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Computational method for analysis of polyethylene biodegradation

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Abstract

In a previous study concerning the biodegradation of polyethylene, we proposed a mathematical model based on two primary factors: the direct consumption or absorption of small molecules and the successive weight loss of large molecules due to β -oxidation. Our model is an initial value problem consisting of a differential equation whose independent variable is time. Its unknown variable represents the total weight of all the polyethylene molecules that belong to a molecular-weight class specified by a parameter. In this paper, we describe a numerical technique to introduce experimental results into analysis of our model. We first establish its mathematical foundation in order to guarantee its validity, by showing that the initial value problem associated with the differential equation has a unique solution. Our computational technique is based on a linear system of differential equations derived from the original problem. We introduce some numerical results to illustrate our technique as a practical application of the linear approximation. In particular, we show how to solve the inverse problem to determine the consumption rate and the β -oxidation rate numerically, and illustrate our numerical technique by analyzing the GPC patterns of polyethylene wax obtained before and after 5 weeks cultivation of a fungus, *Aspergillus* sp. AK-3. A numerical simulation based on these degradation rates confirms that the primary factors of the polyethylene biodegradation posed in modeling are indeed appropriate.

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Keywords: Biodegradation; Polyethylene; *Aspergillus*; β -oxidation

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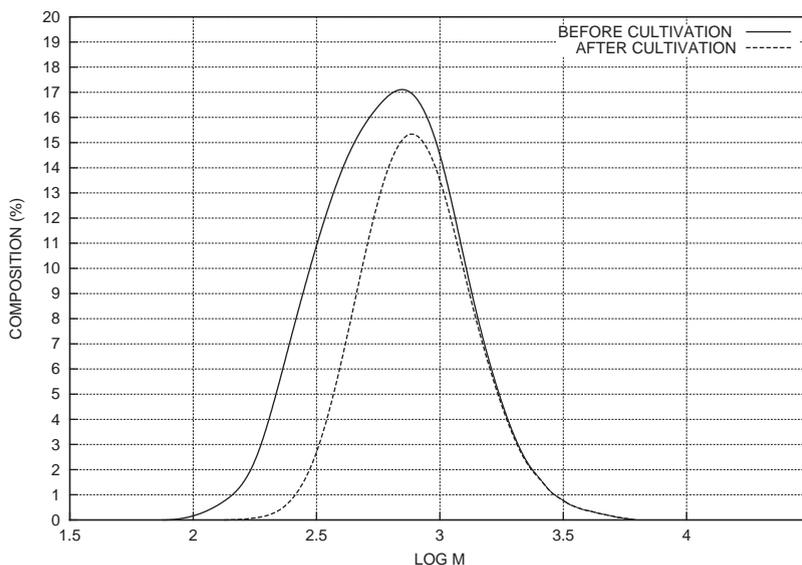


Fig. 1. Weight distributions of PEwax before and after cultivation of *Aspergillus* sp. AK-3.

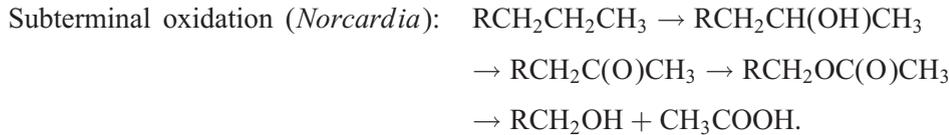
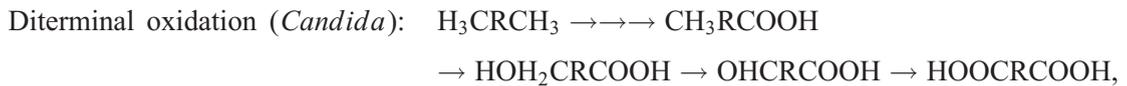
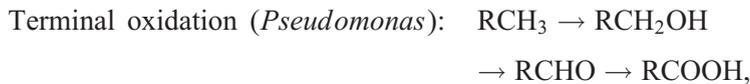
1. Introduction

