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Isomorphisms of Lattices of Closed Convex Sets, II

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1. INTRODUCTION

We continue our study, begun in [1], of the isomorphisms of lattices of closed convex sets. We describe the isomorphisms between lattices of closed balanced convex subsets or between lattices of bounded closed balanced convex subsets of Hausdorff topological vector spaces. Moreover, we give a general isomorphism theorem for the lattice of all closed convex subsets of a Hausdorff locally convex space.

2. PRELIMINARIES

Let us define our notation. Let E be a real vector space. If $a, b \in E$, $[a: b]$ will denote the closed segment with end points a and b; if $a \neq b$, $(a: b)$ will be the straight line through a and b.

If $A \subset E$, c(A) [resp. ce(A)] will denote the convex [resp. convex-balanced] hull of A. If E is a real topological space, $\bar{\mathrm{c}}(A)$ [resp. $\overline{\mathrm{ce}}(A)$] denotes the closed convex [resp. convex-balanced] hull of $A \subset E$.

3. THE SEMILATTICE $\mathscr{P}_2{}^b(E)$ AND ITS ISOMORPHISMS

3.1. DEFINITION. Let E be a real vector space of dimension greater than 1. $\mathscr{P}_2{}^b(E)$ will denote the set whose elements are $\varnothing$, $\{0\}$, the balanced closed segments of E and the balanced convex 2-polytopes (i.e., $\mathscr{P}_2{}^b(E)$ is the set of all balanced convex k-polytopes, $-1 \leqslant k \leqslant 2$).

3.2. $\mathscr{P}_2{}^b(E)$, *ordered by inclusion, is a semilattice whose operation is the set intersection.*

Indeed, any finite intersection of convex polytopes of dimension not greater

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than 2 is empty or is a convex polytope of the same kind and any intersection of balanced sets is balanced.

3.3. *Remark.* The minimum of $\mathscr{P}_2{}^b(E)$ is $\varnothing$ and $\{0\}$ is its unique atom. If $a_1, \ldots, a_r$ are in the same plane, $[-a_i: a_i] \in \mathscr{P}_2{}^b(E)$ $(i = 1, \ldots, r)$ have a supremum

$$\bigvee_{i=1}^{r} [-a_i: a_i] = \mathrm{c}\left(\bigcup_{i=1}^{r} [-a_i: a_i]\right) = \mathrm{ce}\left(\bigcup_{i=1}^{r} \{a_i\}\right).$$

3.4. *Let E_1 and E_2 be $\mathbb{R}$-vector spaces such that* $\dim E_1 > 2$ *and* $\dim E_2 > 1$. *If φ is a semilattice isomorphism from $\mathscr{P}_2{}^b(E_1)$ to $\mathscr{P}_2{}^b(E_2)$, there is a vector isomorphism f from E_1 to E_2 such that, for any $P \in \mathscr{P}_2{}^b(E)$,*

$$\varphi(P) = f\left[\mathrm{ce}\left(\bigcup_{i=1}^{r} \lambda_{a_i}[-a_i: a_i]\right)\right],$$

when the a_i $(i = 1, \ldots, r)$ are the extreme points of P, and where λ_{a_i} is a real depending on a_i, such that $\lambda_{a_i} = \lambda_{-a_i}$.

Since $\{0\}$ is an atom of $\mathscr{P}_2{}^b(E_1)$, $\varphi(\{0\})$ is an atom of $\mathscr{P}_2{}^b(E_2)$; hence

$$\varphi(\{0\}) = \{0\}.$$

Let us show that, for any $a \in E_1 \backslash \{0\}$,

$$\varphi([-a: a]) = [-a': a'], \qquad a' \in E_2 \backslash \{0\}.$$

We shall use a reductio ad absurdum. If, for a certain $a \in E_1$, $\varphi([-a: a])$ is not a closed segment, $\dim {}^l\varphi([-a: a]) = 2$ (for $A \subset E$, ${}^lA$ is the affine hull of A).

Since $\varphi([-a: a])$ is convex and balanced, there are two balanced closed segments $[-b': b']$, $[-c': c']$ $(b' \neq 0,\ c' \neq 0)$, included in $\varphi([-a: a])$, whose intersection is $\{0\}$. The sets $\varphi^{-1}([-b': b'])$ and $\varphi^{-1}([-c': c'])$ are closed balanced convex subsets of $[-a: a]$, hence are balanced closed segments.

