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# Properties of convergence for $\omega, q$-Bernstein polynomials ☆

Heping Wang

*Department of Mathematics, Capital Normal University, Beijing 100037, People's Republic of China*

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**Abstract**

In this paper, we discuss properties of the $\omega, q$-Bernstein polynomials $B_n^{\omega,q}(f;x)$ introduced by S. Lewanowicz and P. Woźny in [S. Lewanowicz, P. Woźny, Generalized Bernstein polynomials, BIT 44 (1) (2004) 63–78], where $f \in C[0,1]$, $\omega, q > 0$, $\omega \neq 1, q^{-1}, \dots, q^{-n+1}$. When $\omega = 0$, we recover the $q$-Bernstein polynomials introduced by [G.M. Phillips, Bernstein polynomials based on the $q$-integers, Ann. Numer. Math. 4 (1997) 511–518]; when $q = 1$, we recover the classical Bernstein polynomials. We compute the second moment of $B_n^{\omega,q}(t^2;x)$, and demonstrate that if $f$ is convex and $\omega, q \in (0,1)$ or $(1,\infty)$, then $B_n^{\omega,q}(f;x)$ are monotonically decreasing in $n$ for all $x \in [0,1]$. We prove that for $\omega \in (0,1)$, $q_n \in (0,1]$, the sequence $\{B_n^{\omega,q_n}(f)\}_{n\geqslant 1}$ converges to $f$ uniformly on $[0,1]$ for each $f \in C[0,1]$ if and only if $\lim_{n\to\infty} q_n = 1$. For fixed $\omega, q \in (0,1)$, we prove that the sequence $\{B_n^{\omega,q}(f)\}$ converges for each $f \in C[0,1]$ and obtain the estimates for the rate of convergence of $\{B_n^{\omega,q}(f)\}$ by the modulus of continuity of $f$, and the estimates are sharp in the sense of order for Lipschitz continuous functions.


*Keywords:* $\omega, q$-Bernstein polynomials; Limit $\omega, q$-Bernstein operators; Rate of convergence; Modulus of continuity

## 1. Introduction

Let $q > 0$. For any non-negative integer $k$, the *$q$-integer* $[k]_q$ is defined by

$$[k]_q := 1 + q + \dots + q^{k-1} \quad (k = 1, 2, \dots), \qquad [0]_q := 0$$

and the *$q$-factorial* $[k]_q!$ by

$$[k]_q! := [1]_q[2]_q \dots [k]_q \quad (k = 1, 2, \dots), \qquad [0]_q! := 1.$$

For integers $k, n$ with $0 \leqslant k \leqslant n$, the *$q$-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

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*E-mail address:* wanghp@mail.cnu.edu.cn.

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We also use the following standard notations:

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{s=0}^{k-1}\big(1-aq^s\big), \quad (a;q)_\infty := \prod_{s=0}^{\infty}\big(1-aq^s\big) \quad (0<q<1).$$

Clearly,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

In [11], Phillips proposed the $q$-Bernstein polynomials: for each positive integer $n$, and $f \in C[0,1]$, the $q$-Bernstein polynomial of $f$ is

$$B_{n,q}(f;x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}.$$

Note that for $q = 1$, $B_{n,q}(f,x)$ is the classical Bernstein polynomial $B_n(f,x)$:

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}.$$

In [4], S. Lewanowicz and P. Woźny introduced the generalized Bernstein polynomials (we call the $\omega, q$-Bernstein polynomials): for $f \in C[0,1]$, the $\omega, q$-Bernstein polynomials of $f$ are

$$B_n^{\omega,q}(f;x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) B_k^n(u;\omega|q) \quad \big(u = \omega + (1-\omega)x\big), \tag{1.1}$$

where $q$ and $\omega$ are positive real parameters, $\omega \neq 1, q^{-1}, \ldots, q^{-n+1}$, and

$$\begin{aligned} B_k^n(u;\omega|q) &:= \frac{1}{(\omega;q)_n}\begin{bmatrix} n \\ k \end{bmatrix}_q u^k\big(\omega u^{-1};q\big)_k (u;q)_{n-k} \\ &= \frac{1}{(\omega;q)_n}\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1}\big(u-\omega q^j\big)\prod_{s=0}^{n-k-1}\big(1-uq^s\big) \quad (k = 0,1,\ldots,n) \end{aligned} \tag{1.2}$$

are the basic $\omega, q$-Bernstein polynomials. When $\omega = 0$, we recover the $q$-Bernstein polynomials; when $q = 1$, we recover the classical Bernstein polynomials.

In recent years, the $q$-Bernstein polynomials have been investigated intensively by a number of authors (see [3,7, 8,10–12] and references therein [13–17]). However, there are very few works about the $\omega, q$-Bernstein polynomials as far as we know, since the study of the $\omega, q$-Bernstein polynomials is more difficult than that of the $q$-Bernstein polynomials. It should be mentioned here that various properties of the basic $\omega, q$-Bernstein polynomials have been studied in [4] and [5]. Also, we can define the generalized Bézier curve and de Casteljau algorithm, which can be used for evaluating $\omega, q$-Bernstein polynomials iteratively (see [4]) and are very useful for computer-aided geometric design. In this paper, it is our main aim to investigate properties of convergence of the $\omega, q$-Bernstein polynomials for $\omega, q \in (0,1)$ or $\omega, q \in (1,\infty)$. Our results demonstrate that in general properties of convergence for the $\omega, q$-Bernstein polynomials are very similar to those for the $q$-Bernstein polynomials but essentially different from those for the classical Bernstein polynomials.

Throughout the paper, we always assume that $f$ is a continuous real function on $[0,1]$, $\omega, q > 0$, $\omega \neq 1, q^{-1}, \ldots, q^{-n+1}$, $u = \omega + (1-\omega)x$ for $x \in [0,1]$. Denote by $C[0,1]$ (or $C^n[0,1]$, $1 \leqslant n \leqslant \infty$) the space of all continuous (corresponding, $n$ times continuously differentiable) real-valued functions on $[0,1]$ equipped with the uniform norm $\|\cdot\|$. The expression $g_n(x) \rightrightarrows g(x)$ $[x \in [0,1]; n \to \infty]$ denotes convergence of $g_n$ to $g$ uniformly in $x \in [0,1]$ as $n \to \infty$; $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $A(n) \gg B(n)$, and $A(n) \ll B(n)$ means that there exists a positive constant $c$ independent of $n$ such that $A(n) \leqslant cB(n)$.