Polyolefins are in general hardly biodegradable in natural environments and regarded as chemically inert polymers. However, recent studies have shown that they can be biodegraded although the process may be slow in general. Potts et al. [7] found that several microorganisms utilized linear paraffin molecules of molecular weight 500 or below. Haines and Alexander [3] showed oxidation of *n*-alkanes up to tetratetracontane ($C_{44}H_{90}$, mass of 618) in 20 days. By measurement of $^{14}CO_2$ generation, Albertsson et al. [1] confirmed the slow degradation of polyethylene (PE). Kawai et al. [4] used commercially available polyethylene wax (PEwax) as the sole carbon and energy source for soil microorganisms. They showed that microbial consortium KH-12 obtained from soil samples degraded PEwax, which was confirmed by significant weight loss (30–50%). Those PE-utilizing bacteria were obtained by screening of soil bacteria that grew on photofragmented PE or PEwax. Nine strains were isolated, five of which were identified as *Pseudomonas*, *Sphingomonas*, *Stenotrophomonas* and *Acinetobacter* species. PE-utilizing fungi were also isolated from empty PE capsules that had been made subject to soil burial tests. These were identified as *Aspergillus*, *Penicillium* and *Acremonium*.

Commercially available PEwax was used as the sole carbon source in experiments for cultivation of isolated bacteria and fungi. After cultivation, PEwax was recovered from a medium by filtration with a membrane filter of pore size 10 μm . The weight distributions with respect to the molecular weight before and after cultivation of microorganisms were analyzed with high-temperature gel-permeation chromatography (GPC). Fig. 1 shows the weight distributions of PEwax before and after 5-week cultivation of *Aspergillus* sp. AK-3. The abscissa represents the logarithm of the molecular weight, and the ordinate represents the percent weight. These GPC profiles show that PE molecules of approximate molecular weight 3000 or lower were consumed by microorganisms and that the entire distribution was shifted higher. As far as these qualitative features are concerned, no significant

differences were found between the weight distributions of PEwax obtained after cultivation of bacteria and fungi. These experimental results suggest the following fact: the smaller the PE molecules are, the faster they are consumed by the microorganisms. The consumption of small molecules should be a common factor between bacteria and fungi in the mechanism of PE biodegradation, but its rates may differ from one group of degraders to another. Furthermore, it is not the only factor and another factor is β -oxidation which, we describe next.

Besides the fast consumption of small molecules, there is another possible factor that arises from the theoretical aspects in the mechanism of PE biodegradation. *n*-Alkane is known to be metabolized via the following three oxidative pathways:



As PE is considered as one of *n*-alkanes, it should be subjected to carboxylation via one of these oxidative pathways. PE molecules carboxylated via these metabolic processes become structurally analogous to fatty acid, and become subject to β -oxidation. In a β -oxidation process, a PE molecule loses its components piece by piece as acetic acid (molecular weight of 60) at its terminal in one cycle of β -oxidation.

In view of these experimental and theoretical aspects of PE biodegradation, we propose the following scenario: sufficiently small PE molecules are directly consumed by microorganisms, and large PE molecules lose their components piece by piece via β -oxidation until they become sufficiently small to be absorbed directly by cells from a culture medium. A mathematical model based on this scenario was obtained in Kawai et al. [5]. In this paper, we introduce a computational technique to analyze the model. In order to guarantee its validity, we first establish the mathematical foundation of the model in Section 2. In particular, we show that the initial value problem associated with the differential equation has a unique solution. Our computational technique is based on a linear problem derived from the original initial value problem. In Section 3, we illustrate our computational technique as a practical application of the linear approximation. We show how the consumption rate and the β -oxidation rate can be determined from experimental data using the linear approximation. We introduce a numerical result obtained from numerical analysis of the experimental data of Fig. 1.