As φ is an isomorphism,

$$\begin{aligned}&\varphi^{-1}([-b': b']) \cap \varphi^{-1}([-c': c'])\\ &\quad = \varphi^{-1}([-b': b'] \cap [-c': c']) = \varphi^{-1}(\{0\}) = \{0\};\end{aligned}$$

hence $\varphi^{-1}([-b': b']) = \{0\}$ or $\varphi^{-1}([-c': c']) = \{0\}$. This leads to a contradiction:

$$[-b': b'] = (\varphi \circ \varphi^{-1})([-b': b']) = \{0\}$$

or

$$[-c': c'] = (\varphi \circ \varphi^{-1})([-c': c']) = \{0\}.$$

Let D be a straight line through 0:

$$D = \bigcup_{a \in D} [-a\colon a],$$

so

$$\varphi(D) = \bigcup_{a \in D} \varphi([-a\colon a]).$$

It is easy to see that the segments $\varphi([-a\colon a])$ are nested. Hence their union is contained in a unique straight line D' through 0. One can check that φ^{-1} maps every balanced closed segment of D' in D. So,

$$\Phi(D) = D'$$

defines a bijection from the set of all straight lines through 0 in E_1 to the set of all straight lines through 0 in E_2 (use φ^{-1} to see that Φ is a surjection).

Let D_1, D_2, and D_3 be straight lines through 0 in E_1 such that D_1 and D_2 are distinct and D_3 is included in the affine hull of $D_1 \cap D_2$. We shall show that $\Phi(D_1) \neq \Phi(D_2)$ and $\Phi(D_3) \subset {}^{l}[\Phi(D_1) \cup \Phi(D_2)]$.

Let $a \in D_1 \backslash \{0\}$, $b \in D_2 \backslash \{0\}$, and $c \in D_3 \backslash \{0\}$. Since Φ is bijective, $\Phi(D_1) \neq \Phi(D_2)$. Moreover, $D_3 \subset {}^{l}(D_1 \cup D_2)$ is equivalent to

$$[-c\colon c] \cap \mathrm{c}([-a\colon a] \cup [-b\colon b]) \neq \{0\},$$

so, we have

$$\varphi([-c\colon c]) \cap \mathrm{c}(\varphi([-a\colon a] \cup \varphi[-b\colon b])) \neq \varphi(\{0\}) = \{0\},$$

and

$$\Phi(D_3) \cap \mathrm{c}[\Phi(D_1) \cup \Phi(D_2)] \neq \{0\};$$

that is, $\Phi(D_3) \subset {}^{l}[\Phi(D_1) \cup \Phi(D_2)]$.

All the assumptions of Lemma A of Mackey [3, p. 245] are satisfied, so there is a vector isomorphism $f\colon E_1 \to E_2$ such that

$$f(D) = \Phi(D),$$

for all straight lines D through 0 in E_1.

Then, for every balanced closed segment in E_1,

$$\varphi([-a\colon a]) = \lambda_a f([-a\colon a]).$$

If $a = 0$, it is trivial, since $\varphi(\{0\}) = \{0\}$. If $a \neq 0$,

$$\varphi([-a\colon a]) \subset \Phi({}^{l}[-a\colon a]) = f({}^{l}[-a\colon a]) = {}^{l}f([-a\colon a]);$$

since $\varphi([-a: a])$ is a balanced closed segment not reduced to a point, $\varphi([-a: a]) = \lambda_a f([-a: a])$. Obviously, $\lambda_{-a} = \lambda_a$.

Let $P \in \mathscr{P}_2^b(E_1)$: $P = (\bigcup_{i=1}^r \{a_i\})$, where the a_i are the extreme points of P. It follows that

$$P = \mathrm{c}\left(\bigcup_{i=1}^r [-a_i: a_i]\right) = \bigvee_{i=1}^r [-a_i: a_i]$$

(this supremum exists in $\mathscr{P}_2^b(E_1)$, since the involved segments are in the same plane).

Hence,

$$\begin{aligned}\varphi(P) &= \bigvee_{i=1}^r \varphi([-a_i: a_i]) = \mathrm{ce}\left[\bigcup_{i=1}^r \lambda_{a_i} f([-a_i: a_i])\right] \\ &= \left\{\sum_{i=1}^r \alpha_i \lambda_{a_i} f([-a_i: a_i]) : \alpha_i \in \mathbb{R}\ (i = 1,..., r), \sum_{i=1}^r |\alpha_i| \leqslant 1\right\} \\ &= \left\{f\left(\sum_{i=1}^r \alpha_i \lambda_{a_i}[-a_i: a_i]\right) : \alpha_i \in \mathbb{R}\ (i = 1,..., r), \sum_{i=1}^r |\alpha_i| \leqslant 1\right\} \\ &= f\left[\mathrm{ce}\left(\bigcup_{i=1}^r \lambda_{a_i}[-a_i: a_i]\right)\right].\end{aligned}$$

3.5. *Let E_1 and E_2 be $\mathbb{R}$-vector spaces of dimension greater than* 1.

The semilattices $\mathscr{P}_2^b(E_1)$ and $\mathscr{P}_2^b(E_2)$ are isomorphic if and only if the $\mathbb{R}$-vector spaces E_1 and E_2 are isomorphic.

