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NOISE TERMS IN DECOMPOSITION SOLUTION SERIES

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Abstract—The appearance of noise terms in some decomposition solutions is investigated.

INTRODUCTION

In applications solved by the decomposition method, the appearance of noise terms sometimes makes it necessary to compute more terms to observe the self-cancellations and separate solution terms from the terms whose sum vanishes in the limit. We investigate the phenomena further here [1–4].

DISCUSSION

Consider an inhomogeneous equation in the form $[L_x + L_y]u = g$. L_x represents a differential with respect to x and L_y represents a differential with respect to y. We can solve for $L_x u$ or for $L_y u$. Using decomposition, the first alternative yields

$$L_x u = g - L_y u, \qquad L_x^{-1} L_x u = L_x^{-1} g - L_x^{-1} L_y u, \qquad u = \phi_x + L_x^{-1} g - L_x^{-1} L_y \sum_{n=0}^{\infty} u_n \quad (1)$$

where $L\phi_x = 0$ and $u_0 = \phi_x + L_x^{-1} g$ and $u_{m+1} = -L_y^{-1} L_x u_m$. The second alternative yields

$$L_y u = g - L_x u, \qquad L_y^{-1} L_y u = L_y^{-1} g - L_y^{-1} L_x u, \qquad u = \phi_y + L_y^{-1} g - L_y^{-1} L_x \sum_{n=0}^{\infty} u_n \quad (2)$$

where $L\phi_y = 0$, $u_o = \phi_y + L_y^{-1} g$. We have shown in [3] that the solution to either equation in the x-dimension solution or the y-dimension solution (we call these "partial solutions") are equal except for special cases discussed in [3,4].

Calculating the components $u_n = -L_y^{-1} L_x u_{n-1}$ in (1) and rearranging, we have

$$u = \sum_{m=0}^{\infty} (-L_x^{-1} L_y)^m (\phi_x + L_x^{-1} g).$$

Proceeding in the same manner with Equation (2)

$$u = \sum_{m=0} (-L_y^{-1} L_x)^m (\phi_y + L_y^{-1} g).$$

Though it has previously been shown [3,4] that the partial solutions are sufficient except for easily identifiable special cases—that either one is the solution, so that the addition used in earlier work is unnecessary, we will consider both procedures for effect on noise terms. Thus,

$$u = \left(\frac{1}{2}\right) \sum_{m=0}^{\infty} \{(-L_x^{-1} L_y)^m \phi_x + (-L_y^{-1} L_x)^m \phi_y + (-L_x^{-1} L_y)^m L_x^{-1} g + (-L_y^{-1} L_x)^m L_y^{-1} g\}$$

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Let $u = u_p + u_h$ where u_p is the so-called particular integral and u_h is the homogeneous solution. We write

$$u_h = \left(\frac{1}{2}\right) \sum_{m=0}^{\infty} \{(-L_x^{-1}L_y)^m \phi_x + (-L_y^{-1}L_x)^m \phi_y\},$$

$$u_p = \left(\frac{1}{2}\right) \sum_{m=0}^{\infty} \{(-L_x^{-1}L_y)^m L_x^{-1}g + (-L_y^{-1}L_x)^m L_y^{-1}g\}.$$

We clearly must have $(L_x + L_y)u_h = 0$ and $(L_x + L_y)u_p = g$ (due to the principle of linear superposition).

PROOF.

$$\begin{aligned}(L_x + L_y)\left(\frac{1}{2}\right) &\sum_{m=0}^{\infty} \{(-L_x^{-1}L_y)^m \phi_x + (-L_y^{-1}L_x)^m \phi_y\} \\ &= \left(\frac{1}{2}\right)\Bigg\{L_x\phi_x + \sum_{m=1}^{\infty}(-1)(L_y)(-L_x^{-1}L_y)^{m-1}\phi_x \\ &\quad + L_y\phi_y + \sum_{m=1}^{\infty}(-1)(L_x)(-L_y^{-1}L_x)^{m-1}\phi_y \\ &\quad + L_y \sum_{m=0}^{\infty}(-L_x^{-1}L_y)^m\phi_x + L_x \sum_{m=0}^{\infty}(-L_y^{-1}L_x)^m\phi_y\Bigg\}.\end{aligned}$$

Since $L_x\phi_x = 0$ and $L_y\phi_y = 0$, and also

$$-\sum_{m=1}^{\infty} L_y(-L_x^{-1}L_y)^{m-1}\phi_x + \sum_{m=0}^{\infty} L_y(-L_x^{-1}L_y)^m\phi_x = 0$$

and

$$-\sum_{m=1}^{\infty} L_x(-L_y^{-1}L_x)^{m-1}\phi_y + \sum_{m=0}^{\infty} L_x(-L_y^{-1}L_x)^m\phi_y = 0,$$

we have $(L_x + L_y)u_h = 0$, i.e., u_h is the homogeneous solution.