2. Solutions of the PE degradation model

In this section, we first present the mathematical model of PE biodegradation proposed in a previous study, and then establish its mathematical validity by showing that the initial value problem associated with the model has a unique solution. We call the PE molecules of molecular weight M

simply M -molecules. We denote the total weight of M -molecules present at time t by $w(t, M)$ (mg). The following mathematical model of PE biodegradation was proposed by Kawai et al. [5].

$$\frac{dx}{dt} = -\rho(M)x - \beta(M)x + \beta(M+L)\frac{M}{M+L}y. \quad (1)$$

Here $x = w(t, M)$, and $y = w(t, M+L)$, the total weight of $(M+L)$ -molecules. L represents the amount which a PE molecule loses in one cycle of β -oxidation. L is equal to 60 since a molecule loses acetic acid (CH_3COOH) at its terminal in one cycle of β -oxidation. $\rho(M)$ (1/week) represents the rate of decrease per week in the total weight of M -molecules due to the direct consumption by microorganisms. $w(t, M)$ can be regarded as proportional to the number of bonds between two-carbon units in all the M -molecules present at time t , and it is known that the reduction rate of the number of bonds per unit time is proportional to the number of bonds itself in a random degradation [8]. $\beta(M)$ (1/week) represents the rate of the weight shift from the class of M -molecules to the class of $(M-L)$ -molecules due to β -oxidation. Thus, the second and the third terms of the right-hand side of (1), respectively, represent the weight decrease and the weight increase per unit time due to β -oxidation. Let

$$\alpha(M) = \rho(M) + \beta(M),$$

then (1) becomes the following simplified form:

$$\frac{dx}{dt} = -\alpha(M)x + \beta(M+L)\frac{M}{M+L}y. \quad (2)$$

Since $\rho(M)$ represents the weight ratio of M -molecules consumed per unit time, it should be a nonincreasing function of M , since small molecules are consumed faster by microorganisms. Similarly, the β -oxidation rate $\beta(M)$ should be a nonincreasing function for the following reason. The reciprocal of $\beta(M)$ can be regarded as characteristic time it takes for one cycle of β -oxidation to be completed, and it should be independent of the molecular size, for β -oxidation takes place at terminals of molecules. On the other hand, β -oxidation takes place inside cells, and PE molecules must penetrate through the membranes to become subject to it. The rate of such molecular penetration should decrease as M increases, and so β -oxidation should practically be less effective for large PE molecules. For these reasons, $\beta(M)$ should be a nonincreasing function of M and a constant function for all small M , and moreover, there should be a value M_{\max} such that $\beta(M) = 0$ for $M \geq M_{\max}$. In addition, we assume that $\rho(M)$ and $\beta(M)$ are continuous functions. Then $\alpha(M)$ is also a continuous nonincreasing function. Moreover,

$$0 \leq \beta(M) \leq \alpha(M).$$

The solution $x = w(t, M)$ is subject to the initial condition

$$w(0, M) = f(M), \quad (3)$$

where $f(M)$ (mg) represents the initial weight distribution. There should be a fixed constant b such that

$$f(M) = 0 \quad \text{for } M \geq b.$$

We assume that $f(M)$ is also a continuous function. Differential equation (2) and initial condition (3) form an initial value problem, and we discuss the problem of the existence and the uniqueness of its solution in the rest of this section.

The variation-of-constants formula [2] for a nonhomogeneous linear system leads to the following integral equation which is equivalent to this initial value problem:

$$w(t, M) = e^{-\alpha(M)t} f(M) + \frac{\beta(M + L)M}{M + L} \int_0^t e^{-\alpha(M)(t-s)} w(s, M + L) ds. \tag{4}$$

We show that there is a unique solution of this integral equation, and that the solution $w(t, M)$ is a continuous function of the pair (t, M) . Such a function necessarily satisfies (2) and (3), i.e., if $w(t, M)$ is a continuous function that satisfies (4), $x = w(t, M)$ and $y = w(t, M + L)$ satisfy Eq. (2), and $w(t, M)$ satisfies initial condition (3) for any fixed but arbitrary M .

