Universal representations of Lie algebras
by coderivations

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Abstract

A class of representations of a Lie superalgebra (over a commutative superring) in its symmetric algebra is studied. As an application we get a direct and natural proof of a strong form of the Poincaré–Birkhoff–Witt theorem, extending this theorem to a class of nilpotent Lie superalgebras. Other applications are presented. Our results are new already for Lie algebras.

Résumé

Une classe des représentations d’une superalgèbre de Lie (sur un superanneau commutatif) dans son algèbre symétrique est étudiée. Comme application on obtient une démonstration naturelle et directe d’une version forte du théorème de Poincaré–Birkhoff–Witt, qui étend ce théorème à une classe de superalgèbres de Lie nilpotentes. D’autres applications sont introduite. Les résultats obtenus sont nouveaux aussi pour les algèbres de Lie.

Keywords: Lie superalgebra; Coderivation; Poincaré–Birkhoff–Witt theorem

Mots-clés : Superalgèbre de Lie ; Coderivation ; Théorème de Poincaré–Birkhoff–Witt

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1. Introduction

We consider a commutative superring \( K = K_0 + K_1 \), and for all Lie superalgebra \( g \) over \( K \) we study the representations of \( g \) on its symmetric algebra \( S(g) \) which are by coderivations and universal.

The symmetric algebra \( S(g) \) has a natural structure of coalgebra, so we have a notion of coderivation of \( S(g) \). A representation \( \rho \) of \( g \) in \( S(g) \) is called by coderivations if \( \rho(a) \) is a coderivation of \( S(g) \) for all \( a \in g \). We focus on representations \( \rho \) by coderivations which are universal. This means informally that \( \rho \) is given by a formula independent of \( g \) (see Definition 5.1).

To each formal power series \( \varphi = c_0 + c_1 t + \cdots \in K_0[[t]] \) we associate a family \( \Phi(a) \equiv \Phi^a \) of coderivations of \( S(g) \) depending linearly of \( a \in g \) (see formula (15)).

We show that \( \Phi \) is an universal representation by coderivations if and only if
\[
\frac{\varphi(t + u) - \varphi(u)}{t} + \varphi(u) \frac{\varphi(t + u) - \varphi(t)}{u} + \varphi(t + u) = 0
\]
in \( K_0[[t, u]] \).

We show that, for Lie algebras over a \( \mathbb{Q} \)-algebra, all universal representations are of this form (Theorem 7.2).

The most interesting case is when the constant term \( c_0 \) is equal to 1. In this case, it is a simple matter to solve the functional equation, because we show (Theorem 5.1) that it is equivalent to the functional equation for the exponential function. This last equation has a non-trivial solution exactly when \( K \) contains \( \mathbb{Q} \). In this case the unique solution of (1) is the generation function for Bernoulli numbers
\[
\varphi = \varphi_1 \equiv \frac{t}{e^t - 1}.
\]

Let \( N \geq 2 \) be an integer. If we restrict to an \( N \)-nilpotent Lie superalgebra \( g \) over \( K \), we get similar results. To a truncated power series \( \varphi \in K_0[[t]]/t^N \) is associated a family of coderivations \( \Phi^a \) depending linearly of \( a \in g \). We show that \( \Phi \) is an universal representation by coderivations if and only if \( \varphi \) verifies Eq. (1) in \( K_0[[t, u]]/I_N \), where \( I_N \) is the ideal generated by \( \{t^i u^{N-1-i} \mid 0 \leq i \leq N - 1 \} \). There exists a solution with \( \varphi(0) = 0 \) exactly when \( \frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{N} \in K \), and in this case the unique solution is \( \varphi_1 \) mod \( t^N \).

We explain the relation of these results with the Poincaré–Birkhoff–Witt theorem. Let \( U(g) \) be the enveloping algebra of \( g \) and assume that \( K \supseteq \mathbb{Q} \). We use the representation obtained using the function \( \frac{t}{e^t - 1} \) to define a symbol map \( \sigma : U(g) \to S(g) \). We show that \( \sigma \) is an inverse for the symmetrization \( \beta : S(g) \to U(g) \), which gives a natural and direct proof of the fact that \( \beta \) is an isomorphism.

Let \( g \) be an \( N \)-nilpotent Lie superalgebra over a commutative superring containing \( \frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{N} \). Also in this case there is a canonical symbol map \( \sigma : U(g) \to S(g) \) which is an isomorphism. For \( N = 2 \) this is due to M. El-Agawany and A. Micali [6]. The case \( N \geq 3 \) is new.

Eq. (1) is a particular case of an equation studied in Section 6. As an application of this "more general equation" we study the universal representations by coderivations of \( g \times g \) on \( S(g) \).
Let \( g \) be any Lie superalgebra over a superring. The enveloping algebra \( U(g) \) also has a natural structure of coalgebra. Assume that the Poincaré–Birkhoff–Witt theorem is verified. Using the isomorphism \( \sigma^{-1} \), we show that an universal representation by coderivations \( \Phi : g \to \text{End}(S(g)) \) gives an universal representation by coderivations \( F : g \to \text{End}(U(g)) \).

We get a family of representations interpolating the left and the right regular representation and the adjoint representation of \( g \) in \( U(g) \).

2. Lie superalgebras over a superring

In this section we recall the basic definitions and examples used in the text, they are from super linear algebra [8].

We say that \( K \) is a superring if it is an unitary ring graded over \( \mathbb{Z}/2\mathbb{Z} \). We denote by \( K_0 \) and \( K_1 \) the subgroups of elements with even and odd degree, for each non-zero homogeneous element \( a \in K \) we denote by \( p(a) \) its degree. We have \( 1 \in K_0 \).

The superring \( K \) is called commutative if \( ab = (-1)^{p(a)p(b)}ba \) for all homogeneous and non-zero \( a, b \in K \), and \( a^2 = 0 \) for \( a \in K_1 \).

Convention 2.1. Each time we use the symbol \( p(a) \) for an element \( a \) of a graded group occurring in a linear expression, it is implicitly assumed that it is non-zero and homogeneous. Moreover the expression is extended by linearity. For example, the expression above will be written as \( ab = (-1)^{p(b)p(a)}ba \) for any \( a, b \in K \).

From now to the end of this section, \( K \) will be a fixed commutative superring. We denote by \( K_0^* \subseteq K \) the subgroup of invertible elements of \( K_0 \).

Definition 2.1. A commutative group \( (M, +) \) graded over \( \mathbb{Z}/2\mathbb{Z} \) is a \( K \)-module if it is equipped with a bilinear application \( M \times K \to M \) such that, for any \( \alpha, \beta \in K \) and \( m \in M \) we have

\[
(m\alpha)\beta = m(\alpha\beta), \quad p(m\alpha) = p(m) + p(\alpha).
\]

We denote by \( M_0 \) and \( M_1 \) the \( K_0 \)-submodules composed of even and odd elements.

In a \( K \)-module \( M \) we use the notation \( am := m\alpha(-1)^{p(\alpha)p(m)} \), for any \( m \in M \) and \( \alpha \in K \). Let \( N \) be another \( K \)-module. A map \( f : M \to N \) is a morphism of \( K \)-modules if \( f(m\alpha) = f(m)\alpha \) for any \( m \in M \) and \( \alpha \in K \).

Definition 2.2. We say that \( A \) is a \( K \)-superalgebra if it a \( K \)-module equipped with a distributive application \( A \times A \to A \) such that

\[
p(a \cdot b) = p(a) + p(b), \quad (a \cdot b)a = a \cdot (ba) = (-1)^{p(b)p(a)}(aa) \cdot b
\]

for any \( a, b \in A \) and \( \alpha \in K \). We say that \( A \) is commutative if \( a \cdot b = (-1)^{p(a)p(b)}b \cdot a \) for \( a, b \in A \), and \( c^2 = 0 \) for \( c \in A_1 \).
Let $A$ and $B$ be two $K$-superalgebras. A map $f : A \rightarrow B$ is said a morphism of $K$-superalgebras if it is a morphism of $K$-modules such that

$$p(f(a)) = p(a) \quad \text{and} \quad f(a \cdot b) = f(a) \cdot f(b), \quad \forall a, b \in A.$$ 

**Notation 2.1.** Let $A$ be a $K$-superalgebra and $a \in A$. We denote by $a_L : A \rightarrow A$ the left multiplication by $a$, and by $a_R : A \rightarrow A$ the right multiplication by $a$:

$$a_R(b) = (-1)^{p(a)p(b)} b \cdot a, \quad \forall a, b \in A.$$

The following is our definition of Lie superalgebra.

**Definition 2.3.** Let $g$ be a $K$-superalgebra such that its product $[\cdot, \cdot] : g \times g \rightarrow g$ verifies

1. $[X, Y] = -(-1)^{p(X)p(Y)} [Y, X], \quad \forall X, Y \in g,$
2. $[X, X] = 0, \quad \forall X \in g_0,$
3. $[[X, Y], Z] = [X, [Y, Z]] - (-1)^{p(Y)p(X)} [Y, [X, Z]], \quad \forall X, Y, Z \in g,$
4. $[Y, [Y, Y]] = 0, \quad \forall Y \in g_1.$

Such $g$ is called a Lie $K$-superalgebra.

The product in a Lie superalgebra is called Lie product or Lie bracket, and (4) is the Jacobi identity.

**Remark 2.1.** If $2 \in K$ is invertible (3) follows from (2). If $3 \in K$ is invertible (5) follows from (2) and (4).

As explained in [2], if $g_1 \neq \{0\}$ and $2 \in K$ is not invertible, Definition 2.3 is not the right one, but it is sufficient for the porpoise of this text.

We end this section with some useful examples.

**Example 2.1.** If $A$ is a commutative superring, $A[[z]]$ denotes the set formal series in $s$, with coefficients in $A$. It inherits the graduation $(A[[z]])_0 = A_0[[z]], (A[[z]])_1 = A_1[[z]]$ and a natural structure of commutative superring.

**Example 2.2.** Let $M, N$ be two $K$-modules.

(a) $\text{Hom}(M, N)$ is the group of functions $F : M \rightarrow N$ which are morphisms of $K$-modules. It is graded in the following way: $F$ is even if $F(M_0) \subseteq N_0$ and $F(M_1) \subseteq N_1$, $F$ is odd if $F(M_0) \subseteq N_1$ and $F(M_1) \subseteq N_0$. Moreover, $\text{Hom}(M, N)$ is a $K$-module by $F \alpha : v \mapsto (-1)^{p(v)p(\alpha)} F(v)\alpha$, for all $\alpha \in K, v \in M$.

(b) $M \otimes N$ is the $K$-module generated by $\{v \otimes w; v \in M, w \in N, \}$ with relations

$$v_1 \otimes w_1 + v_2 \otimes w_2 = v_1 \otimes w + v_2 \otimes w, \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \quad (v \otimes w)\alpha = v \otimes w\alpha = (-1)^{p(w)p(\alpha)} v\alpha \otimes w, \quad \forall \alpha \in K,$$
and graduation $p(v \otimes w) = p(v) + p(w)$. 
The tensor algebra of $M$ is $T(M) := \mathbb{K} + M + (M \otimes M) + (M \otimes M \otimes M) + \cdots$ with product $(v_1 \otimes \cdots \otimes v_i) \cdot (v_{i+1} \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n$, for all $i, n \in \mathbb{N}$. It is an associative $\mathbb{K}$-superalgebra.