To show $(L_x + L_y)u_p = g$,

$$\begin{aligned}(L_x + L_y)u_p &= (L_x + L_y)\left(\frac{1}{2}\right) \sum_{m=0}^{\infty} \{(-L_x^{-1}L_y)^m L_x^{-1}g + (-L_y^{-1}L_x)^m L_y^{-1}g\} \\ &= \left(\frac{1}{2}\right)\Bigg\{L_xL_x^{-1}g + \sum_{m=1}^{\infty}(-1)(L_y)(-L_x^{-1}L_y)^{m-1}L_x^{-1}g \\ &\quad + L_yL_y^{-1}g + \sum_{m=1}^{\infty}(-1)(L_x)(-L_y^{-1}L_x)^{m-1}L_y^{-1}g \\ &\quad + L_y \sum_{m=0}^{\infty}(-L_x^{-1}L_y)^m L_x^{-1}g + L_x \sum_{m=0}^{\infty}(-L_y^{-1}L_x)^m L_y^{-1}g\Bigg\}.\end{aligned}$$

Since

$$-\sum_{m=1}^{\infty} L_y(-L_x^{-1}L_y)^{m-1}L_x^{-1}g + \sum_{m=0}^{\infty} L_y(-L_x^{-1}L_y)^m L_x^{-1}g = 0$$

and

$$-\sum_{m=1}^{\infty} L_x(-L_y^{-1}L_x)^{m-1}L_y^{-1}g + \sum_{m=0}^{\infty} L_x(-L_y^{-1}L_x)^m L_y^{-1}g = 0$$

we have

$$(L_x + L_y)u_p = g,$$

i.e., u_p is the particular integral. Since the equation is linear,

$$(L_x + L_y)(u_p + u_h) = (L_x + L_y)u = g.$$

EXAMPLE—INHOMOGENEOUS CASE

Consider $u_x + u_y = g$. Let $L_x = \partial/\partial x$, $L_y = \partial/\partial y$, $g = -(x+y)$. Suppose $u(0,y) = u(x,0) = 0$

$$L_x u = g - L_y u, \qquad u = \phi_x + L_x^{-1} g - L_x^{-1} L_y u.$$

Since $\phi_x = u(0,y) = 0$,

$$u = L_x^{-1} g - L_x^{-1} L_y u, \qquad u_0 = L_x^{-1} g = \frac{-x^2}{2} - xy$$
$$u_1 = -L_x^{-1} L_y u_0 = \frac{x^2}{2}, \qquad u_2 = -L_x^{-1} L_y u_1 = 0, \qquad u_{m \geq 2} = 0.$$

The solution there is $u = -xy$ using the partial solution in the x-dimension. The terms $-x^2/2 + x^2/2$ are the noise terms.

If we calculate the y partial solution we get,

$$u = \frac{-y^2}{2} - xy + \frac{y^2}{2} = -xy,$$

i.e., the same solution but different self-cancelling noise terms.

If $g = 0$ (the homogeneous case) and we are given initial conditions, e.g., $u(0,y) = y$ and $u(x,0) = x$, we can write the x partial solution immediately. We write

$$L_x u = -L_y u, \qquad L_x^{-1} L_x u = -L_x^{-1} L_y u, \qquad u = u(0,y) = u_0, \qquad u = u_0 - L_x^{-1} L_y \sum_{n=0}^{\infty} u_n.$$

Thus,

$$u_0 = y, \qquad u_1 = -x, \qquad u_{m>1} = 0, \qquad u = y - x.$$

No noise terms are present. The y partial solution gives the same result.