## 2. Statement of results

It follows from the definition of $B_n^{\omega,q}(f;x)$ that the operators $B_n^{\omega,q}$ on $C[0,1]$ which map from $f \in C[0,1]$ to $B_n^{\omega,q}(f)$ are positive linear operators if $\omega, q \in (0,1)$ or $\omega, q \in (1,\infty)$. Also the $\omega, q$-Bernstein polynomials possess the end-point interpolation property, that is,

$$B_n^{\omega,q}(f;0) = f(0), \qquad B_n^{\omega,q}(f;1) = f(1). \tag{2.1}$$

In Section 3, we shall discuss some fundamental properties of the $\omega, q$-Bernstein polynomials. According to the theory of approximation by positive linear operators (see [2, pp. 277–281]), the moments $B_n^{\omega,q}(t^r;x)$ $(r = 0,1,2)$ are of particular importance, so we shall compute the moments $B_n^{\omega,q}(t^r;x)$ $(r = 0,1,2)$. Using the identity (see [1, §10, Exercise 9])

$$(ab;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k (b;q)_k (a;q)_{n-k},$$

and induction on $n$, S. Lewanowicz and P. Woźny proved in [4, Eqs. (2.4) and (2.5)] that

$$\sum_{i=0}^{n} B_i^n(x;\omega|q) = 1 \quad \text{(partition of unity)}, \tag{2.2}$$

and

$$\sum_{i=0}^{n} \frac{[i]_q}{[n]_q} B_i^n(x;\omega|q) = \frac{x-\omega}{1-\omega}. \tag{2.3}$$

That is,

$$B_n^{\omega,q}(at+b;x) = ax+b. \tag{2.4}$$

Hence, the $\omega, q$-Bernstein polynomials reproduce linear functions. About $B_n^{\omega,q}(t^2;x)$, we have the following result:

**Theorem 1.** *For $x \in [0,1]$, $q \neq 1$ and $n \geqslant 1$,*

$$B_n^{\omega,q}\big(t^2;x\big) = \frac{x(1-q)(1-\omega q^n) + qx^2(1-\omega)(1-q^{n-1})}{(1-\omega q)(1-q^n)}. \tag{2.5}$$

It is easy to know that for each $f \in C[0,1]$, $B_n^{\omega,q}(f;x)$ is a polynomial of degree $\leqslant n$. However, if $f$ is a polynomial, the following strong assertion holds:

**Theorem 2.** *If $f$ is a polynomial of degree $m$, then $B_n^{\omega,q}(f;x)$ is a polynomial of degree $\leqslant \min\{m,n\}$.*

Next we show when the function $f$ is convex, the $\omega, q$-Bernstein polynomials $B_n^{\omega,q}(f;x)$ are monotonic in $n$ for $\omega, q \in (0,1)$ or $\omega, q \in (1,\infty)$, as in the classical case and in the case $q \in (0,1)$, $\omega = 0$.

**Theorem 3.** *Let $f$ be continuous and convex on $[0,1]$. If $\omega, q \in (0,1)$ or $\omega, q \in (1,\infty)$, then for all $n \geqslant 2$ and all $x \in [0,1]$,*

$$B_{n-1}^{\omega,q}(f;x) \geqslant B_n^{\omega,q}(f;x). \tag{2.6}$$

*The inequality holds strictly for $0 < x < 1$ unless $f$ is linear in each of intervals between consecutive knots $[r]_q/[n-1]_q$, $0 \leqslant r \leqslant n-1$, in which case we have the equality $B_{n-1}^{\omega,q}(f;x) = B_n^{\omega,q}(f;x)$.*

The following theorem allows us to reduce the case $\omega, q > 1$ to the case $\omega, q \in (0,1)$. So, in the following section, we consider the case $\omega, q \in (0,1)$ only.

**Theorem 4.** *Let $f \in C[0,1]$, $g(x) := f(1-x)$. Then for any $x \in [0,1]$,*

$$B_n^{\omega,q}(f;x) = B_n^{1/\omega,1/q}(g;1-x). \tag{2.7}$$

In Section 4, we shall discuss properties of convergence for the $\omega, q$-Bernstein polynomials. Based on Theorem 1, we have the following approximating theorem.

**Theorem 5.** *Let $q_n \in (0,1]$, $\omega \in (0,1)$. Then the sequence $\{B_n^{\omega,q_n}(f)\}$ converges to $f$ uniformly on $[0,1]$ for each $f \in C[0,1]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

**Remark 1.** This is a generalization of the following result which is given in [4] without proof. If $0 < q_n, \omega < 1$, $\lim_{n\to\infty} q_n = 1$, then for any $f \in C[0,1]$, the sequence $\{B_n^{\omega,q_n}(f)\}$ converges to $f$ uniformly on [0, 1].

Theorem 5 implies that if $\omega, q \in (0,1)$ is fixed, $\{B_n^{\omega,q}(f,x)\}$ may not be approximating for some continuous functions. In Section 4, we also discuss properties of convergence for the $\omega, q$-Bernstein polynomials for fixed $\omega, q \in (0,1)$. For $f \in C[0,1]$ and $\omega, q \in (0,1)$, we set

$$B_\infty^{\omega,q}(f;x) := \begin{cases} \sum_{k=0}^{\infty} f\big(1-q^k\big) B_k^\infty(u;\omega|q), & 0 \leqslant x < 1, \\ f(1), & x = 1 \end{cases} \quad \big(u = \omega + (1-\omega)x\big), \tag{2.8}$$

where

$$B_k^\infty(u;\omega|q) := \frac{u^k(\omega u^{-1};q)_k (u;q)_\infty}{(q;q)_k(\omega;q)_\infty}. \tag{2.9}$$

The operators $B_\infty^{\omega,q}$ on $C[0,1]$ which map from $f$ to $B_\infty^{\omega,q}(f)$ are positive linear operators and are called the limit $\omega, q$-Bernstein operators. When $\omega = 0$, the limit $\omega, q$-Bernstein operators reduce to the limit $q$-Bernstein operators (see [3,7,9,13]). From the $q$-binomial theorem (see [1, p. 488]), we know that

$$\sum_{k=0}^{\infty} B_\infty^k(u;\omega|q) = \sum_{k=0}^{\infty} \frac{u^k(\omega u^{-1};q)_k (u;q)_\infty}{(q;q)_k(\omega;q)_\infty} = 1, \tag{2.10}$$

which means

$$B_\infty^{\omega,q}(1;x) = 1.$$

For $f \in C[0,1]$, $t > 0$, we define the modulus of continuity $\omega(f,t)$ and the second modulus of smoothness $\omega_2(f,t)$ as follows:

$$\omega(f,t) := \sup_{\substack{|x-y|\leqslant t \\ x,y\in[0,1]}} \big|f(x) - f(y)\big|;$$

$$\omega_2(f,t) := \sup_{0<h\leqslant t} \sup_{x\in[0,1-2h]} \big|f(x+2h) - 2f(x+h) + f(x)\big|.$$

