrm{Lat}\,\Delta; \\ F_1 \leq F \subseteq A \cup D, \\ l(F) \text{ balanced}}} |\mu_\Delta(F_1, F)|\, 2^{c(F)} o(\Gamma/F). \tag{8.13}$$

This is less complicated than it looks, since it can be expected to have few outer summands; moreover the inner summands, being positive, do not cancel each other.

8.5. *Sign-symmetric graphs*

Let Γ be an ordinary graph, $U \subseteq N = N(\Gamma)$, and Σ be $\pm\Gamma$ with the nodes in U filled. By Theorem 3.1 we have $o_k^*(\Sigma) = 2^n o_k(\Gamma)$; this together with the balanced expansion formulas in Section 8.1 yields

$$o(\Sigma) = \sum_{\substack{W \supseteq U; \\ W^c \text{ stable in } \Gamma}} (-1)^{n-\#W} 2^{\#W} o(\Gamma : W)$$

and

$$o_k(\Sigma) = \sum_{W \subseteq N} (-1)^{i(W^c)} 2^{\#W} o_{k-i(W^c)}(\Gamma : W),$$

where $i(W^c)$ = the number of isolated nodes of $\Gamma : W^c$ which lie outside U. These results are treated in more detail in [8] (where the language is that of arrangements of hyperplanes and the proof avoids most of the machinery we have employed here).

8.6. *The classical root system arrangements*

The most important elementary examples are $\Sigma = +K_n$, $\pm K_n^\cdot$ and $\pm K_n$. Then $H[\Sigma]$ is the arrangement of hyperplanes R^* dual to the root system $R = A_{n-1}$, B_n or C_n, and D_n, respectively. Thus the chambers of these root systems can be counted by means of the formulas of the previous subsection. (The results for these examples are very well known.)

8.7. *Root system subarrangements containing* A^*_{n-1}

If we are interested in the number of regions of such an arrangement of hyperplanes, we must calculate $o(\Sigma)$ where Σ is a signed graph containing $+K_n$, that is $\Sigma = \pm K_n \setminus (-E)$. This brings us to Theorem 6.1 and matchings. First let us consider $\Sigma^\cdot$, whose arrangement $H[\Sigma^\cdot]$ contains all the coordinate hyperplanes as well as A^*_{n-1}. From (6.1) and (8.2) we have

$$o(\Sigma^\cdot) = \sum_{i=0}^{[n/2]} (-1)^i 2^{n-i} (n-i)!\, m_i(E).$$

If we leave out the coordinate hyperplanes not corresponding to vertices in $U \subseteq N$, we have from (8.4) the formula

$$o(\Sigma) = \sum_{i=0}^{[n/2]} (-1)^i 2^{n-i} (n-i)! \left\{ m_i(E) - \sum_{v \notin U} m_{i-1}(E:\{v\}^c) \right\}.$$

Formulas for $o^*_k(\Sigma)$ can be deduced from Corollary 6.2.

8.8. *Root system subarrangements deficient in* A^*_{n-1}

Suppose an arrangement consists of $B^*_n = H[\pm K^\cdot_n]$ *except* for certain hyperplanes belonging to A^*_{n-1}. Then it equals $H[\Sigma^\cdot]$ for $\Sigma = \pm K_n \setminus (+D)$. Let Γ be the complementary graph $K_n \setminus D$. Then from Corollary 6.3 along with (8.2),

$$o(\Sigma^\cdot) = \sum_{k=0}^{n} (-1)^{n-k} 2^k k!\, \psi_k(\Gamma).$$

8.9. *All-negative graphs*

As another example consider the all-negative graph $-\Gamma$ (whose arrangement $H[-\Gamma]$ consists only of hyperplanes with equations $x_i + x_j = 0$) and its full version $-\Gamma^\cdot$. Based on Theorem 5.2 and equations (8.4) and (8.2):

$$o(-\Gamma) = \sum_{W^c \text{ stable}} \sum_{k=0}^{\#W} (-1)^{\#W-k} 2^k o_k(\Gamma : W),$$

$$o(-\Gamma^\cdot) = \sum_{k=0}^{n} (-1)^{n-k} 2^k o_k(\Gamma).$$

Recalling that $o((K_n : W)/F) = c(F)!$ for $F \in \text{Lat}\, K_n \cong \Pi_n$, one can easily calculate

$$o(-K^\cdot_n) = \sum_{k=0}^{n} (-1)^k 2^k k!\, S(n,k)$$

and

$$o(-K_n) = \sum_{k=0}^{n} (-1)^k 2^k k!\, [S(n,k) - nS(n-1,k)],$$

$S(n,k)$ being the Stirling number of the second kind.