**Example 2.3.** Let $M$ a $\mathbb{K}$-module, $\text{End}(M) := \text{Hom}(M, M)$ is a Lie $\mathbb{K}$-superalgebra with bracket $[F, G] = F \circ G - (-1)^p(F)p(G)G \circ F$, for any $F, G \in \text{End}(M)$.

### 3. Symmetric algebras

In all this section $\mathbb{K}$ is a commutative superring. Let $M$ be a $\mathbb{K}$-module, we recall the definition of its symmetric algebra $S(M)$. The tensor algebra $T(M)$ contains the ideal $I$ generated by \{ $v \otimes w - (-1)^{p(v)p(w)}w \otimes v, u \otimes u \mid v, w \in M, u \in M_1$ \}, and we define $S(M) := T(M)/I$. It is a commutative and associative $\mathbb{K}$-superalgebra. We have $S(M) = \mathbb{K} \oplus \bigoplus_{n=1}^{\infty} S^n(M)$, where $S^n(M)$ is the $\mathbb{K}$-module generated by products of $n$ elements of $M$.

#### 3.1. Formal functions

We recall that $S(M)$ has a natural structure of cocommutative Hopf superalgebra, and in particular it is a coalgebra. This means that $S(M)$ is equipped with three morphisms of superalgebras $\Delta : S(M) \to S(M) \otimes S(M)$, $\varepsilon : S(M) \to \mathbb{K}$, $\delta : S(M) \to S(M)$, such that

\begin{align*}
(id \otimes \Delta) \circ \Delta &= (\Delta \otimes id) \circ \Delta, \quad (6) \\
\text{Mult} \circ (id \otimes \delta) \circ \Delta &= \text{Mult} \circ (\delta \otimes id) \circ \Delta = \varepsilon, \quad (7) \\
\text{Mult} \circ (id \otimes \varepsilon) \circ \Delta &= \text{Mult} \circ (\varepsilon \otimes id) \circ \Delta = id, \quad (8) \\
\Delta &= \sigma \circ \Delta, \quad (9)
\end{align*}

where Mult : $S(M) \otimes S(M) \to S(M)$ is the multiplication of $S(M)$, $\sigma : S(M) \otimes S(M) \ni W \otimes Z \mapsto (-1)^{p(W)p(Z)}Z \otimes W$ is the exchange operator. We call $\delta$ an antipode, and each even morphism of $\mathbb{K}$-modules verifying (6) is called an associative comultiplication. We refer to (9) saying that $\Delta$ is cocommutative.

For any $X \in M$ we have

\[ \Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad \delta(X) = -X. \]

To give formulas for $\Delta$ we introduce the following notation. Let $n \in \mathbb{N}$ and $\Sigma_n$ be the group of permutations of $n$ elements. For any $s \in \Sigma_n$ and $X_1, \ldots, X_n \in M$, let $\alpha(X_{s(1)}, \ldots, X_{s(n)}) \in \{1, -1\}$ be the sign such that $\alpha(X_{s(1)}, \ldots, X_{s(n)})X_{s(1)} \cdots X_{s(n)} = X_1 \cdots X_n$ in $S(M)$. If $X \in M_0$

\[ \Delta(X^n) = \sum_{j=0}^{n} \binom{n}{j} X^j \otimes X^{n-j}, \quad \forall n \geq 0, \quad (10) \]
and, if \( X_1, \ldots, X_n \in M \)

\[
\Delta(X_1 \cdots X_n) = \sum_{j=0}^{n} \sum_{1 \leq p_1 < \cdots < p_{j} \leq n} \alpha(\vec{X}) X_{p_1} \cdots X_{p_j} \otimes X_1 \cdots \hat{X}_{p_1} \cdots \hat{X}_{p_j} \cdots X_n,
\]

where \( \alpha(\vec{X}) := \alpha(X_{p_1}, \ldots, X_{p_j}, X_1, \ldots, \hat{X}_{p_1}, \ldots, \hat{X}_{p_j}, \ldots, X_n) \).

We denote \( S(M)^* := Hom(S(M), \mathbb{K}) \). Because of \( S(M) \) is a coalgebra, \( S(M)^* \) is a commutative superalgebra and it is called the algebra of formal power series over \( M \).

More generally, if \( N \) is a \( \mathbb{K} \)-module, \( Hom(S(M), N) \) is called the space of formal functions on \( M \) with values in \( N \). Each \( X \in N \) defines a “constant function” of \( Hom(S(M), N) \): it is the function such that \( 1 \mapsto X \) and \( S^n(M) \mapsto \{0\} \) for \( n \neq 0 \). We have the following structure of \( S(M)^* \)-module:

\[
F \cdot \phi := \text{Mult} \circ (F \otimes \phi) \circ \Delta \text{ for } \phi \in S(M)^* \text{ and } F \in Hom(S(M), N).
\]

It is called the derivation in the direction \( Y \).

**Remark 3.1.** By definition, \( \partial(Y)(X) = 0 \) for any \( X \in N \).

When \( N = M \), \( Hom(S(M), M) \) is called the space of formal vector fields over \( M \). The identity of \( M \) extends to a morphism of \( \mathbb{K} \)-modules \( x_M : S(M) \to M \) by \( S^n(M) \mapsto \{0\} \) for \( n \neq 1 \). It is called the generic point of \( M \), and it will be denoted by \( x \) when there is no risk of confusion.

**Remark 3.2.** We have \( \partial(Y)(x) = Y \) for any \( Y \in M \).

Let \( A \) be a \( \mathbb{K} \)-superalgebra. In \( Hom(S(M), A) \) we have the following structure of \( S(M)^* \)-superalgebra: \( F \cdot G := \text{Mult} \circ (F \otimes G) \circ \Delta \), for any \( F, G \in Hom(S(M), A) \).

**Remark 3.3.** For any \( Y \in M \), \( \partial(Y) \) is a derivation of \( Hom(S(M), A) \).

We have seen that \( A \subseteq Hom(S(M), A) \), moreover \( A \) is a \( \mathbb{K} \)-subsuperalgebra of \( Hom(S(M), A) \). If \( A \) is associative \( Hom(S(M), A) \) is associative, because \( \Delta \) verifies (6). If \( A \) is unitary \( Hom(S(M), A) \) is unitary, with the unit given by \( \varepsilon : S(M) \to W \mapsto 1 \epsilon(W) \in A \). If \( A \) is commutative \( Hom(S(M), A) \) is commutative, because \( \Delta \) is a cocommutative comultiplication.

In the particular case \( A = S(M) \), it is a tradition to denote by \( * \) the product of \( Hom(S(M), S(M)) \). In this case \( \delta \in Hom(S(M), S(M)) \) and identities (7), (8) give

\[
\delta \ast id = id \ast \delta = \varepsilon.
\]

**Lemma 3.1.** If \( g \) is a Lie \( \mathbb{K} \)-superalgebra, \( Hom(S(M), g) \) is a Lie \( S(M)^* \)-superalgebra.

### 3.2. Coderivations of a symmetric algebra

Let \( A \) be a \( \mathbb{K} \)-module equipped with a comultiplication \( \Delta \).
Definition 3.1. A coderivation of $A$ is a morphism of $K$-modules $\Phi: A \to A$ such that $\Delta \circ \Phi = (\Phi \otimes \text{id} + \text{id} \otimes \Phi) \circ \Delta$.

To describe the coderivations of $S(M)$, we introduce the $K$-module $P(S(M)) := \{ W \in S(M) | \Delta(W) = 1 \otimes W + W \otimes 1 \}$. Its elements are called the primitive elements of $S(M)$. By definition of $\Delta$, $M \subseteq P(S(M))$.

Let $\varphi: S(M) \to P(S(M))$ be a morphism of $K$-modules, we define $\Phi := \text{id}^* \varphi: S(M) \to S(M)$.

Theorem 3.1 (Theorem 1 [11]). The map $\Phi$ is the unique coderivation of $S(M)$ such that $\delta^* \Phi = \varphi$.

3.3. The generic point of a Lie superalgebra

Let $(g, [\cdot, \cdot])$ be a Lie $K$-superalgebra. For any $X \in g$, we denote by $\text{ad} X$ the application $[X, \cdot]: g \to g$.

Let $t$ and $u$ be two even commuting variables. For any $r,q \in \mathbb{N}$ we introduce the notation $(tr_q : [Y,Z])_X := [(\text{ad} X)^r(Y), (\text{ad} X)^q(Z)]$, $\forall X \in g_0$, $\forall Y,Z \in g$.

By linearity it is extended to all polynomials in $K[t,u]$.

Lemma 3.2. For all $q \in K[z]$, $X \in g_0$ and $Y,Z \in g$, we have $q(\text{ad} X)([Y,Z]) = (q(t+u) : [Y,Z])_X$.

Proof. It is sufficient to consider $q(z) = z^k$ with $k \geq 1$. It means that it is sufficient to show that $(\text{ad} X)^k([Y,Z]) = \sum_{p=0}^{k} \binom{k}{p} [(\text{ad} X)^p(Y), (\text{ad} X)^{k-p}(Z)]$, $k \geq 1$.

This identity means that $\text{ad} X$ is an even derivation, which follows from the Jacobi identity. $\square$

We introduce $g_x := \text{Hom}(S(g), g)$. Each $X \in g$ is identified to its image in $g_x$. As seen above (Section 3.1), the comultiplication $\Delta$ of $S(g)$ and the bracket for $g$ allow to define the bilinear application $[F, G] := [\cdot, \cdot] \circ (F \otimes G) \circ \Delta$, for any $F, G \in g_x$. We have seen also that $g_x$ is a Lie $S(g)^*$-superalgebra and $g \subseteq g_x$ is a Lie $K$-subsuperalgebra. Let $x \in g_x$ be the generic point of $g$.

Remark 3.4. For any $n \in \mathbb{N} \setminus \{0\}$, $(\text{ad} x)^n: g_x \to g_x$ is a $S(g)^*$-morphism. In particular, if $Y \in g$, $(\text{ad} x)^n(Y): S(g) \to g$ is the map such that, for any $p \geq 0$ and $X_1, \ldots, X_p \in g$

$$X_1 \cdots X_p \mapsto \begin{cases} 0, & p = 0, \\ \sum_{s \in \Sigma_n} \alpha(Y, X_s) \text{ad} X_{s(1)} \circ \cdots \circ \text{ad} X_{s(n)}(Y), & p = n, \end{cases}$$

where $\alpha(Y, X_s) := (-1)^{s(Y) + \sum_{i=1}^{n} s(X_i)} \alpha(X_{s(1)}, \ldots, X_{s(n)})$.

If $n = 0$, $(\text{ad} x)^0(Y) := Y \in g_x$. 