**Theorem 6.** *Let $0 < \omega, q < 1$. Then for each $f \in C[0,1]$ the sequence $\{B_n^{\omega,q}(f;x)\}$ converges to $B_\infty^{\omega,q}(f;x)$ uniformly on $[0,1]$. Furthermore,*

$$\big\|B_n^{\omega,q}(f) - B_\infty^{\omega,q}(f)\big\| \leqslant C_q \omega\big(f, q^n\big). \tag{2.11}$$

*The above estimate is sharp in the following sense of order: for each $\alpha$, $0 < \alpha \leqslant 1$, there exists a function $f_\alpha(x)$ which belongs to the Lipschitz class $\mathrm{Lip}\,\alpha := \{f \in C[0,1] \mid \omega(f,t) \ll t^\alpha\}$ such that*

$$\big\|B_n^{\omega,q}(f_\alpha) - B_\infty^{\omega,q}(f_\alpha)\big\| \gg q^{\alpha n}. \tag{2.12}$$

**Theorem 7.** *Let $0 < \omega, q < 1$. Then*

$$\left\| B_n^{\omega,q}(f) - B_\infty^{\omega,q}(f) \right\| \leqslant c\omega_2\left(f, \sqrt{q^n}\right). \tag{2.13}$$

*Furthermore,*

$$\sup_{\omega,q\in(0,1)} \left\| B_n^{\omega,q}(f) - B_\infty^{\omega,q}(f) \right\| \leqslant c\omega_2\left(f, n^{-1/2}\right), \tag{2.14}$$

*where $c$ is an absolute constant.*

**Remark 2.** From (2.14) we know that for each $f \in C[0,1]$, $\lim_{n\to\infty} B_n^{\omega,q}(f;x) = B_\infty^{\omega,q}(f;x)$ uniformly in $x \in [0,1]$ and $\omega, q \in (0,1)$.

**Remark 3.** Results similar to Theorems 6 and 7 for the $q$-Bernstein polynomials were obtained in [16] and [14], respectively. Note that when $f(x) = x^2$, we have

$$\left\| B_n^{\omega,q}(f) - B_\infty^{\omega,q}(f) \right\| \asymp q^n \asymp \omega_2\left(f, \sqrt{q^n}\right).$$

Hence, the estimate (2.13) is sharp in the following sense: the sequence $\sqrt{q^n}$ in (2.13) cannot be replaced by any other sequence decreasing to zero more rapidly as $n \to \infty$. However, (2.13) is not sharp for the Lipschitz class $\mathrm{Lip}\,\alpha$ ($\alpha \in (0,1]$) in the sense of order. This, combining with Theorem 6, shows that in the case $0 < \omega, q < 1$, the modulus of continuity is more appropriate to describe the rate of convergence for the $\omega, q$-Bernstein polynomials than the second modulus of smoothness.

**Remark 4.** In the case $\omega, q \in (0,1)$, from (2.11) we conclude that the rate of convergence $\|B_n^{\omega,q}(f) - B_\infty^{\omega,q}(f)\|$ has the order $q^n$ for each $f \in C^1[0,1]$ versus at most $1/n$ for the classical Bernstein polynomials. From (2.14) we know that the rate of convergence $\|B_n^{\omega,q}(f) - B_\infty^{\omega,q}(f)\|$ can be dominated by $c\omega_2(f, n^{-1/2})$ uniformly with respect to $\omega, q \in (0,1)$.

**Remark 5.** The constant $c$ in (2.13) is an absolute constant and does not depend on $q$, however, the constant $C_q$ in (2.11) depends on $q$, and tends to $+\infty$ as $q \to 1-$. Hence, (2.13) does not follow from (2.11).

In Section 5, we shall discuss properties of the limit $\omega, q$-Bernstein operators. For any $f \in C[0,1]$, since the function $B_\infty^{\omega,q}(f;x)$ is the uniform limit of the sequence $\{B_n^{\omega,q}(f;x)\}$, we know $B_\infty^{\omega,q}(f;x) \in C[0,1]$. When $f$ is a polynomial of degree $m$, $B_\infty^{\omega,q}(f;x)$ is a polynomial of degree $\leqslant m$. We show the following strong assertion.

**Theorem 8.** *If $f$ is a polynomial of degree $m$, then $B_\infty^{\omega,q}(f;x)$ is also a polynomial of degree $m$.*

Let $\omega, q \in (0,1)$ be fixed. We want to describe the class of continuous functions satisfying the condition

$$\lim_{n\to\infty} B_n^{\omega,q}(f;x) = f(x) \quad \text{for } x \in [0,1].$$

We know that the limit $\omega, q$-Bernstein operators are positive linear operators on $C[0,1]$ and reproduce linear functions. Also from Theorem 1 and Remark 2, we get

$$B_\infty^{\omega,q}\left(t^2;x\right) = \frac{x(1-q) + qx^2(1-\omega)}{(1-\omega q)} > x^2, \quad x \in (0,1). \tag{2.15}$$

We show the following result.

**Theorem 9.** *Let $L$ be a positive linear operator on $C[0,1]$ which reproduces linear functions. If $L(t^2,x) > x^2$ for all $x \in (0,1)$, then $L(f) = f$ if and only if $f$ is linear.*

**Corollary.** *Let $0 < \omega, q < 1$ be fixed and let $f \in C[0,1]$. Then $B_\infty^{\omega,q}(f;x) = f(x)$ for all $x \in [0,1]$ if and only if $f$ is linear.*

**Remark 6.** From Corollary and Theorem 6, we know for fixed $\omega, q \in (0,1)$ and $f \in C[0,1]$, the sequence $\{B_n^{\omega,q}(f;x)\}$ does not approximate $f(x)$ unless $f$ is linear. This is completely in contrast to the classical Bernstein polynomials, which $\{B_n(f;x)\}$ approximates $f(x)$ for any $f \in C[0,1]$.