In order to prove that (4) has a unique solution, we reformulate the problem as a fixed point problem of a mapping defined on a complete metric space. Let a be a fixed constant, and consider the vector space of all bounded continuous functions $u(t, M)$ defined on the set Ω :

$$\Omega = \{(t, M) \mid t \geq 0, M \geq a\}. \tag{5}$$

The vector space of all bounded real-valued functions becomes a Banach space when it is equipped with the norm

$$\|u\| = \sup\{|u(t, M)|\}$$

[9], and the closed subspace consisting of all functions u such that $u(t, M) = 0$ for $M \geq b$ is also a Banach space. We consider the closed subset S of the Banach space consisting of all functions u that satisfy

$$u(0, M) = f(M).$$

Define the mapping $v = \Phi(u)$ which takes a function $u(t, M)$ in S to another function $v(t, M)$ by

$$v(t, M) = e^{-\alpha(M)t} f(M) + \beta(M + L) \frac{M}{M + L} \int_0^t e^{-\alpha(M)(t-s)} u(s, M + L) ds. \tag{6}$$

Note that a fixed point of Φ satisfies integral equation (4). We show that this mapping has a unique fixed point in S .

We summarize some basic properties of the mapping Φ in the following lemma. Its proof is given in the Appendix. Here we define

$$\|f\|_0 = \sup_{M \geq a} \{|f(M)|\}.$$

Lemma 1. *Suppose that u , u_1 , and u_2 are arbitrary functions in S .*

(a) $v = \Phi(u)$ is a continuous function defined on Ω that satisfies

$$(b) \quad v(0, M) = f(M), \tag{7}$$

$$(c) \quad \|\Phi(u)\| \leq \|f\|_0 + \frac{b}{b + L} \|u\|,$$

$$\|\Phi(u_1) - \Phi(u_2)\| \leq \frac{b}{b + L} \|u_1 - u_2\|.$$

Lemma 1(a) and (b) show that Φ maps S into S , and Lemma 1(c) shows that it is a contraction mapping. Since S is a complete metric space, there is a unique fixed point $w(t, M)$ of the mapping Φ , and it is a solution of integral equation (4) in S . It follows that $x = w(t, M)$ and $y = w(t, M + L)$ satisfy differential equation (2), and that $w(t, M)$ satisfies initial condition (3). We conclude this section with the following summary concerning the existence and the uniqueness of the solution of (4).

Proposition 1. *Integral equation (4) has a unique solution $w(t, M)$ that is continuous in set (5). It is the unique solution of initial value problem (2), (3).*

The unique existence of the solution of initial value problem (2), (3) follows also from the standard theory of ordinary differential equations, simply by considering a system for the K -dimensional vector

$$[w(t, M), w(t, M + L), \dots, w(t, M + (K - 1)L)].$$

Here integer K satisfies the condition: $M + iL \in [a, b]$ ($i = 0, 1, 2, \dots, K - 1$), and $M + KL > b$. There are infinitely many such systems, and their infinitely many solutions. The proposition indicates that those solutions can be represented by the continuous function $w(t, M)$.

3. Derivation of an approximate linear system

In Section 2, we showed that initial value problem (2), (3) has the unique solution, which is summarized in Proposition 1. However, that result does not show how the solution can be constructed. It does not even show how an approximate solution can be obtained. In the present section, we show how an approximate solution can be obtained. We also show how the consumption rate and the β oxidation rate can be determined from the experimental results introduced in Section 2. Let

$$\Delta M = \frac{b - a}{N}$$

and consider the $N + 1$ equally spaced points M_i ($i = 0, 1, 2, \dots, N$) given by

$$M_i = a + i\Delta M.$$

We may assume that there is a positive integer n such that

$$L = n\Delta M$$

for otherwise the interval $[a, b]$ can be enlarged slightly for ΔM to satisfy this condition without changing the value of the solution in the original domain. Then