Let \( q = c_0 + c_1 t + c_2 t^2 + \cdots \in K[[t]] \) and \( Y \in g \). As a consequence of Remark 3.4, we can define

\[
q(ad_x)(Y) := c_0 Y + c_1 (ad_x)(Y) + c_2 (ad_x)^2(Y) + \cdots.
\]

It is the morphism of \( K \)-modules from \( S(g) \) to \( g \) such that, for any \( n \in \mathbb{N} \), its restriction to \( S^n(g) \) is \( c_n(ad_x)^n(Y) \).

**Remark 3.5.** In the Lie superalgebra \( g_x \), we consider the formula (12) with \( X = x \). This gives the formula of a morphism of \( K \)-modules from \( S(g) \) to \( g \). Let \( Y, Z \in g \), for any \( n \in \mathbb{N} \) and \( X_1, \ldots, X_n \in g \), this morphism is given by

\[
(t^r u^q : [Y, Z])_x(X_1 \cdots X_n)
= \begin{cases} 
0, & n \neq r + q, \\
\sum_{s \in \Sigma_n} \alpha(X_s)[ad X_{s(1)} \circ \cdots \circ ad X_{s(r)}(Y), ad X_{s(r+1)} \circ \cdots \circ ad X_{s(n)}(Z)], & n = q + r,
\end{cases}
\]

where the coefficients \( \alpha(X_s) \) are given by

\[
\alpha(X_s) := (-1)^{p(X_1 + \cdots + X_s)p(Z) + p(Y)p(X_s(1) + \cdots + X_s(q))} \alpha(X_s(1), \ldots, X_s(r+q)).
\]

As above this allows to define \((p(t, u) : [Y, Z])_x\) for any formal power series \( p \in K[[t, u]] \). The following theorem plays a key role in this text.

**Theorem 3.2.** Let \( Y, Z \in g \) and \( q(z) \in K[[z]] \). In \( g_x \) we have

\[
\partial(Y)(q(ad_x)(Z)) = (-1)^{p(q)p(Y)} \left( \frac{q(t + u) - q(u)}{t} : [Y, Z] \right).
\]

**Proof.** We only need to consider the case \( q(z) = z^k \), with \( k \geq 0 \). For \( k = 0 \) the statement follows from Remark 3.1. We recall that \( \partial(Y) \) is a derivation. By induction over \( k \) and by the Jacobi identity in \( g_x \), we get

\[
\partial(Y)((ad_x)^{k+1}(Z)) = \partial(Y)((x, (ad_x)^k(Z))]
= [Y, (ad_x)^k(Z)] + [x, \partial(Y)((ad_x)^k(Z))]
= (u^k : [Y, Z]) + ad_x \left( \frac{(u + t)^k - u^k}{t} : [Y, Z] \right)_x
= (u^k : [Y, Z])_x + \left( (u + t)^k - u^k \right)_x : [Y, Z].
\]
4. Functional equations associated to coderivations

Let \( K \) be a commutative superring and \( \varphi(z) = \sum_j z^j/c_j \in K[[z]] \). For any Lie \( K \)-superalgebra \( g \) and \( a \in g \), we define the formal vector field

\[
\varphi^a := \varphi(\text{ad} x)(a) \in \mathfrak{g}_c.
\]

(13)

We recall from Remark 3.4 that

\[
\varphi^a(X_1 \cdots X_n) = (-1)^p(a)p(X_1 + \cdots + X_n) \sum_{\sigma \in \Sigma_n} c_{n\alpha}(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) \text{ad} X_{\sigma(1)} \circ \cdots \circ \text{ad} X_{\sigma(n)}(a)
\]

for any \( n \in \mathbb{N} \) and \( X_1, \ldots, X_n \in g \). In particular, if \( X \in g_0 \) we get

\[
\varphi^a(1) = c_0(a),
\]

\[
\varphi^a(X^n) = n!c_n(\text{ad} X)^n(a), \quad \forall n \geq 1.
\]

(14)

Remark 4.1. By Lemma 3.2 we have

\[
\varphi^{[a,b]} = (\varphi(t + u)[a,b])_x.
\]

By Theorem 3.1, to the formal vector field \( \varphi^a \) we associate the coderivation

\[
\Phi^a := \text{id} \ast \varphi^a \equiv \text{Mult} \circ (1 \otimes \varphi^a) \circ \Delta.
\]

(15)

For any \( n \in \mathbb{N} \) and \( X_1, \ldots, X_n \in g \) we have

\[
\Phi^a(X_1 \cdots X_n) = \sum_{j=0}^n \sum_{1 \leq p_1 < \cdots < p_j \leq n} a(\tilde{X}, \tilde{p}) X_{p_1} \cdots X_{p_j} \varphi^a(X_1 \cdots \tilde{X}_{p_1} \cdots \tilde{X}_{p_j} \cdots X_n),
\]

where

\[
a(\tilde{X}, \tilde{p}) := (-1)^{p(\varphi^a)p(X_{p_1} + \cdots + X_{p_j})} a(X_{p_1}, \ldots, X_{p_j}, X_1, \ldots, X_{\tilde{p}_1}, \ldots, \tilde{X}_{p_j}, \ldots, X_n).
\]

In particular, if \( X \in g_0 \) we get

\[
\Phi^a(X^n) = \sum_{j=0}^n \binom{n}{j} X^j \cdot \varphi^a(X^{n-j}).
\]

Let \( \psi \in K[[t]] \), \( b \in g \), and let \( \psi^b : S(g) \to S(g) \) be the associated coderivation.

Remark 4.3. By definition, for any \( Y \in g \) we have \( \text{id} \ast Y = Y^L \).
Lemma 4.1. For any $Y \in \mathfrak{g}$ we have:

(i) $\Phi^a \circ Y^L = id \ast (\phi^a \ast Y - \frac{(\phi(t+u) - \phi(t))}{u} \psi(u) : [a, Y])_x$.

(ii) $\Phi^a \circ \Psi^b = id \ast (\psi^a \ast \psi^b - (-1)^{p(a)p(\psi)} \frac{(\phi(t+u) - \phi(t))}{u} \psi(u) : [a, b])_x$.

Proof. (i) From the fact that $\Delta$ is, in particular, a morphism of algebras, and from Remark 4.3 we have $\Phi^a \circ Y^L = id \ast (\phi^a \circ Y - \phi^a \circ Y^L) + (-1)^{p(\phi)p(Y)} Y \ast \phi^a$. As $\ast$ is commutative, this shows that $\Phi^a \circ Y^L = id \ast (\phi^a \circ Y + (-1)^{p(\phi)p(Y)} Y \ast \phi^a)$, so Theorem 3.2 gives the desired formula.

(ii) Let us consider the Lie superalgebra $\mathfrak{g}_x$ and its generic point $y \in Hom(S(\mathfrak{g}_x), \mathfrak{g}_x)$. If $X_1, \ldots, X_n \in \mathfrak{g} \subset \mathfrak{g}_x$ we have $\Phi^a \circ \Psi^b(X_1 \cdots X_n) = \Phi^a \circ \Psi^b(X_1 \cdots X_n)$, so it is sufficient to prove the statement for $\mathfrak{g}_x$. By definition and by Remark 4.3 we have $\Phi^a \circ \Psi^b = \Phi^a \circ \Psi^b \circ (id \ast \psi(ad y)(b)) = \Phi^a \circ \psi(ad y)(b)^L$, so from case i, we get

$\Phi^a \circ \Psi^b = \Phi^a \circ (id \ast \psi(ad y)(b)) = \Phi^a \circ \psi(ad y)(b)^L$.

By definition

$\frac{(\psi(t+u) - \psi(t))}{u} \psi(u) : [a, \psi(ad y)(b)]_y$.

so the proof is finished. □

Theorem 4.1.

$[\Phi^a, \Psi^b] = id \ast (-1)^{p(\psi)p(a)} \left( \frac{(\psi(t+u) - \psi(t))}{u} \psi(u) - \frac{(\psi(t+u) - \psi(t))}{t} \psi(u) : [a, b] \right)_x$.

Proof. Let

$\omega(t, u) := -\frac{(\psi(t+u) - \psi(t))}{u} \psi(u) - \frac{(\psi(t+u) - \psi(t))}{t} \psi(u)$.

we denote by $\Omega^{[a, b]}$ the coderivation corresponding to $(-1)^{p(\psi)p(a)}(\omega(t, u) : [a, b])$. By Theorem 3.1 we want to show that $[\Phi^a, \Psi^b] = \Omega^{[a, b]}$. From Lemma 4.1 we get

$[\Phi^a, \Psi^b] = id \ast (-1)^{p(\psi)p(a)} (\frac{(\psi(t+u) - \psi(t))}{u} \psi(u) : [a, b] + \psi^a \ast \psi^b)$.
\[ + \text{id} \ast \left( (-1)^{p(b)p(\psi)p(\Phi^b)} \left( \frac{\psi(t + u) - \psi(t)}{u} \varphi(a) : [b, a] \right) \right) \]

\[ - (-1)^{p(\Phi^b)p(b)p(\psi)} \psi^b \ast \varphi^a \]

\[ = -(-1)^{p(\psi)p(\Phi^b)} \text{id} \ast \left( \left( (-1)^{p(\psi)p(\Phi^b)} \frac{\psi(t + u) - \psi(t)}{u} \varphi(a) : [b, a] \right) \right) \]

\[ = \text{id} \ast (-1)^{p(\psi)p(\Phi^b)} (\omega(t, u) : [a, b]). \]

To prove the next theorem we need some preliminaries, which we state in a form that will be useful later.

**Definition 4.1.** Let \( N \geq 1 \) be an integer. A Lie superalgebra \( g \) is said to be \( N \)-nilpotent if we have \( \text{ad} X_1 \circ \cdots \circ \text{ad} X_N = 0 \) for any \( X_1, \ldots, X_N \in g \).

**Remark 4.4.** For \( N = 1 \) we have a commutative Lie superalgebra, for \( N = 2 \) we have a Lie superalgebra of Heisenberg type.

**Lemma 4.2.** For any \( N \geq 2 \), there exists an \( N \)-nilpotent Lie \( \mathbb{K} \)-superalgebra \( g_N \), equipped with an infinite family of even elements \( \{ \alpha, \beta, X_1, X_2, \ldots \} \) such that

\[ \bigcup_{r,s \geq 0, 0 \leq r+s \leq N-2} \{ \left[ \text{ad} X_{i(1)} \circ \cdots \circ \text{ad} X_{i(r)}(\alpha), \text{ad} X_{i(r+1)} \circ \cdots \circ \text{ad} X_{i(r+s)}(\beta) \right] ; i(1), \ldots, i(r+s) \in \mathbb{N} \} \]

is contained in a basis.

**Proof.** We start by considering the free Lie algebras \( \mathfrak{h} \) over \( \mathbb{Z} \), with an infinite family of generators \( \alpha, \beta, X_1, X_2, \ldots \). By properties of free Lie algebras \([3, \text{Proposition 10, p. 26}]\) we know that \( \mathfrak{h} \) is free, and that

\[ \bigcup_{r,s \geq 0} \{ \left[ \text{ad} X_{i(1)} \circ \cdots \circ \text{ad} X_{i(r)}(\alpha), \text{ad} X_{i(r+1)} \circ \cdots \circ \text{ad} X_{i(r+s)}(\beta) \right] ; i(1), \ldots, i(r+s) \in \mathbb{N} \} \]

is contained in a basis of \( \mathfrak{h} \).