At last, we discuss approximating property of the limit $\omega, q$-Bernstein operators.

**Theorem 10.** *Let $\omega \in (0,1)$ be fixed. Then for any $f \in C[0,1]$, $\{B_\infty^{\omega,q}(f)\}$ converges to $f$ uniformly on $[0,1]$ as $q \to 1-$.*

## 3. Proofs of Theorems 1–4

**Proof of Theorem 1.** First we prove the following recurrence formula:

$$B_{n+1}^{\omega,q}\big(t^2;x\big) = \frac{qx^2(1-\omega)(1-q^n) + x(1-q)(1-\omega q^n) + q\omega(1-q^n)^2 B_n^{\omega,q}(t^2;x)}{(1-q^{n+1})(1-\omega q^n)}. \tag{3.1}$$

From [4, Eq. (2.1)], we have

$$\frac{[i+1]_q}{[n+1]_q} B_{i+1}^{n+1}(u;\omega|q) = \frac{u-\omega q^i}{1-\omega q^n} B_i^n(u;\omega|q). \tag{3.2}$$

Thus,

$$\begin{aligned}
B_{n+1}^{\omega,q}\big(t^2;x\big) &= \sum_{i=0}^{n} \frac{[i+1]_q^2}{[n+1]_q^2} B_{i+1}^{n+1}(u;\omega|q) = \sum_{i=0}^{n} \frac{1-q^{i+1}}{1-q^{n+1}} \frac{u-\omega q^i}{1-\omega q^n} B_i^n(u;\omega|q) \\
&= \sum_{i=0}^{n} \frac{u-(uq+\omega)q^i + \omega q^{2i+1}}{(1-q^{n+1})(1-\omega q^n)} B_i^n(u;\omega|q) \\
&= \sum_{i=0}^{n} \frac{(u+\omega q - uq - \omega) + (uq+\omega-2\omega q)(1-q^i) + \omega q(1-q^i)^2}{(1-q^{n+1})(1-\omega q^n)} B_i^n(u;\omega|q).
\end{aligned}$$

Using (2.2), (2.3) and $u = \omega + (1-\omega)x$, by direct computation we get

$$\begin{aligned}
B_{n+1}^{\omega,q}\big(t^2;x\big) &= \frac{(u+\omega q - uq - \omega) + (uq+\omega-2\omega q)(1-q^n)x + \omega q(1-q^n)^2 B_n^{\omega,q}(t^2;x)}{(1-q^{n+1})(1-\omega q^n)} \\
&= \frac{(1-q)(1-\omega)x + ((\omega+x-\omega x)q + \omega - 2\omega q)(1-q^n)x + q\omega(1-q^n)^2 B_n^{\omega,q}(t^2;x)}{(1-q^{n+1})(1-\omega q^n)} \\
&= \frac{qx^2(1-\omega)(1-q^n) + x(1-q)(1-\omega q^n) + q\omega(1-q^n)^2 B_n^{\omega,q}(t^2;x)}{(1-q^{n+1})(1-\omega q^n)},
\end{aligned}$$

which proves (3.1). Now we show Theorem 1. We use induction on $n$. It follows from the definition of $B_n^{\omega,q}(f;x)$ that

$$B_1^{\omega,q}\big(t^2;x\big) = x.$$

Using (3.1) we get

$$B_2^{\omega,q}\big(t^2;x\big) = \frac{x(1-q)(1-\omega q^2) + qx^2(1-\omega)(1-q)}{(1-\omega q)(1-q^2)},$$

which means (2.5) is true for $n = 2$. Assume (2.5) holds for a certain $n$. Then by (3.1) and induction assumption, we get

$$\begin{aligned}
&B_{n+1}^{\omega,q}\big(t^2;x\big)\\
&\quad=\frac{1}{(1-\omega q)\big(1-q^{n+1}\big)\big(1-\omega q^n\big)}\Big((1-\omega q)\big(x(1-q)\big(1-\omega q^n\big)+qx^2(1-\omega)\big(1-q^n\big)\big)\\
&\qquad+\omega q\big(1-q^n\big)\big(x(1-q)\big(1-\omega q^n\big)+qx^2(1-\omega)\big(1-q^{n-1}\big)\big)\Big)\\
&\quad=\frac{x(1-q)(1-\omega q^n)((1-\omega q)+\omega q(1-q^n))+qx^2(1-\omega)(1-q^n)((1-\omega q)+\omega q(1-q^{n-1}))}{(1-\omega q)(1-q^{n+1})(1-\omega q^n)}\\
&\quad=\frac{x(1-q)(1-\omega q^{n+1})+qx^2(1-\omega)(1-q^n)}{(1-\omega q)(1-q^{n+1})},
\end{aligned}$$

which proves Theorem 1. □

**Proof of Theorem 2.** Denote by $\Pi_n$ the space of polynomials of degree $\leqslant n$. Obviously, $B_n^{\omega,q_n}(f;x)\in\Pi_n$ for all $f\in C[0,1]$ and $B_n^{\omega,q_n}(t^r;x)\in\Pi_r$ for $r=0,1$ by (2.4), so it suffices to prove the statement that $B_n^{\omega,q_n}(t^r;x)\in\Pi_r$ for $1<r<n$. We use induction on $n$. Let us suppose that the statement is true for $n\geqslant r$, that is, $B_n^{\omega,q}(t^k;x)\in\Pi_k$ for $k\leqslant n$. Then, for $1<r<n+1$, by the definition of $B_{n+1}^{\omega,q_n}(t^r;x)$ and (3.2), we get that

$$\begin{aligned}
&B_{n+1}^{\omega,q}\big(t^r;x\big)\\
&\quad=\sum_{i=0}^{n}\frac{[i+1]_q^{r-1}}{[n+1]_q^{r-1}}\frac{u-\omega q^i}{1-\omega q^n}B_i^n(u;\omega|q)\\
&\quad=\sum_{i=0}^{n}\frac{(1+q[i]_q)^{r-1}((1-\omega)x+\omega(1-q)[i]_q)}{[n+1]_q^{r-1}(1-\omega q^n)}B_i^n(u;\omega|q)\\
&\quad=\sum_{i=0}^{n}\frac{(1-\omega)x+q^{r-1}\omega(1-q)[i]_q^r+\sum_{j=1}^{r-1}\big(\binom{r-1}{j}q^j(1-\omega)x+\binom{r-1}{j-1}q^{j-1}\omega(1-q)\big)[i]_q^j}{[n+1]_q^{r-1}(1-\omega q^n)}B_i^n(u;\omega|q)\\
&\quad=\frac{(1-\omega)x+\omega(q^{r-1}-q^r)[n]_q^rB_n^{\omega,q}(t^r;x)+\sum_{j=1}^{r-1}\big(\binom{r-1}{j}q^j(1-\omega)x+\binom{r-1}{j-1}\omega(q^{j-1}-q^j)\big)[n]_q^jB_n^{\omega,q}(t^j;x)}{[n+1]_q^{r-1}(1-\omega q^n)}.
\end{aligned}$$