$$M_i + L = M_i + n\Delta M = M_{i+n}$$

for $i \leq N - n$. Set $M = M_i$, $x = w(t, M_i)$, and $y = w(t, M_i + L)$ in (2). Then we obtain the following linear system:

$$\frac{dw_i}{dt} = -\alpha_i w_i + \beta_i w_{i+n}, \quad i = 0, 1, 2, \dots, N, \quad (8)$$

where

$$\begin{aligned} \alpha_i &= \alpha(M_i), \\ \beta_i &= \frac{\beta(M_{i+n})M_i}{M_i + L}. \end{aligned} \tag{9}$$

Here we abbreviate $w(t, M_i)$ to w_i , and $w(t, M_{i+n})$ to w_{i+n} . The second term in the right-hand side is absent for $i + n > N$. The functions $w_i = w_i(t)$ ($i = 0, 1, 2, \dots, N$) which satisfy (8) should be subject to the following initial condition, which corresponds to initial condition (3) of the original initial value problem.

$$w_i(0) = f_i \equiv f(M_i), \quad i = 0, 1, 2, \dots, N. \tag{10}$$

Eqs. (8) and (10) form the following initial value problem:

$$\begin{aligned} \frac{dw_i}{dt} &= -\alpha_i w_i + \beta_i w_{i+n} \quad (i = 0, 1, 2, \dots, N), \\ w_i(0) &= f_i. \end{aligned} \tag{11}$$

The solution $w(t, M)$ of initial value problem (2), (3) satisfies (11) at discrete points $M = M_0, M_1, M_2, \dots, M_N$. On the other hand, the solution of (11) should serve as some approximation of $w(t, M)$ even when $w(t, M)$ is nonlinear with respect to M . For $M_i \leq M \leq M_{i+1}$, we define an approximate solution $\tilde{w}(t, M)$ of $w(t, M)$ to be a linear approximation in terms of $w_i(t)$ and $w_{i+1}(t)$ given by

$$\tilde{w}(t, M) \equiv \frac{M_{i+1} - M}{\Delta M} w_i(t) + \frac{M - M_i}{\Delta M} w_{i+1}(t),$$

where $N + 1$ functions $w_i = w_i(t)$ ($i = 0, 1, 2, \dots, N$) satisfy (11).

4. Numerical solution of an inverse problem

Now, we present some numerical results that we obtained using a method based on the linear approximation. In particular, we show how one can determine the consumption rate $\rho(M)$ and the β -oxidation rate $\beta(M)$. We indicated in Section 2 that $\beta(M)$ should be a nonincreasing function, and that it should be a constant function for all small M . In particular, we may assume that there is \tilde{M} such that $\beta(M)$ is constant for $M \leq \tilde{M}$. On the other hand, since microorganisms cannot absorb large PE molecules, we may assume that $\rho(M) = 0$ for $M \geq \tilde{M}$, i.e., $\alpha(M) = \beta(M)$ for $M \geq \tilde{M}$, for $\alpha(M) = \rho(M) + \beta(M)$.

Under these assumptions, we can determine $\alpha(M)$ and $\beta(M)$ numerically by analyzing the data concerning the weight distributions obtained before and after cultivation of microorganisms. Let $g(M)$ be the function which represents the final weight distribution, and let

$$g_i = g(M_i), \quad i = 0, 1, 2, \dots, N.$$

Now given some approximate values α_i of $\alpha(M_i)$, and an approximate value $\tilde{\alpha}$ of $\alpha(\tilde{M})$, e.g., the linear interpolation of (M_i, α_i) and (M_{i+1}, α_{i+1}) at $M = \tilde{M}$ for $M_i \leq \tilde{M} < M_{i+1}$, we set approximate

values λ_i of $\beta(M_i)$ by

$$\lambda_i = \begin{cases} \tilde{\alpha}, & M_i \leq \tilde{M}, \\ \alpha_i, & M_i > \tilde{M}. \end{cases}$$