Let \( I_N \) be the ideal of \( \mathfrak{h} \) generated by \( \{ \text{ad} x_1 \circ \cdots \circ \text{ad} x_N(Y) : x_1, \ldots, x_N, Y \in \mathfrak{h} \} \). The quotient \( \mathfrak{h}_N := \mathfrak{h}/I_N \) is an \( N \)-nilpotent Lie superalgebra over \( \mathbb{Z} \) and the family

\[ f_N := \bigcup_{0 \leq r+s \leq N-2} \{ \left[ \text{ad} X_{i(1)} \circ \cdots \circ \text{ad} X_{i(r)}(\alpha), \text{ad} X_{i(r+1)} \circ \cdots \circ \text{ad} X_{i(r+s)}(\beta) \right] ; i(1), \ldots, i(r+s) \in \mathbb{N} \} \]

is contained in a basis of \( \mathfrak{h}_N \).
We define \( g_N := h_N \otimes K \). It is an \( N \)-nilpotent Lie superalgebra over \( K \) and \( f_N \) is contained in a basis of \( g_N \). □

**Lemma 4.3.** Let \( \omega(t,u) = \sum_{i,j=0}^{\infty} c_{ij} t^i u^j \in K[[t,u]] \) and \( N \geq 2 \). If for any \( N \)-nilpotent \( K \)-superalgebra \( g \) we have \( (\omega(t,u) : [a,b])_s = 0, \forall a, b \in g \), then \( c_{ij} = 0 \) for any \( 0 \leq i + j < N - 2 \).

**Proof.** We consider the case \( g = g_N \), where \( g_N \) is the \( N \)-nilpotent Lie superalgebra of Lemma 4.2. Choosing \( a = \alpha \) and \( b = \beta \) we get

\[
(\omega(t,u), [\alpha, \beta])_s = \sum_{i+j=0}^{N-2} c_{i,j} (t^i u^j : [\alpha, \beta])_s.
\]

Let \( 0 \leq p \leq N - 2 \), Remark 3.5 gives

\[
(\omega(t,u), [\alpha, \beta])_s (X_1 \cdots X_p) = \sum_{i=0}^{p} c_{i,p-i} \sum_{s \in \Sigma_p} \left[ \text{ad} X_{s(1)} \circ \cdots \circ \text{ad} X_{s(i)}(\alpha), \text{ad} X_{s(i+1)} \circ \cdots \circ \text{ad} X_{s(p)}(\beta) \right].
\]

As \( (\omega(t,u), [\alpha, \beta])_s (X_1 \cdots X_p) \) is zero, Lemma 4.2 gives that \( c_{i,p-i} = 0 \) for any \( i = 0, \ldots, p \). As \( 0 \leq p \leq N - 2 \), the proof is finished. □

Let \( \lambda \in K[[t]] \). For any \( a \in g \) we consider the formal vector field \( \lambda^a \) (see formula (13)) and the corresponding coderivation \( A^a : S(g) \rightarrow S(g) \) (see formula (15)).

**Theorem 4.2.** Let \( \varphi, \psi, \lambda \in K_0[[t]] \). For any Lie \( K \)-superalgebra \( g \) we have

\[
[\varphi^a, \psi^b] = A^{[a,b]}, \quad \forall a, b \in g
\]

if and only if \( \varphi, \psi, \lambda \) verify

\[
\left( - \frac{\varphi(t + u) - \varphi(t)}{u} \psi(u) - \varphi(t) \frac{\psi(t + u) - \psi(t)}{t} \right) = \lambda(t + u)
\]

in \( K_0[[t,u]] \).

**Proof.** Let \( \omega(t,u) := -\frac{\varphi(t + u) - \varphi(t)}{u} \psi(u) - \varphi(t) \frac{\psi(t + u) - \psi(u)}{t} - \lambda(t + u) \). Using Theorem 4.1 and Remark 4.2, we see that (16) is equivalent to \( \text{id} \ast (\omega(t,u) : [a,b])_s = 0, \forall a, b \in g \). By Theorem 3.1, this identity is equivalent to \( (\omega(t,u) : [a,b])_s = 0, \forall a, b \in g \).

We get immediately that the functional equation is sufficient. To show the converse, it is sufficient to apply Lemma 4.3 to any \( N \)-nilpotent Lie superalgebra \( g_N \), with \( N \geq 2 \). We get that in \( \omega(t,u) \) the coefficients of degree \( N - 2 \) are zero, for any \( N \geq 2 \). In particular \( \omega(t,u) = 0 \). □

**Theorem 4.3.** Let \( \varphi \in K[[t]] \). For any Lie \( K \)-superalgebra \( g \), we have

\[
[\varphi^a, \varphi^b] = \varphi^{[a,b]}, \quad \forall a, b \in g
\]
if and only if \( \varphi \) has even coefficients (\( \varphi \in \mathbb{K}_0[[t]] \)) and verifies
\[
\left( -\frac{\varphi(t+u) - \varphi(t)}{u} \varphi(u) - \varphi(t) \frac{\varphi(t+u) - \varphi(u)}{t} \right) = \varphi(t+u).
\]

**Proof.** As \( p(\Phi_a) \equiv p(\varphi) + p(a) \), the identity (17) needs \( p(\varphi) = 0 \). Theorem 4.3 follows from Theorem 4.2. \( \square \)

5. Universal representations

Let \( \mathbb{K} \) be a commutative superring, \( \varphi(t) = \sum_j t^j c_j \in \mathbb{K}_0[[t]] \), and \( \mathfrak{g} \) be a Lie \( \mathbb{K} \)-superalgebra. We consider map \( \Phi : \mathfrak{g} \ni a \mapsto \Phi^a \in \text{Hom}(S(\mathfrak{g}), S(\mathfrak{g})) \) defined in (15). From Theorem 4.3 we know that \( \Phi \) is a representation for any Lie \( \mathbb{K} \)-superalgebras \( \mathfrak{g} \), if and only if \( \varphi \in \mathbb{K}_0[[t]] \) verifies the functional equation (18). Before looking for solutions of this functional equation, we introduce the notion of universal representation.

Let \( M(\mathfrak{g}) \) be the symmetric algebra or the enveloping algebra of \( \mathfrak{g} \) (defined in Section 8).

**Definition 5.1.** Assume that
(i) for any \( \mathbb{K} \)-Lie superalgebra \( \mathfrak{g} \), we have a representation \( \Phi_{\mathfrak{g}} : \mathfrak{g} \to \text{End}(M(\mathfrak{g})) \).
(ii) For any \( a \in \mathfrak{g} \) and any morphism of Lie \( \mathbb{K} \)-superalgebras \( f : \mathfrak{g} \to \mathfrak{h} \) the following diagram, with \( \tilde{f} : M(\mathfrak{g}) \to M(\mathfrak{h}) \) the induced morphism of algebras, is commutative:
\[
\begin{array}{ccc}
M(\mathfrak{g}) & \xrightarrow{\Phi_{\mathfrak{g}}^a} & M(\mathfrak{g}) \\
\downarrow f & & \downarrow f \\
M(\mathfrak{h}) & \xrightarrow{\Phi_{\mathfrak{h}}^f} & M(\mathfrak{h})
\end{array}
\]
Then we say that \( \Phi \) is an universal representation in the category of Lie \( \mathbb{K} \)-superalgebras.

Let \( N \in \{2, 3, 4, \ldots \} \). In an analogous way we define the universal representations in the category of \( N \)-nilpotent Lie \( \mathbb{K} \)-superalgebras.

For any commutative superring \( \mathbb{K} \) we introduce
\[
\varphi_0(t) := -t \in \mathbb{K}_0[[t]].
\]
(19)
For any \( c \in \mathbb{K}_0^\times \), if \( \mathbb{K} \supseteq \mathbb{Q} \), we introduce also
\[
\varphi_c(t) = \frac{t}{e^t/c - 1} \in \mathbb{K}_0[[t]].
\]
(20)
All these series verify \( \varphi_c(0) = c \).

**Lemma 5.1.** Let \( \mathbb{K}_0 \) be a commutative field. The solutions of Eq. (18) which lie in \( \mathbb{K}_0[[t]] \) and such that \( \varphi(0) = 0 \), are \( \varphi = 0 \) and \( \varphi = \varphi_0 \).

**Proof.** If \( \varphi(0) = 0 \), the limit \( \lim_{u \to 0} \) applied to Eq. (18) gives \( \varphi(t)(1 + \varphi(t)/t) = 0 \), so \( \varphi(t) = 0 \) or \( \varphi(t) = -t \) because \( \mathbb{K}_0[[t]] \) is a domain. \( \square \)
Theorem 5.1. (i) Let $\varphi \in \mathbb{K}_0[[t]]$ be a solution of Eq. (18). If the constant term $\varphi(0) = c$ is invertible, then $f(t) := \frac{\varphi(t)}{\varphi(0)}$ satisfies
$$
\begin{align*}
&f(t) \cdot f(u) = f(t + u), \\
&f(0) = 1, \\
&f'(0) = 1/c.
\end{align*}
$$

(ii) System (21) has solutions if and only if $\mathbb{K}_0$ contains $\mathbb{Q}$. In this case the unique solution is $e^{t/c} \in \mathbb{K}_0[[t]]$.

(iii) Let $\mathbb{K} \supseteq \mathbb{Q}$ and $c \in \mathbb{K}_0^\times$. The unique solution of (18) in $\mathbb{K}_0[[t]]$ verifying $\varphi(0) = c$ is $\varphi_c(t)$.

Proof. (i) We recall that $c$ is invertible if and only if the series $\varphi$ is invertible, so we write Eq. (18) as
$$
\begin{align*}
&\varphi(t) + t \varphi(t) \cdot \varphi(u) + u \varphi(u) = \varphi(t + u) + t + u.
\end{align*}
$$
We have $\frac{\varphi(t) + t}{\varphi(t)} = 1 + \frac{t}{c} + \cdots$.

(ii) Let $f = 1 + \frac{t}{c} + \sum_{k=2}^{\infty} f_k t^k$. System (21) gives $f'(t) = \frac{1}{c} f(t)$, so $f_2 = \frac{1}{c}$ and $kf_k = \frac{1}{c} f_{k-1}$ for any $k \geq 3$. By induction we get that $k$ is invertible and $f_k = \frac{1}{k! c^k}$ for any $k \geq 2$.

(iii) When $f(t) = e^{t/c}$, we get $\varphi(t) = \varphi_c(t)$.  

\[ \Box \]

Remark 5.1. The Bernoulli numbers $\{b_k \in \mathbb{Q}, k \in \mathbb{N}\}$ are defined by the generating series
$$
\varphi_1(z) \equiv \frac{1}{e^z - 1} = \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k.
$$
For example $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{6}$. Let $c \in \mathbb{K}_0^\times$, the fact that $\varphi_c(t) \in \mathbb{K}_0[[t]]$ verifies identity (18) can be written in the following way:
$$
\forall k \geq 0, \quad 0 = b_k + \sum_{p=0}^{k-1} \binom{k-1}{p} b_{k-p-1} + \sum_{l=0}^{k} \binom{k}{l} b_{k+1-l}.
$$
We have shown the following theorem

Theorem 5.2. The map $\Phi_0 : \mathfrak{g} \to \text{Hom}(S(\mathfrak{g}), S(\mathfrak{g}))$ associated to $\varphi_0$ is a representation by coderivations.