By the induction assumption we obtain $B_{n+1}^{\omega,q}(t^r;x)\in\Pi_r$. Theorem 2 is proved. □

**Remark 7.** If $r\leqslant n$, then $\deg B_n^{\omega,q}(t^r;x)=r$. Indeed, from [4, Lemma 2.3], we know that $B_i^n(u;\omega|q)$, $i=0,1,\dots,n$, form a basis in the space $\Pi_n$. Since the rank of the matrix

$$\begin{pmatrix}
1 & 1 & 1 & \cdots & 1\\
0 & \frac{[1]_q}{[n]_q} & \frac{[2]_q}{[n]_q} & \cdots & \frac{[n]_q}{[n]_q}\\
0 & \big(\frac{[1]_q}{[n]_q}\big)^2 & \big(\frac{[2]_q}{[n]_q}\big)^2 & \cdots & \big(\frac{[n]_q}{[n]_q}\big)^2\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \big(\frac{[1]_q}{[n]_q}\big)^r & \big(\frac{[2]_q}{[n]_q}\big)^r & \cdots & \big(\frac{[n]_q}{[n]_q}\big)^r
\end{pmatrix}$$

is $r+1$, we get the polynomials $B_n^{\omega,q}(t^j;x)$, $j=0,1,\dots,r$, are linearly independent. Using the fact that $B_n^{\omega,q}(t^j;x)\in\Pi_j$, we get that $B_n^{\omega,q}(t^r;x)$ is a polynomial of degree $r$ for $r\leqslant n$.

**Proof of Theorem 3.** The proof is very similar to the one of Theorem 7.3.4 in [12, pp. 270–271]. From [4, Lemma 2.1(v)], we know that

$$B_i^{n-1}(u;\omega|q)=\frac{[n-i]_q}{[n]_q}B_i^n(u;\omega|q)+\left(1-\frac{[n-i-1]_q}{[n]_q}\right)B_{i+1}^n(u;\omega|q).$$

We have

$$
\begin{aligned}
&B_{n-1}^{\omega,q}(f;x) - B_n^{\omega,q}(f;x) \\
&\quad = \sum_{i=0}^{n-1} f\left(\frac{[i]_q}{[n-1]_q}\right) B_i^{n-1}(u;\omega|q) - \sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right) B_i^n(u;\omega|q) \\
&\quad = \sum_{i=0}^{n-1} f\left(\frac{[i]_q}{[n-1]_q}\right)\left(\frac{[n-i]_q}{[n]_q} B_i^n(u;\omega|q) + \left(1 - \frac{[n-i-1]_q}{[n]_q}\right) B_{i+1}^n(u;\omega|q)\right) - \sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right) B_i^n(u;\omega|q) \\
&\quad = \sum_{i=1}^{n-1}\left(\frac{[n-i]_q}{[n]_q} f\left(\frac{[i]_q}{[n-1]_q}\right) + \left(1 - \frac{[n-i]_q}{[n]_q}\right) f\left(\frac{[i-1]_q}{[n-1]_q}\right) - f\left(\frac{[i]_q}{[n]_q}\right)\right) B_i^n(u;\omega|q) \\
&\quad =: \sum_{i=1}^{n-1} a_i B_i^n(u;\omega|q).
\end{aligned}
$$

Clearly, if $\omega, q \in (0,1)$ or $(1,\infty)$, then $B_i^n(u;\omega|q) \geqslant 0$ for all $x \in [0,1]$. It suffices to show that each $a_i$ is non-negative. For $1 \leqslant i \leqslant n-1$, put $t_0 = \frac{[i-1]_q}{[n-1]_q}$, $t_1 = \frac{[i]_q}{[n-1]_q}$ and $\lambda = \frac{[n-i]_q}{[n]_q}$. Then

$$
a_i = \lambda f(t_1) + (1-\lambda) f(t_0) - f\big(\lambda t_1 + (1-\lambda) t_0\big) \geqslant 0,
$$

since the function $f$ is convex. Thus $B_{n-1}^{\omega,q}(f;x) \geqslant B_n^{\omega,q}(f;x)$. The inequality will be strict for $0 < x < 1$ unless each $a_i = 0$ which can only occur when $f$ is linear in each of the intervals between consecutive knots $[i]/[n-1]$, $0 \leqslant i \leqslant n-1$, when we have $B_{n-1}^{\omega,q}(f;x) = B_n^{\omega,q}(f;x)$ for $0 \leqslant x \leqslant 1$. This completes the proof. □

**Proof of Theorem 4.** It is easy to know from (1.1) that

$$
B_n^{\omega,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[n-k]_q}{[n]_q}\right) B_{n-k}^n(u;\omega|q), \quad u = \omega + (1-\omega)x.
$$

Let us put $u' = 1/\omega + (1 - 1/\omega)(1-x) = 1 + (1/\omega - 1)x$. Note that

$$
\begin{aligned}
&\begin{bmatrix} n \\ n-k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q = q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q}, \\
&(\omega;q)_n = q^{n(n-1)/2} \omega^n (-1)^n (1/\omega;1/q)_{1/q}, \\
&\prod_{j=0}^{n-k-1} \big(u - \omega q^j\big) = (-1)^{n-k} \omega^{n-k} q^{(n-k)(n-k-1)/2} \prod_{j=0}^{n-k-1} \big(1 - u' q^{-j}\big),
\end{aligned}
$$

and

$$
\prod_{s=0}^{k-1} \big(1 - u q^s\big) = (-1)^k \omega^k q^{k(k-1)} \prod_{s=0}^{k-1} \big(u' - \omega^{-1} q^{-s}\big),
$$

we get by (1.2)

$$
\begin{aligned}
B_{n-k}^n(u;\omega|q) &= \frac{1}{(\omega;q)_n} \begin{bmatrix} n \\ n-k \end{bmatrix}_q \prod_{j=0}^{n-k-1} \big(u - \omega q^j\big) \prod_{s=0}^{k-1} \big(1 - u q^s\big) \\
&= \frac{1}{(1/\omega;1/q)_n} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} \prod_{s=0}^{k-1} \big(u' - \omega^{-1} q^{-s}\big) \prod_{j=0}^{n-k-1} \big(1 - u' q^{-j}\big) \\
&= B_k^n(u';1/\omega|1/q).
\end{aligned}
$$