We assume that the final weight distribution is reached at $t = T$. At $t = T$, g_i , ($i = 0, 1, 2, \dots, N$) should satisfy

$$g_i = e^{-\alpha_i T} f_i + \int_0^T e^{-\alpha_i(T-s)} \beta_i w_{i+n}(s) ds.$$

This is equivalent to the following equation:

$$\alpha_i = \frac{1}{T} \log \left(\frac{1}{g_i} \left\{ f_i + \int_0^T e^{\alpha_i s} \beta_i w_{i+n}(s) ds \right\} \right). \tag{12}$$

Now, we can redefine the approximate values of $\alpha_i = \alpha(M_i)$ ($i = 0, 1, 2, \dots, N$) by evaluating the right-hand side of this equation once we numerically compute $w_i(t)$ ($0 \leq t \leq T$, $i = 0, 1, 2, \dots, N$) and the integral. The numerical computation of α_i ($i = 0, 1, 2, \dots, N$) can be iterated to improve the approximation.

In order to evaluate the integral on the right-hand side of Eq. (12), we must find the solution of the initial value problem first. We solved the initial-value problem numerically using a fourth-order Adams–Bashforth–Moulton predictor–corrector method in PECE mode in conjunction with the fourth-order Runge–Kutta method to generate starting values of approximate solutions [6]. The system appears to be nonstiff, and any of the Adams-type methods would serve adequately.

In the experiment described in Section 1, 500 mg of PEwax was present initially, and we assume that

$$\int_a^b f(M) dM = 500 \text{ mg}.$$

We set $T = 5$, for the cultivation time was 5 weeks. Then $f(M)$ and $g(M)$ can be determined from the data shown in Fig. 1. We obtained $\alpha(M)$ and $\beta(M)$ numerically using the method described above. In this numerical study, $a \approx 75$, $b \approx 6310$, $\tilde{M} = 1500$, and $N = 10\,080$. We set $\Delta t = 0.005$. We also assumed that $\alpha(M)$ was represented by the exponential function

$$e^{c_1 M + c_2}$$

for small M , and evaluated c_1 and c_2 numerically with a least square approximation at every step of the iteration. Fig. 2 shows the graphs of $\alpha(M)$ and $\beta(M)$ obtained after 1000 times of iterative computations.

The final values of c_1 and c_2 were approximately equal to -0.046912 and 9.046223 , respectively. In order to start the iteration, we set the following initial approximation:

$$\lambda_i \equiv \alpha_i, \quad \alpha_i = \begin{cases} 1, & M_i \leq 1500, \\ \frac{3000 - M_i}{1500}, & M_i > 1500. \end{cases}$$

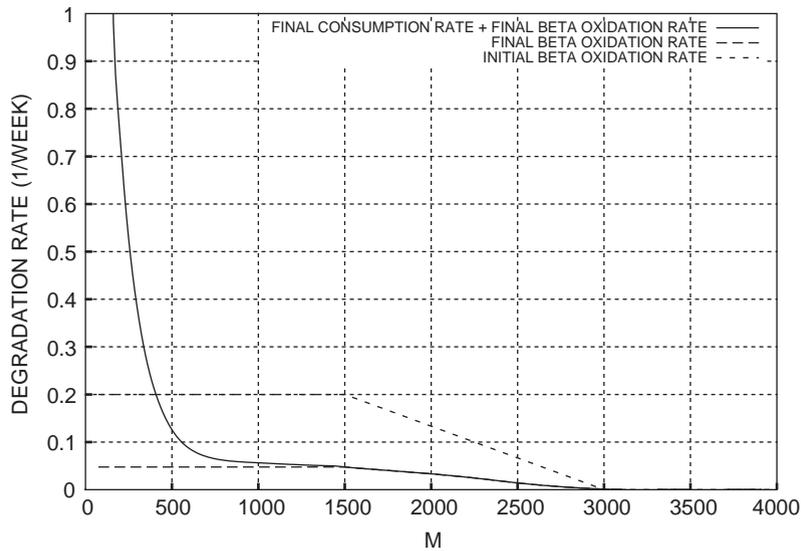


Fig. 2. Numerically evaluated consumption rate and β -oxidation rate of *Aspergillus* sp. AK-3 cultivated on PEwax.