Remark 5.2. Let $a \in \mathfrak{g}$. The map $\Phi_0^a$ is in the same time a derivation and a coderivation of $S(\mathfrak{g})$: for any $X_1, \ldots, X_n \in \mathfrak{g}$ we have
$$
\Phi_0^a(X_1 \cdots X_n) = \sum_{j=1}^{n} (-1)^{p(a)p(X_1+\cdots+X_{j-1})} X_1 \cdots \Phi_0^a(X_j) \cdots X_n.
$$
It is the only derivation of $S(\mathfrak{g})$ such that $\Phi_0^a(X) = [a, X]$ for $X \in \mathfrak{g}$, so $\Phi_0$ is the adjoint representation of $\mathfrak{g}$ in $S(\mathfrak{g})$. 

Theorem 5.3. Let $\mathbb{K} \supseteq \mathbb{Q}$. For any $c \in \mathbb{K}_0^\times$, the series $\varphi_c(z) \in \mathbb{K}_0[[z]]$ gives a representation by coderivations $\Phi_c : g \to \text{Hom}(S(g), S(g))$.

Let $g$ and $h$ be two Lie $\mathbb{K}$-superalgebras, $f : g \to h$ be a morphism of Lie $\mathbb{K}$-superalgebras. It extends to a morphism of $\mathbb{K}$-superalgebras $\overline{f} : S(g) \to S(h)$.

Remark 5.3 (Functorial property). By Remark 4.1, for any $a \in g$ and $c \in \mathbb{K}_0^\times$, the following diagram commutes

$$
\begin{array}{ccc}
S(g) & \xrightarrow{\Phi_c} & S(g) \\
\downarrow{f} & & \downarrow{f} \\
S(h) & \xrightarrow{\overline{\Phi}_c} & S(h).
\end{array}
$$

In particular, $\Phi_0$ and $\Phi_c$ with $c \in \mathbb{K}_0^\times$ are universal representations by coderivations.

5.1. The case of nilpotent Lie superalgebras

We give an analogue of Theorem 5.3 for $\mathbb{K}$ not necessarily containing $\mathbb{Q}$.

Let $N \geq 2, g$ an $N$-nilpotent Lie superalgebra over $\mathbb{K}$, $a$ and $b \in g$.

Remark 5.4. (i) The notation $\varphi(\text{ad } x)(a) \in g_x$ is well-defined for $\varphi \in \mathbb{K}_0[t]/tN$ a truncated polynomial with coefficients in $\mathbb{K}$.

(ii) The notation $(\rho(t, u) : [a, b])_x \in g_x$ is well-defined if $\rho(t, u) \in \mathbb{K}_0[t, u]/I_N$, where $I_N$ is the ideal generated by $\{t^iu^j, i + j \geq N - 1\}$.

To a truncated polynomial $\varphi(t) \in \mathbb{K}_0[t]/tN$, we associate by formulas (13) and (15), a family of coderivations still denoted by $\Phi_a : S(g) \to S(g)$, $a \in g$.

Theorem 5.4. For any $g$ an $N$-nilpotent Lie superalgebra over $\mathbb{K}$ the map $\Phi : g \ni a \mapsto \Phi^a \in \text{Hom}(S(g), S(g))$ is a representation by coderivations, if and only if $\varphi$ verifies

$$
\varphi(u)\frac{\varphi(t + u) - \varphi(t)}{u} + \varphi(t)\frac{\varphi(t + u) - \varphi(u)}{t} = -\varphi(t + u)
$$

in $\mathbb{K}_0[t, u]/I_N$.

Proof. The direct part of the following theorem is a particular case of Theorem 4.2. Let us prove the converse. Let

$$
(\omega(t, u) : [a, b])_x = 0, \forall a, b \in g.
$$

Moreover, for an $N$-nilpotent Lie superalgebra, this reduces to

$$
(\omega(t, u) \mod I_N : [a, b])_x = 0, \forall a, b \in g.
$$

Using Lemma 4.3 we see that $\omega(t, u) \mod I_N = 0$. \(\blacksquare\)
Example 5.1. Let $\mathbb{K}_0$ be a field.
(i) Let $N = 2$ and $\frac{1}{2} \in \mathbb{K}_0$. We look for $\varphi(t) = c_0 + c_1 t$ mod $t^2$ solution of $2c_0c_1 = -c_0$. We get $\varphi(t) = c_1 t$ or $\varphi(t) = c_0 - \frac{1}{2}t$.
(ii) Let $N = 3$ and $\frac{1}{2}, \frac{1}{3} \in \mathbb{K}_0$. We look for $\varphi(t) = c_0 + c_1 t + c_2 t^2$ mod $t^3$ solution of $2c_0c_1 + (3c_0c_2 + c_1^3)(u + t) = -c_0 - c_1(t + u)$. We get $\varphi(t) = c_2 t^2$, or $\varphi(t) = -t + c_2 t^2$, or $\varphi(t) = c_0 - \frac{1}{2}t + \frac{1}{12}c_1 t^2$ with $c_0 \neq 0$.

Lemma 5.2. Let $N \geq 2$. Eq. (22) has solutions in $\mathbb{K}[t]/t^N$ with $\varphi(0) \in \mathbb{K}_0^\times$, if and only if $\frac{1}{2}, \ldots, \frac{1}{N} \in \mathbb{K}$. In this case the unique solution such that $\varphi(0) = c \in \mathbb{K}_0^\times$ is

$$\varphi_c(t) \text{ mod } t^N.$$ 

Proof. We look for $\varphi \in \mathbb{K}_0[t]/t^N$ such that $1 + \frac{1}{\varphi(0)} \in \mathbb{K}_0[t]/t^{N+1}$ solves system (21) in $\mathbb{K}[t,1/u]/I_{N+2}$.

System (21) has solutions in $\mathbb{K}_0[t,u]/I_{N+2}$ exactly when $2, \ldots, N$ are invertible in $\mathbb{K}$. In this case the unique solution is $c^N/c$ mod $t^{N+1}$ with $c \in \mathbb{K}_0^\times$. It means that $\varphi(t) = \varphi_c(t)t^N$. □

We have shown that

Theorem 5.5. Let $\mathbb{K} \ni \frac{1}{2}, \ldots, \frac{1}{N}$. For any $c \in \mathbb{K}_0^\times$, the truncated polynomial $\varphi_c(t) \in \mathbb{K}_0[t]/t^n$ gives a representation by coderivations $\Phi_c : g \to \text{Hom}(S(g), S(g))$.

Remark 5.5. Let $\mathbb{K}$ be a field, $p$ its characteristic. When $2 \leq N < p$, the previous theorem applies.

5.2. Some properties of the representations $\Phi_c$

This section applies to the case $\mathbb{K} \ni \mathbb{Q}$ and $g$ any Lie $\mathbb{K}$-superalgebra, and to the case $\mathbb{K} \ni \frac{1}{2}, \ldots, \frac{1}{N}$ with $N \geq 2$ and $g$ an $N$-nilpotent Lie $\mathbb{K}$-superalgebra.

Remark 5.6. If $c \in \mathbb{K}_0^\times$, the representation $g \ni a \mapsto \Phi_c^a \in \text{Hom}(S(g), S(g))$ is faithful because $\Phi_c^a(1) = c \cdot a$.

Remark 5.7. Theorem 4.2 gives that $[\Phi_0^a, \Phi_c^b] = \Phi_c^{[a,b]}$ for any $a, b \in g$, $c \in \mathbb{K}_0^\times \cup \{0\}$.

Theorem 5.6. Each representation $\Phi_c$, $c \in \mathbb{K}_0^\times$, is equivalent to $\Phi_1$.

Proof. Let $c \in \mathbb{K}_0^\times$. We consider the map $f_c : S(g) \to S(g)$ such that $f_c(X_1 \cdots X_n) = c^n X_1 \cdots X_n$ for all $X_1, \ldots, X_n \in g$. We have $f_c^{-1} \circ \Phi_c^a \circ f_c = \Phi_1^a$ for any $a \in g$. □
6. A more general equation

Let $\mathbb{K}$ be a field of characteristic zero, $t$ and $u$ be two commutating variables. We classify the triples of formal series $(\varphi, \psi, \rho)$ such that $\varphi, \psi, \rho \in \mathbb{K}[[t]]$ and

$$
\varphi(t) \frac{\psi(t + u) - \psi(u)}{t} + \frac{\varphi(t + u) - \varphi(t)}{u} \psi(u) = \rho(t + u).
$$

(23)

This is motivated by Theorem 4.2 and it is clear that Eq. (18) is a particular case of Eq. (23). The classification is contained in Theorems 6.2 and 6.3.

Remark 6.1. Applying the limit $t \to 0$ we get

$$
\varphi(0) \psi'(u) + \frac{\varphi(u) - \varphi(0)}{u} \psi(u) = \rho(u).
$$

(24)

Remark 6.2. Applying limits $t \to 0$ and $u \to 0$ to Eq. (23) we get

$$
\varphi(0) \psi'(u) - \psi(0) \varphi'(u) = \frac{\varphi(0) \psi(u) - \psi(0) \varphi(u)}{u}.
$$

Thus there exists $a \in \mathbb{K}$ such that

$$
\varphi(0) \psi(u) - \psi(0) \varphi(u) = au.
$$

(25)

As the formal series $\rho$ is determined by (24), it is natural to ask if Eq. (23) can be reduced to an equation for the couple $(\varphi, \psi)$. To get such an equation we introduce $p(t), q(t) \in \mathbb{K} + \mathbb{K}[[t]]$ such that

$$
\varphi(t) = tp(t), \quad \psi(t) = tq(t).
$$

Theorem 6.1. The pair $(p(t), q(t))$ gives a solution of (23) if and only if

$$
q'(u) \left\{ p(t + u) - p(t) \right\} = p'(t) \left\{ q(t + u) - q(u) \right\}.
$$

(26)

Proof. Eq. (23) becomes

$$
\varphi(t) \left\{ p(t + u)q(t + u) - uq(u) \right\} + q(u) \left\{ (t + u)p(t + u) - tp(t) \right\} = \rho(t + u).
$$

We recall that a function $f(t, u)$ is a function of $t + u$ if and only if $\frac{\partial}{\partial t} f(t, u) - \frac{\partial}{\partial u} f(t, u) = 0$. We apply this fact to

$$
f(t, u) := p(t) \left\{ (t + u)q(t + u) - uq(u) \right\} + q(u) \left\{ (t + u)p(t + u) - tp(t) \right\}
$$

and we get Eq. (26).

Formula (26) is very elegant. However, we will use it only through the following remark.

Remark 6.3. If the pair $(p, q)$ is a solution without poles of (26) then $q'(u)(p(u) - p(0)) = 0$. In particular $p$ or $q$ is constant.