On the other hand,

$$f\left(\frac{[n-k]_q}{[n]_q}\right)=f\left(1-\frac{[k]_{1/q}}{[n]_{1/q}}\right)=g\left(\frac{[k]_{1/q}}{[n]_{1/q}}\right).$$

Therefore,

$$B_n^{\omega,q}(f;x)=\sum_{k=0}^{n}g\left(\frac{[k]_{1/q}}{[n]_{1/q}}\right)B_k^n(u';1/\omega|1/q)=B_n^{1/\omega,1/q}(g;1-x).$$

The proof of Theorem 4 is complete. □

## 4. Proofs of Theorems 5–7

**Proof of Theorem 5.** Since the $\omega,q$-Bernstein operators $B_n^{\omega,q_n}$ are positive linear operators and reproduce linear functions, the well-known Korovkin theorem (see [2, pp. 8–9]) implies that $B_n^{\omega,q_n}(f;x)\rightrightarrows f(x)$ $[x\in[0,1];\ n\to\infty]$ for any $f\in C[0,1]$ if and only if

$$B_n^{\omega,q_n}\left(t^2;x\right)\rightrightarrows x^2\quad\left[x\in[0,1];n\to\infty\right].\tag{4.1}$$

From Theorem 1, we get

$$\frac{x(1-x)}{[n]_q}\leqslant B_n^{\omega,q}\left(t^2;x\right)-x^2=\frac{(1-\omega q^n)x(1-x)}{(1-\omega q)[n]_q}\leqslant\frac{x(1-x)}{(1-\omega)[n]_q}.$$

Hence, (4.1) is equivalent to the condition $\lim_{n\to\infty}[n]_{q_n}=\infty$, which is equivalent to the condition $\lim_{n\to\infty}q_n=1$ (see [13]). Theorem 5 is proved. □

**Proof of Theorem 6.** The proof is similar to the one of Theorem 1 in [16]. It follows directly from (2.1) and (2.8) that

$$B_n^{\omega,q}(f;0)=B_\infty^{\omega,q}(f;0)=f(0),\qquad B_n^{\omega,q}(f;1)=B_\infty^{\omega,q}(f;1)=f(1).$$

Obviously, if $\omega,q\in(0,1)$, then $B_k^n(u;\omega|q)\geqslant 0$, $B_k^\infty(u;\omega|q)\geqslant 0$ for all $x\in[0,1]$, $u=\omega+(1-\omega)x$. Hence for all $x\in(0,1)$, by (1.1), (2.8), (2.2) and (2.10) we know that

$$\begin{aligned}&\left|B_n^{\omega,q}(f;x)-B_\infty^{\omega,q}(f;x)\right| &&(4.2)\\&\quad=\left|\sum_{k=0}^{n}f\left([k]/[n]\right)B_k^n(u;\omega|q)-\sum_{k=0}^{\infty}f\left(1-q^k\right)B_k^\infty(u;\omega|q)\right|\\&\quad=\left|\sum_{k=0}^{n}\left(f\left([k]/[n]\right)-f(1)\right)B_k^n(u;\omega|q)-\sum_{k=0}^{\infty}\left(f\left(1-q^k\right)-f(1)\right)B_k^\infty(u;\omega|q)\right|\\&\quad\leqslant\sum_{k=0}^{n}\left|f\left([k]/[n]\right)-f\left(1-q^k\right)\right|B_k^n(u;\omega|q)+\sum_{k=0}^{n}\left|f\left(1-q^k\right)-f(1)\right|\left|B_k^n(u;\omega|q)-B_k^\infty(u;\omega|q)\right|\\&\qquad+\sum_{k=n+1}^{\infty}\left|f\left(1-q^k\right)-f(1)\right|B_k^\infty(u;\omega|q)\\&\quad=:I_1+I_2+I_3. &&(4.3)\end{aligned}$$

First we estimate $I_1$, $I_3$. Since

$$\begin{aligned}&0\leqslant\frac{[k]}{[n]}-\left(1-q^k\right)=\frac{1-q^k}{1-q^n}-\left(1-q^k\right)=\frac{q^n(1-q^k)}{1-q^n}\leqslant q^n,\\&0\leqslant 1-\left(1-q^k\right)=q^k\leqslant q^n\quad(k\geqslant n+1),\end{aligned}$$

we get

$$I_1 \leqslant \omega\big(f, q^n\big) \sum_{k=0}^{n} B_k^n(u; \omega|q) = \omega\big(f, q^n\big) \tag{4.4}$$

and

$$I_3 \leqslant \omega\big(f, q^n\big) \sum_{k=n+1}^{\infty} B_k^{\infty}(u; \omega|q) \leqslant \omega\big(f, q^n\big). \tag{4.5}$$

Now we estimate $I_2$. For $0 \leqslant k \leqslant n$, we note that

$$\begin{aligned} \big|B_k^n(u; \omega|q) - B_k^{\infty}(u; \omega|q)\big| &\leqslant B_k^n(u; \omega|q)\left|1 - \prod_{s=n-k}^{\infty}\big(1 - uq^s\big)\right| \\ &\quad + B_k^{\infty}(u; \omega|q)\left|\prod_{s=n}^{\infty}\big(1 - \omega q^s\big) \prod_{s=n-k+1}^{n}\big(1 - q^s\big) - 1\right|. \end{aligned} \tag{4.6}$$