The graph of the initial approximation of $\beta(M)$ is also shown in Fig. 2. We also computed the differences

$$\sqrt{\sum_{i=0}^N (\alpha_i - \alpha_i^*)^2}, \quad \max_{0 \leq M \leq N} \{|\alpha_i - \alpha_i^*|\}$$

for each pair of successive approximations α_i^* and α_i , and their final values were approximately equal to 0.00002155 and 0.00000877, respectively. Estimated local errors by way of differences between the predictor and the corrector were less than 10^{-15} with the step size chosen.

The numerical integrations could have been carried out with one of many existing ODE solvers of Adams type, that are freely available. If the size $N = 10\,800$ is too large to handle, then one can simply solve the problem dividing it into small uncoupled parts

$$[w(t, M), w(t, M + L), \dots, w(t, M + (K - 1)L)],$$

whose sizes range from 100 to 1000. The system is by no means restricted to use of Adams methods. The method which we used involved 1000 steps. However, other types of ODE solvers including stiff solvers might require fewer steps, and interested readers might pursue the problem in this respect.

Note that the consumption rate becomes almost 0 at $M = 1000$. However, the β -oxidation rate is positive even for $M \geq 2000$ and it eventually becomes 0 at $M = 3000$. This shows that the direct consumption is effective for PE molecules of molecular weight up to 1000, and that β -oxidation is effective for PE molecules of molecular weight up to 3000. Fig. 3 shows a result of numerical simulation based on the degradation rates of Fig. 2. It shows numerically generated weight distributions at every week until the final stage is reached. Note that our numerical result well captured the experimental features. In particular, no apparent difference can be seen between the GPC profile obtained after 5-week cultivation of *Aspergillus* sp. AK-3, and the final result (5 weeks) of the

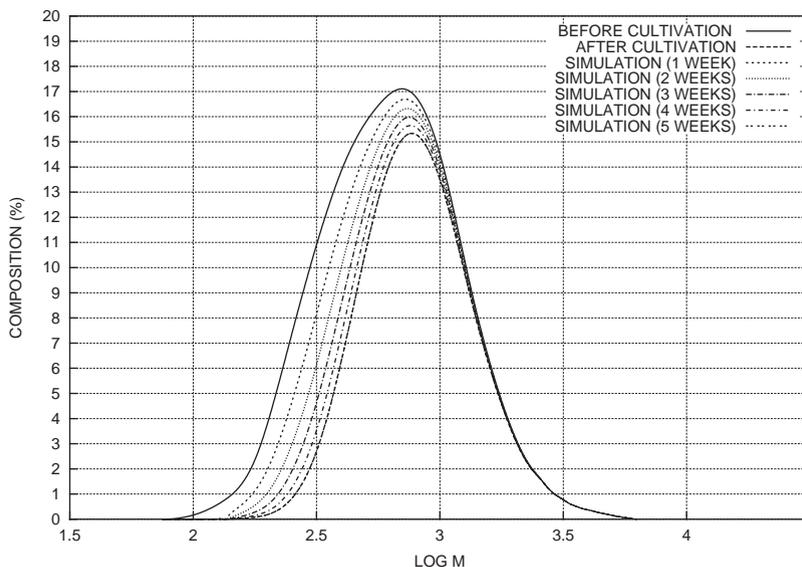


Fig. 3. Transition of the PEwax weight distribution under cultivation of *Aspergillus* sp. AK-3.

numerical simulation. In fact, we found the following error between the experimental result and the computational result:

$$\sqrt{\sum_{i=0}^N (g_i - w_i)^2} < 0.00085239, \quad \max_{0 \leq M \leq N} \{|g_i - w_i|\} < 0.00008758.$$