Theorem 6.2. All triples of series $(\varphi, \psi, \rho)$ verifying (23) and $\varphi(0) \psi(0) = 0$, are given by the following list
(i) \((\varphi(t), \psi(t), \rho(t)) = (\varphi(t), ct, c\varphi(t)), \ c \in K, \ \varphi \in K[[t]]\).

(ii) \((\varphi(t), \psi(t), \rho(t)) = (ct, \psi(t), c\psi(t)), \ c \in K, \ \psi \in K[[t]]\).

**Proof.** It is sufficient to consider the case \(\psi(0) = 0\). From identity (25) we get \(\varphi(0)\psi(u) = au\), with \(a \in K\).

Let \(\varphi(0) \neq 0\), we get \(\psi(u) = cu\) with \(c \in K\). Eq. (23) is verified with \(c\varphi(t) = \rho(t)\), so we have a triple of type (i).

Let \(\psi(0) = \varphi(0) = 0\). From Remark 6.3 we get that \(q(u)\) or \(p(u)\) is constant. It means that we get a triple of type (i) or (ii). \(\Box\)

Now we treat the case \(\varphi(0) \cdot \psi(0) \neq 0\).

**Remark 6.4.** Let \(a, b \in K\). If \((\varphi(t), \psi(t), \rho(t))\) verifies the functional equation (23), then the triple \((a\varphi(t), b\psi(t), a \cdot b\rho(t))\) verifies (23).

Thus it is sufficient to look for series such that \(\psi(0) = \varphi(0) = 1\).

**Remark 6.5.** Let \(a, b \in K\). If \((\varphi, \psi, \rho)\) verifies (23), then the triple \((\varphi(at), \psi(at), a \cdot b\rho(at))\) also verifies it.

By this remark and by identity (25), we can restrict ourselves to look for triples with \(\varphi = \psi\), \(\varphi(0) = 1\), and \(\psi'(0) = 0\).

**Remark 6.6.** Let \(a \in K\). If \((\varphi(t), \psi(t), \rho(t))\) verifies the functional equation (23), then \((\varphi(at), \psi(at), a\rho(at))\) verifies (23).

For any \(c \in K\) we introduce the notation

\[
\theta_c(t) = \sqrt{c} t \coth(\sqrt{c} t) = 1 + \frac{c}{3} t^2 - \frac{c^2}{45} t^4 + \cdots \in K[[t]].
\]

In particular \(\theta_0(t) = 1\).

**Lemma 6.1.** Let \(c \in K\). There exists exactly one triple \((\psi_c, \psi_c, \rho_c)\) such that \(\psi_c(t) = 1 + \sqrt{c} t^2 + o(t^2)\). It is given by \(\psi_c(t) = \theta_c(t)\) and \(\rho_c(t) = ct\).

**Proof.** We consider the left-hand side of (23). As its derivative by \(t\) must be equal to its derivative by \(u\) (see the proof of Theorem 6.1), if \(\psi = \psi'\) we get

\[
\begin{align*}
\psi'(t) \frac{\psi(t+u) - \psi(u)}{u} + \psi(t) \frac{\partial}{\partial t} \left( \frac{\psi(t+u) - \psi(u)}{u} \right) + \psi(u) \frac{\psi'(t+u) - \psi'(t)}{u} \\
= \psi(t) \frac{\psi(t+u) - \psi'(u)}{u} + \psi'(u) \frac{\psi(t+u) - \psi(t)}{u} \\
+ \psi(u) \frac{\partial}{\partial u} \left( \frac{\psi(t+u) - \psi(t)}{u} \right).
\end{align*}
\]
As $\psi(0) = 1$ and $\psi'(0) = 0$, the limit $t \to 0$ gives

$$
-\frac{1}{2} \psi''(u) = \frac{\psi'(u)}{u}(\psi(u) - 1) - \psi(u) \frac{\psi(u) - 1}{u^2}.
$$

(27)

Substituting $\psi(t) = 1 + \frac{c_3}{3} t^2 + \sum_{k=3}^{\infty} c_k t^k$ we get

$$
c_{k+2} = -\frac{2}{k(k+3)} \left( \frac{k c_3 c}{3} + \frac{k-1}{p=3} \sum_{p} c_p c_{k-p+2}(k - p + 1) \right), \quad k \geq 1.
$$

This formula gives $c_3 = 0$. By induction, all coefficients $c_{2j+1}$, $j \geq 2$, are zero. The series $t \coth(t) = 1 + \frac{1}{3} t^2 + \cdots$ is a solution of Eq. (27). By Remark 6.6 also $ct \coth(ct) = 1 + \frac{c_3}{3} t^2 + \cdots$ is a solution of Eq. (27). □

**Theorem 6.3.** The list of solutions of (23) verifying $\psi(0)\varphi(0) \neq 0$ is

$$
\begin{align*}
\varphi(t) &= a \theta_c(t) + bt, \\
\psi(t) &= d \theta_c(t) + et, \\
\rho(t) &= (ae + bd) \theta_c(t) + (adc + be)t
\end{align*}
$$

with $a, b, c, d, e \in \mathbb{K}$ and $a \cdot d \neq 0$.

**Remark 6.7.** We have

$$
\varphi_d + \varphi_{-d} = \varphi_0 \equiv -t, \quad \forall d \in \mathbb{K}^x,
$$

(28)

and $\theta_c(t) \equiv \sqrt{c}(\varphi_{\frac{1}{2c}}(t) - \varphi_{-\frac{1}{2c}}(t)) = 2 \sqrt{c}(\varphi_{\frac{1}{2c}}(t) - \frac{1}{2c} \varphi_0(t))$.

### 6.1. Application to a direct product

Let $\mathbb{K}$ be a commutative superring and $g$ a Lie $\mathbb{K}$-superalgebra.

**Lemma 6.2.** Let $\mathbb{K} \supseteq \mathbb{Q}$ and $g, h \in \mathbb{K}_0^x$. The couple $(\Phi_g, \Phi_h)$ is composed of commuting representations for any Lie $\mathbb{K}$-superalgebra, if and only if $g = -h$.

**Proof.** By Theorem 5.3, $\Phi_g$ and $\Phi_h$ commute for any Lie $\mathbb{K}$-superalgebra if and only if $(\varphi_g, \varphi_h, 0)$ verifies Eq. (23). As the formal series $\varphi_g$ and $\varphi_h$ have invertible coefficients, this is equivalent to the fact that $(\varphi_g, \varphi_h, 0)$ is one of the triples given by Theorem 6.3. Using that

$$
\varphi_c(t) = c \varphi\left( \frac{t}{c} \right), \quad \forall c \in \mathbb{K}_0^x.
$$

we see that $\varphi = \varphi_g$ gives a triple if and only if $a = g$, $b = -\frac{1}{2}$, $c = \frac{1}{4g^2}$. In particular, $(\varphi_g, \varphi_h, 0)$ is a solution of Eq. (23) if and only if

$$
0 = -\frac{1}{2}(g + h)\theta_c + \frac{1}{4}\left( \frac{h}{g} + 1 \right) t.
$$

□
By denote by \( g \times g \) the direct product of \( g \) with its-self. Let \( \rho : g \times g \to \text{Hom}(S(g), S(g)) \) be a representation, it decomposes into the sum of two commuting representations \( \rho_1, \rho_2 : g \to \text{Hom}(S(g), S(g)) \) such that \( \rho(a_1, a_2) = \rho_1(a_1) + \rho_2(a_2) \) for each \( (a_1, a_2) \in g \times g \). We write \( \rho = (\rho_1, \rho_2) \). Theorem 5.3 and Lemma 6.2 give Theorem 6.4.

For any \( K \)-Lie superalgebra \( g \), \((\Phi_0, 0)\) and \((0, \Phi_0)\) are representations by coderivations of \( g \times g \) in \( S(g) \).

Theorem 6.5. Assume that \( K \supseteq \mathbb{Q} \). For any \( K \)-Lie superalgebra \( g \), \((\Phi_c, 0)\) and \((0, \Phi_c)\) whit \( c \in K_0^* \cup \{0\} \), \((\Phi_d, \Phi_{-d})\) with \( d \in K_0^* \), are representations by coderivations of \( g \times g \) in \( S(g) \).

7. Lie algebras

In this paragraph we consider the case of Lie algebras over a \( \mathbb{Q} \)-algebra. This means that we assume that \( K = K_0 \supseteq \mathbb{Q} \) is a commutative ring, and \( g \) a Lie \( K \)-algebra.

We have considered coderivations associated to vector fields on \( g \) of type
\[
\varphi^a = \varphi(\text{ad } x)(a)
\]
with \( a \in g \), \( \varphi \in \mathbb{K}[[t]] \), \( x \in g_x \) the generic point of \( g \). We have seen in Remarks 4.1 and 5.3 that \( \varphi^a \) and the corresponding coderivation \( \Phi^a \) satisfy a functorial property. In this paragraph we prove the converse.

We look for the family of morphisms of \( K \)-modules \( F_g : S(g) \otimes g \to g \) defined for all \( K \)-Lie algebra \( g \) and such that, for all morphisms of Lie \( K \)-algebras \( f : g \to \mathfrak{h} \) the diagram
\[
\begin{array}{ccc}
S(g) \otimes g & \xrightarrow{F_g} & g \\
\downarrow{f \otimes f} & & \downarrow{f} \\
S(\mathfrak{h}) \otimes \mathfrak{h} & \xrightarrow{F_{\mathfrak{h}}} & \mathfrak{h},
\end{array}
\]
where \( f : S(g) \to S(\mathfrak{h}) \) is the algebra-morphism induced by \( f \), commutes.

Theorem 7.1. For each \( n \in \mathbb{N} \), there exists \( c_n \in \mathbb{K} \) such that
\[
F_g(X_1 \cdots X_n \otimes a) = (c_n(\text{ad } x)^n(a))(X_1 \cdots X_n)
\]
for any \( X_1, \ldots, X_n, a \in g \).

Proof. We consider the free Lie \( K \)-algebra \( \mathfrak{h} \) with generators \( x_1, \ldots, x_{n+1} \). Let \( Y := F_g(x_1 \cdots x_n \otimes x_{n+1}) \), \( t \in \mathbb{Q} \). We fix \( i \in \{1, \ldots, n+1\} \). By the universal property of free Lie algebras, the map
\[
f_{t,i} : X_j \mapsto \begin{cases} X_j, & j \neq i, \\ X_{jt}, & j = i, \end{cases}, \quad \forall j = 1, \ldots, n + 1,
\]
extends to a morphism of Lie \( K \)-algebras \( \tilde{f}_{t,i} : \mathfrak{h} \to \mathfrak{h} \). As the diagram (29) associated to this map commutes, we get \( Yt = \tilde{f}_{t,i}(Y) \). We write \( Y = \sum_n Y_{n,i} \) where \( Y_{n,i} \) is a
bracket containing \( n \) times \( x_i \), so \( f_{x,i}(Y) = \sum_n Y_{n,i} t^n \). To get \( YT = \sum_n Y_{n,i} t^n \), we need \( \sum_{n\neq 1} Y_{n,i}(t-t^n) = 0 \) for any \( t \in \mathbb{Q} \). As the family \( \{ Y_{n,i} | Y_{n,i} \neq 0, n \geq 0 \} \) is free (see [3, Proposition 10, p. 26]), we get that \( Y = \sum_{i=1}^{n+1} Y_{i,i} \). This is true for any \( i \in \{i, \ldots, n+1\} \), so \( Y \) is a linear combination of brackets of \( n \) elements, exactly the elements \( x_1, \ldots, x_{n+1} \).