If as in the previous work, we write

$$u = \left(\frac{1}{2}\right)(y-x) - \left(\frac{1}{2}\right)(L_x^{-1} L_y + L_y^{-1} L_x)\, u,$$

we get $u_0 = (1/2)(y-x)$, $u_1 = (1/2)^2(y-x)$, $\ldots$ $u = \sum_{m=0}^{\infty} (1/2)^{m+1}(y-x)$ or a monotonic sequence to $y - x$. We see again that partial solutions should be used.

If we return to the inhomogeneous case and use the earlier technique where all partial solutions are computed, we have

$$u = L_x^{-1} g - L_x^{-1} L_y u, \quad u = L_y^{-1} g - L_y^{-1} L_x u$$
$$\text{or} \quad u = \left(\frac{1}{2}\right)(L_x^{-1} + L_y^{-1})\, g - \left(\frac{1}{2}\right)(L_x^{-1} L_y + L_y^{-1} L_x)\, u$$
$$u_0 = \left(\frac{1}{2}\right)(L_x^{-1} + L_y^{-1})\, g = -xy - \left(\frac{(x^2+y^2)}{4}\right)$$
$$u_1 = \frac{xy}{2} + \frac{(x^2+y^2)}{4},$$
$$u_2 = \frac{-xy}{2} - \frac{(x^2+y^2)}{8},$$
$$u_3 = \frac{xy}{4} + \frac{(x^2+y^2)}{8}$$
$$\vdots$$

It is easy to see the convergence to $-xy$. For an m-term decomposition series $\phi_m = \sum_{i=0}^{m-1} u_i$ with m even

$$\phi_{2m} = -\frac{2^m - 1}{2^m}(xy).$$

If m is odd

$$\phi_{2m+1} = -xy - \frac{1}{2^{m+2}}(x^2 + y^2).$$

Note $\lim_{m\to\infty} \phi_{2m} = \lim_{m\to\infty} \phi_{2m+1} = -xy$. We have a series of self-cancelling noise terms which vanish in the limit, i.e.,

$$u = -xy + \left(\frac{xy}{2} - \frac{xy}{2}\right) + \left(\frac{xy}{4} - \frac{xy}{4}\right) + \cdots + \left(\frac{x^2+y^2}{4} - \frac{x^2+y^2}{4}\right) + \cdots$$

The partial solution procedure involves less computation and it appears easier to separate out the solution from noise which only appears when $g \neq 0$. In both the homogeneous and inhomogeneous examples, the partial solution is simpler—a finite series—and the combination is the same in the limit.

Thus the earlier way of combining equations leads to an infinite series for inhomogeneous problems which is the correct solution in the limit. For comparison, we should compare linear combinations. Consider again the inhomogeneous example and consider the combination of partial solutions

$$u_0 = L_x^{-1} g, \qquad u_0 = L_y^{-1} g, \qquad u_m = (-L_x^{-1} L_y)^m u_0, \qquad u_m = (-L_y^{-1} L_x)^m u_0$$

$$\phi_m = \sum_{n=0}^{m-1} (-L_x^{-1} L_y)^n L_x^{-1} g, \qquad \phi_m = \sum_{n=0}^{m-1} (-L_y^{-1} L_x)^n L_y^{-1} g.$$

Now we take the combination of x-dimension and y-dimension solutions

$$\Phi_m = \left(\frac{1}{2}\right) \sum_{n=0}^{m-1} \left\{(-L_x^{-1} L_y)^n L_x^{-1} + (-L_y^{-1} L_x)^n L_y^{-1}\right\} g.$$

In the limit

$$\lim_{m\to\infty} \Phi_m = \left(\frac{1}{2}\right)(u + u) = u.$$

Both examples demonstrate fast convergence (in two terms), i.e., $\phi_2 = u_0 + u_1 = u$, the exact solution, by the partial solutions technique. The homogeneous case shows no noise while the inhomogeneous case has noise but it is already cancelled by u_1. On the other hand, the solution by the earlier technique has noise terms which cancel only in the limit for the inhomogeneous case, and, for the homogeneous case, we have monotonic convergence in the infinite limit. It appears that noise terms appear only in the inhomogeneous case and that the partial solution technique is much more effective in avoiding noise terms; the older method should be avoided now both because of noise and the greater computation. For example when we observe that a term in u_0 is cancelled by a term in u_1 even though u_1 introduces a further term, we should assume the remaining term is the solution and try substitution.

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