5. Discussion

In our mathematical model, we assumed an initial oxidation to yield carboxylic acid for the hydrocarbon to become subject to β -oxidation. We also assumed that PE molecules of low molecular weight were consumed by microorganisms faster than large ones, which is based on the experimental observation. Fig. 2 indicates that PE molecules of molecular weight over 1000 gradually lose their weight undergoing the β -oxidation process, since $\beta(M)$ is positive even for $M \geq 2000$, and since $\rho(M)$ is almost equal to 0 for $M \geq 1000$. It also indicates that the weight loss continues until their molecular weight becomes less than 1000. One may also speculate that some of PE molecules of weight less than 1000 are directly consumed, and the direct consumption is more effective for smaller molecules. So the numerical result of Fig. 2 well supports the assumption on which our mathematical model is based.

There had not been any results as far as the mathematical results such as the existence and the uniqueness of the associated initial-value problem. In this paper, we gave a solid mathematical foundation of the model. The existence and the uniqueness of the solution are guaranteed by Proposition 1. We also illustrated how one can solve the inverse problem to determine the consumption rate and

the β -oxidation rate given the GPC data of PEwax which were obtained before and after cultivation of *Aspergillus* sp. AK-3. There we demonstrated how the linear problem can be analyzed and how the consumption rate and the oxidation rate can be determined. The correspondence between the experimental result and the numerical result shows that our model is acceptable and that our assumption concerning the mechanism of PE biodegradation is appropriate.

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Appendix

Here we give the proofs of Lemma 1.

Proof of Lemma 1. (a) The continuity of $v = \Phi(u)$ and Eq. (7) follow directly from definition (6) of the mapping Φ .

(b) Suppose that $\alpha(M) = 0$. Then since $\beta(M) \leq \alpha(M)$ and $\beta(M + L) \leq \beta(M)$, $\beta(M + L) = 0$. It follows that $v(t, M) = f(M)$ and that

$$|v(t, M)| = |f(M)| \leq \|f\|_0.$$

Suppose that $\alpha(M) \neq 0$. If $M \geq b$, then $v(t, M) = 0$. If $M < b$, then

$$\begin{aligned} |v(t, M)| &\leq e^{-\alpha(M)t} |f(M)| + \frac{\beta(M + L)M \|u\|}{M + L} \int_0^t e^{-\alpha(M)(t-s)} ds \\ &\leq e^{-\alpha(M)t} |f(M)| + \frac{\beta(M + L)M \|u\|}{\alpha(M)(M + L)} (1 - e^{-\alpha(M)t}) \\ &\leq e^{-\alpha(M)t} \|f\|_0 + \frac{b}{b + L} \|u\| (1 - e^{-\alpha(M)t}) \leq \|f\|_0 + \frac{b}{b + L} \|u\|. \end{aligned}$$

In either case the inequality is valid.

(c) Suppose that $v_1 = \Phi(u_1)$ and that $v_2 = \Phi(u_2)$. Then

$$\begin{aligned} v_1(t, M) - v_2(t, M) &= \frac{\beta(M + L)M}{M + L} \int_0^t e^{-\alpha(M)(t-s)} [u_1(s, M + L) - u_2(s, M + L)] ds. \end{aligned}$$

If $\alpha(M) = 0$ or $M \geq b$, then

$$v_1(t, M) - v_2(t, M) = 0.$$

If $\alpha(M) > 0$ and $M < b$, then

$$|v_1(t, M) - v_2(t, M)| \leq \frac{\beta(M+L)M}{\alpha(M)(M+L)} (1 - e^{-\alpha(M)t}) \|u_1 - u_2\| \leq \frac{b}{b+L} \|u_1 - u_2\|$$

for

$$\frac{\beta(M+L)}{\alpha(M)} \leq 1 \quad \text{and} \quad \frac{M}{M+L} \leq \frac{b}{b+L}.$$

In either case, the inequality is valid. \square

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