Using the Jacobi identity and the fact that the bracket of a Lie algebra is antisymmetric, we show that \( Y \) is a linear combination of brackets of \( n \) elements, exactly the elements \( x_1, \ldots, x_{n+1} \).

Let \( Y = \sum_{i=1}^{n+1} C(Y_{i,i}) \) and \( \theta \) be the map such that \( \theta = \sum_{i=1}^{n+1} \theta(Y_{i,i}) \). Then \( \theta \) extends to a morphism of Lie \( \mathbb{K} \)-algebras (see [3, Proposition 10, p. 26]) we get that the family \( \{ C(Y_{i,i}) \} \) is free. In particular, \( c_s = \sum_{i=1}^{n+1} \theta(Y_{i,i}) \) is free. As a permutation \( s \), we get \( c_s = \sum_{i=1}^{n+1} \theta(Y_{i,i}) \).

Let \( f \) be a map \( \{x_1, \ldots, x_n, x_{n+1}\} \to g \) in a Lie \( \mathbb{K} \)-algebra. From the universal property of free Lie algebras, \( f \) extends to a morphism of Lie \( \mathbb{K} \)-algebras still noted \( f \). Let \( a := f(x_{n+1}), f(x_i) =: X_i \). The commutative diagram for \( f \) gives

\[
F_g(X_1 \cdots X_n \otimes a) = c_n \sum_{j \in \Sigma_n} \sum_{j \in \Sigma_n} \theta(Y_{j,i}) \theta(Y_{j,i})
\]

**Remark 7.1.** If \( \mathbb{K} = \mathbb{K}_0 \), the previous theorem is not valid for a Lie \( \mathbb{K} \)-superalgebra \( g \). For example, if \( \theta_g = \theta \) is the map such that \( \theta(g) = id \) and \( \theta|_{\mathbb{K}} = -id \) then \( g \ni a \mapsto \theta(a) \) has the functorial property expressed in diagram (29).

As an application of Theorem 7.1 we get the following theorems.

**Theorem 7.2.** Assume that \( \mathbb{K} = \mathbb{K}_0 \supseteq \mathbb{Q} \) is a field and \( g = g_0 \) is a Lie \( \mathbb{K} \)-algebra. All universal representations by coderivations \( g \to Hom(S(g), S(g)) \)) are: the zero representation, \( \Phi_c \) with \( c \in \mathbb{K} \).

**Theorem 7.3.** Assume that \( \mathbb{K} = \mathbb{K}_0 \supseteq \mathbb{Q} \) is a field and \( g = g_0 \) is a Lie \( \mathbb{K} \)-algebra. All universal representations by coderivations \( g \times g \to Hom(S(g), S(g)) \)) are: (0, 0), (\( \Phi_c \), 0), (0, \( \Phi_c \)), (\( \Phi_d \), \( \Phi_{-d} \)) with \( c \in \mathbb{K} \) and \( d \in \mathbb{K} \setminus \{0\} \).

8. The Poincaré–Birkhoff–Witt theorem

Let \( \mathbb{K} \) be a commutative superring and \( g \) be a Lie \( \mathbb{K} \)-superalgebra. We assume that \( \frac{1}{2} \in \mathbb{K} \) or that \( \mathbb{K} = \mathbb{K}_0 \) and \( g = g_0 \).

We recall that the enveloping algebra \( U(g) \) is defined as the quotient of the tensor algebra \( T(g) \) by the ideal \( J \) generated by \( \{ a \otimes b - (-1)^{p(a)p(b)} b \otimes a | a, b \in g \} \). The inclusion of \( g \) in \( T(g) \) gives a map \( j: g \to U(g) \). Let \( gr(U(g)) \) be the graded module of \( U(g) \) associated to the filtration \( \{ U_l \} \) with \( U_0 = \mathbb{K} \) and \( U_i \) the \( \mathbb{K} \)-module generated by \( \{ j(X_1) \cdots j(X_l) | l \leq i, X_1, \ldots, X_l \in g \} \). The hypothesis give that it is a commutative superalgebra.
Remark 8.1. If our assumptions are not verified, \( gr(U(g)) \) might not be commutative. For example let us consider \( g = \mathbb{Z}e \) with odd \( e \) and \([e, e] = 0\). As \( j(e)^2 \notin U_1(g) \), \( gr(U(g)) \) is not commutative.

By the universal property of symmetric algebras, \( j \) extends to the algebra-morphism \( \tilde{j} : S(g) \to gr(U(g)) \) such that \( X_1 \cdots X_n \mapsto j(X_1) \cdots j(X_n) \mod U_{n-1} \), for any \( n \in \mathbb{N} \) and \( X_1, \ldots, X_n \in g \). This map is onto.

Definition 8.1. We say that \( g \) verifies the weak Poincaré–Birkhoff–Witt theorem if \( \tilde{j} \) is bijective.

Before giving the next definition we recall that a map \( f : g \to g \) is said to be an automorphism if it is an invertible morphism of Lie \( \mathbb{K} \)-superalgebras. Such a map induces two isomorphisms of \( \mathbb{K} \)-superalgebras: \( \tilde{f} : S(g) \to S(g) \) and \( \bar{f} : U(g) \to U(g) \).

Remark 8.2. A derivation \( g : g \to g \) extends to derivations \( g_1 : U(g) \to U(g) \) and \( g_2 : S(g) \to S(g) \). Moreover, \( g_1 \) and \( g_2 \) are also two coderivations.

Definition 8.2. We say that \( g \) verifies the strong Poincaré–Birkhoff–Witt theorem if it does exist an isomorphism \( \rho \in \text{Hom}(S(g), U(g)) \) such that

(i) \( \rho(S^n(g)) \subseteq U_n(g) \) for any \( n \in \mathbb{N} \),
(ii) the associated graded map \( gr(\rho) \) is \( \tilde{j} \),
(iii) \( \rho \) commutes with any derivation of \( g \) and any automorphism of \( g \).

Remark 8.3. Let \( n \geq 1 \). From (i) and (ii) we have \( \rho(S^n(g)) \oplus U_{n-1} = U_n \). From (ii) we get that \( \rho(S^n(g)) \) is stable by any derivation or automorphism of \( g \). In particular \( S(g) \) and \( U(g) \) are isomorphic for the adjoint representation.

Now we suppose also that

Hypothesis 8.1. \( \mathbb{K} \supseteq \mathbb{Q} \) or \( g \) is \( N \)-nilpotent with \( N \geq 2 \) and \( \frac{1}{2}, \ldots, \frac{1}{N} \in \mathbb{K} \).

From Theorems 5.3 and 5.5 we have a representation \( \Phi_1 : g \to \text{Hom}(S(g), S(g)) \). By the universal property of enveloping algebras, it extends to an algebra-morphism \( \Phi : U(g) \to \text{Hom}(S(g), S(g)) \) such that \( \Phi_1 = \Phi \circ j \). Using \( \Phi \) we construct the map \( \sigma : U(g) \to S(g) \), called the symbol map and defined by

\[
\sigma(u) := \Phi(u)(1), \quad \forall u \in U(g).
\]

Example 8.1. For any \( a_1, a_2, a_3 \in g \) we have

\[
\sigma(1) = 1, \quad \sigma(j(a_1)) = a_1,
\]
Theorem 8.1. \[ \text{onto: for any } n \in \mathbb{N}, \]\[ \text{Proof. (i) By Lemma 8.1 the graded map } \Lambda_{an}(p_n) \text{ we have} \]
\[ \sigma(j(a_1) j(a_2)) = a_1 \cdot a_2 + \frac{1}{2}[a_1, a_2], \]
\[ \sigma(j(a_1) j(a_2) j(a_3)) = a_1 \cdot a_2 \cdot a_3 + \frac{1}{2}([a_1, a_2, a_3] + [a_1, a_2] \cdot a_3 + (-1)^{p(a_1)p(a_2)} a_2 \cdot [a_1, a_3]) \]
\[ + \frac{1}{12}(-1)^{p(a_2)p(a_1)}([a_2, [a_1, a_3]] + [[a_1, a_2], a_3]) + \frac{1}{4}[a_1, [a_2, a_3]]. \]

Lemma 8.1. Let \( \mathbb{K} \) be any commutative superring and \( g \) any Lie \( \mathbb{K} \)-superalgebra. If \( \lambda \in \mathbb{K}[[z]] \) with \( \lambda(0) = 1 \), the coderivations corresponding to \( \lambda \) have the property
\[ (\bigoplus_{j=0}^{n-1} S^j(g), \forall a_1, \ldots, a_n \in g. ) \]

Proof. If \( n = 1 \) the theorem is evident. As \( \Lambda^{an} \circ \cdots \circ \Lambda^{a_1}(1) = \Lambda^{an}(\Lambda^{an-1} \circ \cdots \circ \Lambda^{a_1}(1)), \)
by induction there exists \( p_n \in \bigoplus_{j=0}^{n-2} S^j(g) \) such that \( \Lambda^{an-1} \circ \cdots \circ \Lambda^{a_1}(1) = a_{n-1} \cdots a_1 + p_n). \) As for any \( p \geq 0 \)
\[ \Delta(S^p(g)) \subseteq \bigoplus_{j=0}^p S^j(g) \otimes S^{p-j}(g), \]
we have \( \Lambda^{an}(p_n) \in \bigoplus_{j=0}^{n-2} S^j(g), \) so it is sufficient to show that \( \Lambda^{an}(a_{n-1} \cdots a_1) = a_n \cdots a_1 \in \bigoplus_{j=0}^{n-2} S^j(g). \) This identity follows using \( \Delta(a_{n-1} \cdots a_1) = a_{n-1} \cdots a_1 \otimes 1 \in \bigoplus_{j=0}^{n-2} S^j(g) \otimes S^{n-1-j}(g), \) the definition of \( \Delta, \) identity (14). \( \Box \)

Theorem 8.1. \( ^2 \) Assume Hypothesis 8.1. Then

(i) the map \( \sigma \) is invertible,
(ii) \( g \) verifies the strong Poincaré–Birkhoff–Witt theorem with \( \rho = \sigma^{-1}. \)

Proof. (i) By Lemma 8.1 the graded map \( gr(\sigma) : gr(U(g)) \rightarrow S(g) \) is well-defined and onto: for any \( n \in \mathbb{N} \) we have
\[ gr(\sigma)(a_1 \cdots a_n + U_{n-1}) = \sigma(a_1) \cdots \sigma(a_n), \forall a_1, \ldots, a_n \in g. \]
The inverse of \( gr(\sigma)|_{U_{n-1}/U_{n-2}} \) is \( j|_{S^1(g)}, \) so \( gr(\sigma) \) is one-to-one.