The "if" part is obvious: φ defined by

$$\varphi(P) = f(P), \qquad \forall P \in \mathscr{P}_2^b(E_1),$$

is a semilattice isomorphism, if $f: E_1 \to E_2$ is a vector isomorphism.

Let us prove the "only if" part. If $\dim E_1 > 2$, one need only to use 3.4. If $\dim E_1 = 2$, $\dim E_2 = 2$ (if not, 3.4 applied to φ^{-1} would show that $\dim E_1 > 2$) and E_1 and E_2 are canonically isomorphic.

4. $\mathscr{F}^e(E)$, $\mathscr{B}^e(E)$ AND THEIR ISOMORPHISMS

4.1. DEFINITION. Let E be a topological vector space [resp. a locally convex space]. $\mathscr{F}^e(E)$ [resp. $\mathscr{B}^e(E)$] will denote the set of all closed [resp., bounded closed] balanced convex subsets of E.

Ordered by inclusion, these sets are lattices whose operations of infimum and supremum are, respectively, set intersection and the closed convex hull of set union.

4.2. *Let* E_1 *and* E_2 *be Hausdorff topological* [resp. *locally convex*] $\mathbb{R}$*-vector spaces of dimension greater than* 1, *the dimension of* E_1 *being greater than* 2.

If φ *is a lattice isomorphism from* $\mathscr{F}^e(E_1)$ *to* $\mathscr{F}^e(E_2)$ [resp. $\mathscr{B}^e(E_1)$ *to* $\mathscr{B}^e(E_2)$], $\varphi\,|_{\mathscr{P}_2{}^b(E_1)}$ *is a semilattice isomorphism from* $\mathscr{P}_2{}^b(E_1)$ *to* $\mathscr{P}_2{}^b(E_2)$.

First, $\varphi\,|_{\mathscr{P}_2{}^b(E_1)}(\varnothing) = \varnothing$ and $\varphi\,|_{\mathscr{P}_2{}^b(E_1)}(\{0\}) = \{0\}$.

To show that the balanced closed segments of E_1 are in correspondance with those of E_2, one proceeds as in 3.4, noting that if there is an $a \in E_1$ such that $\varphi([-a\colon a])$ is a straight line through 0, there is a $b \in E_1$ such that $\dim {}^l\varphi([-b\colon b]) \geqslant 2$ (take b such that $[-b\colon b] \supset [-a\colon a]\colon \varphi([-b\colon b]) \supset \varphi([-a\colon a]) = D$).

Since every balanced convex 2-polytope P is the supremum in $\mathscr{B}^e(E_1)$ or $\mathscr{B}^e(E_2)$ of the balanced closed segments whose endpoints are the extreme points of P, the proof is achieved.

4.3. *Let* E_1 *and* E_2 *be Hausdorff topological* $\mathbb{R}$*-vector spaces of dimension greater than* 1, *the dimension of* E_1 *being greater than* 2.

If φ *is a lattice isomorphism from* $\mathscr{F}^e(E_1)$ *to* $\mathscr{F}^e(E_2)$, *there is a vector isomorphism* $f\colon E_1 \to E_2$ *which preserves, as does its inverse, the closed vector subspaces.*

The conjunction of 3.4 and 4.2 shows that there is a vector isomorphism $f\colon E_1 \to E_2$ such that, for every $a \in E_1$,

$$\varphi([-a\colon a]) = \lambda_a f([-a\colon a]).$$

Let D be a straight line through 0 in E_1. We shall first prove that

$$\varphi(D) = \bigvee_{a \in D} \varphi([-a\colon a]).$$

For every $a \in D$,

$$D \supset [-a\colon a],$$

so

$$\varphi(D) \supset \varphi([-a\colon a]);$$

moreover, if $F \in \mathscr{F}^e(E_2)$ is such that

$$F \supset \varphi([-a\colon a]), \qquad \forall a \in D,$$
$$\varphi^{-1}(F) \supset [-a\colon a], \qquad \forall a \in D;$$

hence $\varphi^{-1}(F) \supset D$ and $F \supset \varphi(D)$. This proves the equality stated above.

Then,

$$\varphi(D) = \bar{c}\left[\bigcup_{a\in D} \lambda_a f([-a: a])\right] = \overline{f\left[c\left(\bigcup_{a\in D} \lambda_a[-a: a]\right)\right]};$$

hence

$$\varphi(D) \in f(D).$$

Assume that $\varphi(D) \subsetneq f(D)$: $\varphi(D)$ would be $\varnothing$, $\{0\}$, or a balanced closed segment. In every case this would lead to a contradiction.

So, we have $\varphi(D) = f(D)$.