Using the inequality (see [15])

$$1 - \prod_{s=j}^{\infty}\big(1 - q^s\big) \leqslant \frac{q^j}{q(1-q)} \ln \frac{1}{1-q} \quad (j = 1, 2, \ldots),$$

we get

$$\left|1 - \prod_{s=n-k}^{\infty}\big(1 - q^s u\big)\right| \leqslant 1 - \prod_{s=n-k}^{\infty}\big(1 - q^s\big) \leqslant \frac{q^{n-k}}{q(1-q)} \ln \frac{1}{1-q} \quad \big(u \in [\omega, 1]\big), \tag{4.7}$$

and

$$\begin{aligned} \left|\prod_{s=n}^{\infty}\big(1 - \omega q^s\big) \prod_{s=n-k+1}^{n}\big(1 - q^s\big) - 1\right| &\leqslant \left|1 - \prod_{s=n}^{\infty}\big(1 - q^s\big) \prod_{s=n-k}^{n-1}\big(1 - q^s\big)\right| = \left|1 - \prod_{s=n-k}^{\infty}\big(1 - q^s\big)\right| \\ &\leqslant \frac{q^{n-k}}{q(1-q)} \ln \frac{1}{1-q}. \end{aligned} \tag{4.8}$$

Using (4.6)–(4.8) and the property of modulus of continuity (see [6, p. 20])

$$\omega(f, \lambda t) \leqslant (1 + \lambda)\omega(f, t), \quad \lambda > 0,$$

we get

$$\begin{aligned} I_2 &\leqslant \sum_{k=0}^{n} \omega\big(f, q^k\big)\big|B_k^n(u; \omega|q) - B_k^{\infty}(u; \omega|q)\big| \\ &\leqslant \sum_{k=0}^{n} \omega\big(f, q^n\big)\big(1 + q^{k-n}\big)\frac{q^{n-k}}{q(1-q)} \ln \frac{1}{1-q}\big(B_k^n(u; \omega|q) + B_k^{\infty}(u; \omega|q)\big) \\ &\leqslant \frac{2\omega(f, q^n)}{q(1-q)} \ln \frac{1}{1-q} \sum_{k=0}^{n}\big(B_k^n(u; \omega|q) + B_k^{\infty}(u; \omega|q)\big) \\ &\leqslant \frac{4\omega(f, q^n)}{q(1-q)} \ln \frac{1}{1-q}. \end{aligned} \tag{4.9}$$

From (4.4), (4.5), and (4.9), we conclude that

$$\big\|B_n^{\omega,q}(f) - B_{\infty}^{\omega,q}(f)\big\| \leqslant C_q \omega\big(f, q^n\big),$$

where $C_q = 2 + \frac{4 \ln \frac{1}{1-q}}{q(1-q)}$.

At last we show that the estimate (2.11) is sharp. For each $\alpha$, $0<\alpha\leqslant 1$, suppose that $f_\alpha(x)$ is a continuous function which is equal to zero in $[0,1-q]$ and $[1-q^2,1]$, equal to $(x-(1-q))^\alpha$ in $[1-q,1-q+q(1-q)/2]$, and linear in the rest of $[0,1]$. Then

$$\omega(f_\alpha,t)\asymp t^\alpha,$$

and

$$\left\|B_n^{\omega,q}(f_\alpha;x)-B_\infty^{\omega,q}(f_\alpha;x)\right\|\asymp q^{\alpha n}\left\|B_1^n(u;\omega|q)\right\|\asymp q^{\alpha n}.$$

The proof of Theorem 6 is complete. □

**Remark 8.** From the proof of Theorem 6, we know that the rate of convergence for the $\omega,q$-Bernstein polynomials depends only on the smoothness of the function $f(x)$ at the points $1-q^k$, $k=1,2,\ldots$ (from the right), and at $x=1$.

In order to prove Theorem 7, we need the following result (see [14]):

**Theorem A.** *Let the sequence $(L_n)$ of positive linear operators on $C[0,1]$ satisfy the following conditions*:

(A) *The sequence $(L_n(e_2))$ converges to a function $L_\infty(e_2)$ in $C[0,1]$, where $e_i(x)=x^i$, $i=0,1,2$.*
(B) *The sequence $(L_n(f,x))_{n\geqslant 1}$ is non-increasing for any convex function $f$ and for any $x\in[0,1]$.*

*Then there exists an operator $L_\infty$ on $C[0,1]$ such that $L_n(f)\rightrightarrows L_\infty(f)$ for any $f\in C[0,1]$ as $n\to\infty$. Furthermore,*

$$\left|L_n(f,x)-L_\infty(f,x)\right|\leqslant c\omega_2\left(f,\sqrt{\lambda_n(x)}\right), \tag{4.10}$$

*where $\lambda_n(x)=L_n(e_2,x)-L_\infty(e_2,x)$, $c$ is a constant dependent only on $\|L_1(e_0)\|$.*

**Proof of Theorem 7.** From Theorem 3, we know that the $\omega,q$-Bernstein operators satisfy condition (B). From Theorem 6 we know that for $\omega,q\in(0,1)$,

$$B_n^{\omega,q}(f,x)\rightrightarrows B_\infty^{\omega,q}(f,x)\quad\left[x\in[0,1];\ n\to\infty\right].$$

Also, by (2.5) and (2.15), we get

$$0\leqslant\lambda_n(x):=B_n^{\omega,q}\left(t^2;x\right)-B_\infty^{\omega,q}\left(t^2;x\right)=\frac{q^n(1-q)(1-\omega)x(1-x)}{(1-\omega q)(1-q^n)}\leqslant q^n \tag{4.11}$$

and

$$\sup_{0<\omega,q<1}\lambda_n(x)=\sup_{0<\omega,q<1}\frac{q^n(1-q)(1-\omega)}{(1-q^n)(1-\omega q)}x(1-x)=\frac{x(1-x)}{n}. \tag{4.12}$$

Theorem 7 follows from (4.11), (4.12), and (4.10). □

## 5. Proofs of Theorems 8–10

**Proof of Theorem 8.** It suffices to prove the statement that $B_\infty^{\omega,q}(t^r;x)$ is a polynomial of degree $r$. We use induction on $r$. For $r=0$ or $r=1$ the statement is true, since $B_\infty^{\omega,q}$ reproduce linear functions. Assume that the statement is true for $r\leqslant m$ and consider $B_\infty^{\omega,q}(t^{m+1};x)$. Set

$$\psi(x)=\frac{(u;q)_\infty}{(\omega;q)_\infty}\quad\left(u=\omega+(1-\omega)x\right).$$