(ii) In (i) we have seen, in particular, that \( j = gr(\sigma^{-1}). \) Let \( f : g \rightarrow g \) be an automorphism of \( g. \) The fact that \( \sigma \circ f = f \circ \sigma \) follows from Remark 5.3.

Let \( g \) be a derivation of \( g \) and \( a_1, \ldots, a_n \in g. \) We want to show that \( g_2 \circ \sigma = \sigma \circ g_1, \)
which means
\[ g_2 \circ \Phi^{a_1} \circ \cdots \circ \Phi^{a_n})(1) = \sum_{j=1}^{n}(-1)^{p(g_p(a_1+\cdots+a_{j-1})}[(\Phi^{a_1} \circ \Phi^{g(a_j)} \circ \Phi^{a_n})(1)] \]

\( ^2 \) See the historical note below.
for any \(a_1, \ldots, a_n \in \mathfrak{g}\). By induction, it is sufficient to show that \([g_2, \Phi_{\mathfrak{g}}^a] = \Phi_{\mathfrak{g}^a}\) for any \(a \in \mathfrak{g}\). By definitions \(g_2 \circ \psi_{\mathfrak{g}}^a = \psi_{\mathfrak{g}^a} \circ g_2 + \psi_{\mathfrak{g}}^a\). This gives \([g_2, \Phi_{\mathfrak{g}}^a] = 1 \otimes \psi_{\mathfrak{g}}^a \circ (g_2 \otimes 1 + 1 \otimes g_2 - \Delta \circ g_2) + \Phi_{\mathfrak{g}^a}\). The fact that \(g_2\) is a coderivation ends the proof. \(\square\)

Let \(\beta := \sigma^{-1}\).

**Remark 8.4 (Functorial property).** Let \(f : \mathfrak{g} \rightarrow \mathfrak{h}\) be a morphism of Lie \(\mathbb{K}\)-superalgebras. By Remark 5.3 we get a commuting diagram

\[
\begin{array}{ccc}
S(\mathfrak{g}) & \xrightarrow{\beta} & U(\mathfrak{g}) \\
\downarrow f & & \downarrow f \\
S(\mathfrak{h}) & \xrightarrow{\beta} & U(\mathfrak{h}).
\end{array}
\]

To get formulas for \(\beta\) we use the following lemma.

**Lemma 8.2.** For any \(n \in \mathbb{N}\) and \(a \in \mathfrak{g}_0\) we have \((\Phi_{\mathfrak{g}}^a)^n(1) = a^n\).

**Proof.** If \(n = 1\) the statement is obvious. By induction

\[
(\Phi_{\mathfrak{g}}^a)^{n+1}(1) = \Phi_{\mathfrak{g}}^a \circ (\Phi_{\mathfrak{g}}^a)^n(1) = \Phi_{\mathfrak{g}}^a(a^n) = \sum_{j=0}^{n} \binom{n}{j} a^j \cdot \psi^a(a^{n-j}).
\]

From identity (3) we get \(\psi^a(a^j) = 0\) for \(j \geq 1\), so \((\Phi_{\mathfrak{g}}^a)^{n+1}(1) = a^n \cdot \psi^a(1) = a^{n+1}\). \(\square\)

**Corollary 8.1.** Let \(n \in \mathbb{N}\).

(i) For any \(a \in \mathfrak{g}_0\) we have \(\beta(a^n) = \beta(a)^n = j(a)^n\).

(ii) For any \(a_1, \ldots, a_n \in \mathfrak{g}\),

\[
n! \beta(a_1 \cdots a_n) = \sum_{s \in \Sigma_n} \alpha(a_{s(1)}, \ldots, a_{s(n)}) \beta(a_{s(1)}) \cdots \beta(a_{s(n)}).
\]

From now on \(\beta\) will be called the symmetrization map. If \(\mathbb{K}\) contains \(\mathbb{Q}\), \(\beta\) is the usual symmetrization map. If \(\mathbb{K}\) does not contain \(\mathbb{Q}\), the previous corollary does not give an explicit formula for the symmetrization map. However, in principle we can compute \(\beta(a_1 \cdots a_n)\) (as in the following example) but we do not know a nice formula.

**Example 8.2.** Let \(K = \mathbb{Z}/3\mathbb{Z}\), it contains \(\frac{1}{3}\). Let \(\mathfrak{g}\) be a 2-nilpotent Lie superalgebra over \(K\). For each \(a \in \mathfrak{g}\), \(\Phi^a = a^L + \frac{1}{2} \text{ad} a\). Let \(a_1, a_2, a_3 \in \mathfrak{g}\). From Example 8.1 we get

\[
\sigma(j(a_1)) = a_1,
\]

\[
\sigma(j(a_1) j(a_2)) = a_1 \cdot a_2 + \frac{1}{2}[a_1, a_2].
\]
Theorem 8.2. The symmetrization map verifies comultiplication over $S(g)$.

Proof. We consider the map $g \mapsto g \times g$ such that $X \mapsto (X, X)$. It induces the comultiplication over $S(g)$ and $U(g)$, so Remark 8.4 ends the proof. \qed

Remark 8.5 (Historical note). In the literature you can find proofs of the fact that the symmetrization $\beta$ is an isomorphism of $K$-modules for $K = K_0 \supseteq \mathbb{Q}$, $g$ a Lie $K$-algebra ([5], [3, Exercise 16, p. 78]) or a Lie superalgebra (appendix of [10]). Even in these particular cases, our proof is different and more direct. In particular, we do not have to consider first the special case of free Lie superalgebras. The case of $N$-nilpotent Lie superalgebras was known only for $N = 2$. It was proved by M. El-Agawany and A. Micali (see [6]). Before [5] the strong theorem was known for some class of Lie algebras. For example, if $K$ is a field of characteristic zero, it is due to Poincaré (see [15]). Examples of Lie algebras not verifying the weak Poincaré–Birkhoff–Witt theorem are given in [4, 5, 13].

If $K$ is a field (or more generally if $g$ is a free $K$-module), then any Lie $K$-algebra verifies the weak Poincaré–Birkhoff–Witt theorem (see for example [3]).

This is also true for a Lie $K$-superalgebra if 2 is invertible in $K$ (see [2]). However, if $K$ is a field of finite characteristic, the strong Poincaré–Birkhoff–Witt theorem is usually not satisfied.

8.1. Universal representations in the enveloping algebra

We still assume Hypotheses 8.1. By Theorem 8.1 we can transport each coderivations $\Phi_c$, $c \in K_0^\times \cup \{0\}$, on $U(g)$.

We recall that $U(g)$ is equipped with a natural comultiplication $\Delta'$, such that for $a \in g$ we have $\Delta'(j(a)) = 1 \otimes j(a) + j(a) \otimes 1$.

Theorem 8.2. The symmetrization map verifies $\Delta' \circ \beta = (\beta \otimes \beta) \circ \Delta$. In particular for any $a \in g$ and $c \in K_0^\times \cup \{0\}$, $\beta \circ \Phi^c_{\delta}$ is a coderivation of $U(g)$.

Proof. We consider the map $g \mapsto g \times g$ such that $X \mapsto (X, X)$. It induces the comultiplication over $S(g)$ and $U(g)$, so Remark 8.4 ends the proof. \qed
Let $a \in g$. In $U(g)$ we have $\text{ad} j(a) = j(a)L - j(a)R$ (see Notation 2.1).

**Theorem 8.3.** Let $a \in g$.

(i) $\beta^{-1} \circ \text{ad} j(a) \circ \beta = \Phi_0^a$,

(ii) $\beta^{-1} \circ j(a)L \circ \beta = \Phi_1^a$,

(iii) $\beta^{-1} \circ j(a)R \circ \beta = -\Phi_{-1}^a$.

**Proof.** (i) The map $g \ni X \mapsto [a,X]$ is a derivation of $g$, it extends to the derivations $\text{ad} j(a)$ and $\Phi_0^a$. Theorem 8.1 gives identity (i).

(ii) Let $W \in S(g)$. To show that $\sigma(j(a) \cdot \beta(W)) = \Phi_1^a (W)$ we only need to recall that by definitions we have $\sigma(j(a) \cdot \beta(W)) = \Phi_1^a \circ \sigma(\beta(W)) \equiv \Phi_1^a (W)$.

(iii) As $\text{ad} j(a) = j(a)L - j(a)R$ in $U(g)$, cases (i) and (ii) give $\beta^{-1} \circ j(a)R \circ \beta = -\Phi_{-1}^a$. By identity (28), the coderivation $\Phi_1^a - \Phi_0^a$ is equal to $-\Phi_{-1}^a$. $\square$

**Remark 8.6.** The map $a \mapsto \beta \circ \Phi_1^a \circ \beta^{-1}$ interpolates the regular left representation $a \mapsto j(a)L$ ($c = 1$) and the regular right representation $a \mapsto -j(a)R$ ($c = -1$).

**Theorem 8.4.** Let $K = K_0$ be a field of characteristic zero and $g = g_0$ a Lie $K$-algebra. All universal representations $g \to \text{Hom}(U(g), U(g))$ by coderivations are equivalent to the zero representation, or to the adjoint representation, or to the regular left representation.

**Proof.** Let $F : g \to \text{Hom}(U(g), U(g))$ be a representation by coderivations. We assume that $F$ is not the zero representation. By Theorem 8.2, for any $a \in g$, $G(a) := \beta^{-1} \circ F(a) \circ \beta$ is a coderivation of $S(g)$. In particular $G$ is a representation by coderivations of $g$ in $S(g)$. Using Theorem 7.2, we get that $G$ is one of the representations given in Theorems 5.2 and 5.3. From Theorem 5.6 we get that $G$ is equivalent to $\Phi_1$ or $\Phi_0$. By Theorem 8.3, $g \ni a \mapsto G(a)$ is equivalent to $g \ni a \mapsto \text{ad} j(a)$ or $g \ni a \mapsto j(a)L$. $\square$

Using Theorems 5.6, 7.3, and 8.3, we get

**Theorem 8.5.** Let $K = K_0 \supseteq Q$ be a field and $g = g_0$ a $K$-Lie algebra. We have five classes of equivalence for non-zero universal representations by coderivations $g \times g \to \text{Hom}(U(g), U(g))$: $g \times g \ni (a,b) \mapsto \alpha \text{ad} a + (1 - \alpha) \text{ad} b$, $\alpha \in [0, 1]$,

$g \times g \ni (a,b) \mapsto \alpha j(a)L - (1 - \alpha) j(b)R$, $\alpha \in [0, 1]$,

$g \times g \ni (a,b) \mapsto j(a)L - j(b)R$.

**Remark 8.7.** Let $K$ a field of characteristic zero. We denote by $\pi_1 : S(g) \to g$ the projection over $g$ and we put $P = \pi_1 \circ \beta^{-1}$. By Theorem 3.1, the parts (ii) and (iii) of Theorem 8.3 are equivalent to $P \circ aL = \phi_1^a \circ \beta^{-1}$, $P \circ aR = -\phi_{-1}^a \circ \beta^{-1}$. In [14] and [7] we can find a formula for $P$. 

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Remark 8.8. In [1] and [12] we can find the formula for $β^{-1} \circ a^L \circ β$ which is in Theorem 8.3.

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References