8.10. *Signed complete graphs*

Finally we examine the signed complete graph K_Q whose negative arc set is Q. Equation (8.3) reduces to

$$o_k(K_Q) = o_k^*(K_Q) + \sum_{W_1 \subset N} o_k^*(K_Q : W_1) - \sum_{W_2 \subset N} o_{k-1}^*(K_Q : W_2), \tag{8.14}$$

the ranges of W_1 and W_2 being the sets W for which $K_Q : W^c$ is respectively unbalanced and balanced. Equation (8.4) reduces to

$$o(K_Q) = o^*(K_Q) - \sum_{v \in N} o^*(K_Q : \{v\}^c). \tag{8.15}$$

(If we form Σ from K_Q by filling some nodes, say those in $U \subseteq N$, similar formulas hold but $\Sigma : W^c$ is always unbalanced if $W \not\supseteq U$, and v ranges over U^c.) Thus what is crucial is the evaluation of the o_k^*.

One evaluation is a switching-class 'averaging' formula deduced from (7.3):

$$o_k^*(K_Q) = \sum_{W \subseteq N} o_k^*(P^W).$$

A second kind of evaluation is based on the addition/deletion results (7.7), (7.8), and (7.9). All give nice expressions for $o^*(K_Q)$ in terms of acyclic orientations of ordinary graphs. To illustrate we adapt (7.8) and (7.10) to acyclic orientations. From the former,

$$o^*(K_Q) = \sum_{\varphi \in \Pi_n} (-1)^{\mathrm{rk}(P:\varphi)} o(P/\varphi) \prod_{B \in \varphi} f(i(P:B), c(P:B) - i(P:B)),$$

where $i(P:B)$ is the number of isolated nodes in the induced graph $P:B$. From (7.10), with μ_n denoting the Möbius function of the partition lattice,

$$o^*(K_Q) = \sum_{\varphi \in \Pi_n} (-1)^{\mathrm{rk}(Q:\varphi)} |\mu_n(\pi(Q:\varphi), \varphi)| \, 2^{\#\varphi} o(Q/\varphi).$$

9. In conclusion

We have developed three general varieties of formula for signed graphs: balanced expansion, sign/magnitude (including a combinatorial interpretation), and addition/deletion. But these do not exploit the full range. For instance a convolutional formula for ordinary graphs, provable by an easy coloring argument, extends to signed graphs in two forms:

$$\chi_\Sigma^b(\lambda + \mu) = \sum_{X \subseteq N} \chi_{\Sigma:X}^b(\lambda) \chi_{\Sigma:X^c}^b(\mu) \tag{9.1b}$$

and

$$\chi_\Sigma(\lambda + \mu) = \sum_{X \subseteq N} \chi_{\Sigma:X}(\lambda) \chi_{\Sigma:X^c}^b(\mu) \tag{9.1u}$$

(assuming Σ has no free loops). This and other expressions are treated in the general context of biased graphs (which include voltage graphs) in [7].

The method of counting by magnitudes generalizes to voltage graphs, carrying with it most of Section 2. Addition/deletion formulas, on the other hand, cannot extend without substantial modification because they depend strongly on the fact that in a signed graph there are only two possible labels for an arc. But since they are proved by the method of deletion and contraction they should generalize to the dichromatic polynomial

$$Q_\Sigma(u, v) = \sum_{S \subseteq E(\Sigma)} (uv)^{b(S)} v^{\#S-n},$$

the generalization of Tutte's dichromatic polynomial of a graph [6]. The dichromatic polynomial is investigated in [7] in the context of biased graphs.

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