Now, let $V \subset E_1$ be a closed vector subspace. Since

$$V = \bigcup_{D\in\mathscr{V}} D,$$

where $\mathscr{V}$ is the set of all straight lines through 0 included in V, one can show, proceeding as above, that

$$\varphi(V) = \bigvee_{D\in\mathscr{V}} \varphi(D) = \bar{c}\left[\bigcup_{D\in\mathscr{V}} f(D)\right] = \bar{c}[f(V)] = \overline{f(V)}.$$

Let $x \in \overline{f(V)}$: as $\overline{f(V)}$ is a vector subspace,

$$[-x: x] \subset \overline{f(V)},$$

hence,

$$\varphi^{-1}([-x: x]) \subset \varphi^{-1}[\overline{f(V)}] = \varphi^{-1}[\varphi(V)] = V,$$

or

$$\varphi^{-1}([-x: x]) = [-a: a] \subset V;$$

then,

$$x \in [-x: x] = \varphi([-a: a]) = \lambda_a f([-a: a]) \subset \lambda_a f(V) = f(V).$$

So, we have proved that $f(V)$ is closed: $\varphi(V) = f(V)$ and f transforms every closed vector subspace of E_1 into a closed vector subspace of E_2. The same reasoning can be applied to φ^{-1}. This ends the proof.

4.4. *Let E_1 and E_2 be Hausdorff locally convex $\mathbb{R}$-vector spaces of dimension greater than 1, the dimension of E_1 being greater then 2.*

If the lattices $\mathscr{B}^e(E_1)$ and $\mathscr{B}^e(E_2)$ are isomorphic, the vector spaces E_1 and E_2 are isomorphic.

It suffices to bring together 3.4 and 4.2.

4.5. *Let E_1 and E_2 be Hausdorff locally convex $\mathbb{R}$-vector spaces.*

The lattices $\mathscr{F}^e(E_1)$ and $\mathscr{F}^e(E_2)$ are isomorpic if and only if the topological vector spaces $(E_1, \sigma(E_1, E_1'))$ and $(E_2, \sigma(E_2, E_2'))$ are isomorphic.

The "if" part is obvious. Let us prove the "only if" part. If $\dim E_1 = 0$, 1, or 2, the same holds true for $\dim E_2$, so E_1 and E_2 are canonically isomorphic. If $\dim E_1 \geqslant 3$, Theorem 4.3 asserts that there exists a vector isomorphism $f: E_1 \to E_2$ which preserves, like its inverse, the closed vector subspaces.

Let ${}^tf: E_2{}^* \to E_1{}^*$ be the adjoint mapping of f: ${}^tf$ is an isomorphism. If $y' \in E_2'$ (i.e., if y' is continuous),

$$[{}^tf(y')]^{-1}(\{0\}) = (y' \circ f)^{-1}(\{0\}) = (f^{-1} \circ y'^{-1})(\{0\}) = f^{-1}(H'),$$

where $H' = y'^{-1}(\{0\})$ is a closed hyperplane of E_2, so $[{}^tf(y')]^{-1}(\{0\})$ is a closed hyperplane of E_1 and ${}^tf(y') \in E_1'$. The same can be done for $({}^tf)^{-1} = {}^t(f^{-1})$.

Hence, ${}^tf|_{E_2'}$ is an isomorphism from E_2' to E_1' and Theorem 4(1) of [2, p. 237] asserts that $(E_1, \sigma(E_1, E_1'))$ and $(E_2, \sigma(E_2, E_2'))$ are isomorphic.

4.6. *Let E_1 and E_2 be Hausdorff bornological locally convex $\mathbb{R}$-vector spaces.*

The lattices $\mathscr{F}^e(E_1)$ and $\mathscr{F}^e(E_2)$ are isomorphic if and only if the topological vector spaces E_1 and E_2 isomorphic.

The "if" part is obvious. To prove the "only if" part, we can dispose of dimension 0, 1 and 2 in a trivial way (cf. 4.5). If $\dim E_1 \geqslant 3$, the use of Theorem 4.3 gives rise to a vector isomorphism $f: E_1 \to E_2$ which preserves, like its inverse, the closed hyperplanes. Lemma 5.2 of [1] shows that f is a topological vector isomorphism.

5. Isomorphisms of $\mathscr{F}$(E): A Complement

5.1. Definition. Let E be a topological $\mathbb{R}$-vector space: $\mathscr{F}(E)$ denotes the lattice, ordered by inclusion of all closed convex subsets of E.

5.2. *Let E_1 and E_2 be Hausdorff locally convex $\mathbb{R}$-vector-spaces.*

The lattices $\mathscr{F}(E_1)$ and $\mathscr{F}(E_2)$ are isomorphic if and only if the topological vector spaces $(E_1, \sigma(E_1, E_1'))$ and $(E_2, \sigma(E_2, E_2'))$ are isomorphic.

The proof is just like that of 4.5 except that one uses Theorem 4.3 of [1] instead of 4.3 of this paper.

References

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