By (2.8) and (2.9) we get

$$\begin{aligned}
B_\infty^{\omega,q}\big(t^{m+1};x\big) &= \sum_{k=1}^{\infty}\big(1-q^k\big)^m \frac{\prod_{j=0}^{k-1}(u-\omega q^j)}{(q;q)_{k-1}}\psi(x) \\
&= \sum_{k=0}^{\infty}\big(1-q+q\big(1-q^k\big)\big)^m\big(u-\omega+\omega\big(1-q^k\big)\big)\frac{\prod_{j=0}^{k-1}(u-\omega q^j)}{(q;q)_k}\psi(x) \\
&= \sum_{s=0}^{m-1}\binom{m}{s}q^s(1-q)^{m-s}\sum_{k=0}^{\infty}\big((1-\omega)x\big(1-q^k\big)^s+\omega\big(1-q^k\big)^{s+1}\big)\frac{\prod_{j=0}^{k-1}(u-\omega q^j)}{(q;q)_k}\psi(x) \\
&\quad + \sum_{k=0}^{\infty}\big((1-\omega)q^m x\big(1-q^k\big)^m+\omega q^m\big(1-q^k\big)^{m+1}\big)\frac{\prod_{j=0}^{k-1}(u-\omega q^j)}{(q;q)_k}\psi(x) \\
&= \sum_{s=0}^{m-1}\binom{m}{s}q^s(1-q)^{m-s}\big((1-\omega)xB_\infty^{\omega,q}\big(t^s;x\big)+\omega B_\infty^{\omega,q_n}\big(t^{s+1};x\big)\big) \\
&\quad + (1-\omega)q^m x B_\infty^{\omega,q}\big(t^m;x\big)+\omega q^m B_\infty^{\omega,q}\big(t^{m+1};x\big),
\end{aligned}$$

which means

$$\begin{aligned}
B_\infty^{\omega,q}\big(t^{m+1};x\big) &= \frac{1}{1-\omega q^m}\big((1-\omega)q^m x B_\infty^{\omega,q}\big(t^m;x\big) \\
&\quad + \sum_{s=0}^{m-1}\binom{m}{s}q^s(1-q)^{m-s}\big((1-\omega)xB_\infty^{\omega,q}\big(t^s;x\big)+\omega B_\infty^{\omega,q}\big(t^{s+1};x\big)\big)\big).
\end{aligned}$$

By the induction assumption this is a polynomial of degree $m+1$. $\square$

**Proof of Theorem 9.** It suffices to prove that $f$ is linear if $L(f)=f$. Let $g(x)=f(x)-f(0)-(f(1)-f(0))x$. Then $g(0)=g(1)=0$ and $Lg=g$, since $L$ reproduces linear functions. We will prove $g=0$. Assume $g\neq 0$. Without loss of generalization we may assume that there exist an $x_0\in(0,1)$ such that $g(x_0)>0$. Then, for some $\alpha<0$, $\alpha/4>\alpha(x_0-1/2)^2-g(x_0)$. Now $h(x)=\alpha(x-1/2)^2-g(x)$ is continuous on [0, 1] and $h(0)=h(1)>h(x_0)$. Let $m$ be the minimum of $h$ on [0, 1], and suppose it is assumed at $\xi$ with $\xi\in(0,1)$. Then for all $x\in[0,1]$, $\alpha(x-1/2)^2-g(x)\geqslant m=\alpha(\xi-1/2)^2-g(\xi)$. Hence, $g(x)\leqslant\alpha(x-\xi)^2+\beta(x-\xi)+g(\xi)$ for some $\beta$ and therefore,

$$L(g,\xi)\leqslant\alpha L\big((t-\xi)^2,\xi\big)+\beta L\big((t-\xi),\xi\big)+g(\xi)=\alpha\big(L\big(t^2,\xi\big)-\xi^2\big)+g(\xi).$$

Since $L(g,\xi)=g(\xi)$ and $L(t^2,\xi)-\xi^2>0$, we get $\alpha\geqslant 0$, which leads to a contradiction. Hence, $g=0$ and therefore, $f$ is linear. $\square$

**Proof of Theorem 10.** The proof is standard. We know that the limit $\omega,q$-Bernstein operators are positive linear operators on $C[0,1]$ and reproduce linear functions. Also, by (2.15)

$$B_\infty^{\omega,q_n}\big(t^2;x\big)\rightrightarrows x^2 \quad \big[x\in[0,1];\ q\to 1-\big].$$

Theorem 10 follows from the Korovkin theorem. $\square$

## References


[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge Univ. Press, Cambridge, 1999.
[2] R.A. Devore, G.G. Lorentz, Constructive Approximation, Springer-Verlag, 1993.
[3] A. Il'inskii, S. Ostrovska, Convergence of generalized Bernstein polynomials, J. Approx. Theory 116 (2002) 100–112.
[4] S. Lewanowicz, P. Woźny, Generalized Bernstein polynomials, BIT 44 (1) (2004) 63–78.
[5] S. Lewanowicz, P. Woźny, Dual generalized Bernstein basis, J. Approx. Theory 128 (2) (2006) 129–150.
[6] G.G. Lorentz, Bernstein Polynomials, Math. Expo., vol. 8, University of Toronto Press, Toronto, 1953.
[7] S. Ostrovska, $q$-Bernstein polynomials and their iterates, J. Approx. Theory 123 (2003) 232–255.
[8] S. Ostrovska, On the $q$-Bernstein polynomials, Adv. Stud. Contemp. Math. 11 (2) (2005) 193–204.
[9] S. Ostrovska, On the improvement of analytic properties under the limit $q$-Bernstein operator, J. Approx. Theory 138 (2006) 37–53.

[10] S. Ostrovska, The approximation by $q$-Bernstein polynomials in the case $q \downarrow 1$, Arch. Math. 86 (3) (2006) 282–288.
[11] G.M. Phillips, Bernstein polynomials based on the $q$-integers, Ann. Numer. Math. 4 (1997) 511–518.
[12] G.M. Phillips, Interpolation and Approximation by Polynomials, Springer-Verlag, 2003.
[13] V.S. Videnskii, On some classes of $q$-parametric positive operators, Oper. Theory Adv. Appl. 158 (2005) 213–222.
[14] H. Wang, Korovkin-type theorem and application, J. Approx. Theory 132 (2) (2005) 258–264.
[15] H. Wang, Voronovskaya type formulas and saturation of convergence for $q$-Bernstein polynomials for $0 < q < 1$, J. Approx. Theory 145 (2) (2007) 182–195.
[16] H. Wang, F. Meng, The rate of convergence of $q$-Bernstein polynomials for $0 < q < 1$, J. Approx. Theory 136 (2) (2005) 151–158.
[17] H. Wang, X. Wu, Saturation of convergence for $q$-Bernstein polynomials in the case $q > 1$, J. Math. Anal. Appl. 337 (1) (2008) 744–750.