AN UNSTABLE ADAMS SPECTRAL SEQUENCE

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§1. INTRODUCTION

Using the lower central series of a semisimplicial group, Curtis [4] has defined for each space $X$ a spectral sequence whose $E^1$-term depends only on $H_\ast(X)$ and which, for $X$ simply connected, converges to $\pi_\ast X$. The object of this note is to define (in §2) a mod-$p$ version of Curtis's spectral sequence and to show that

(i) the $E^1$-term is a $\mathbb{Z}_p$-module which depends only on $H_\ast(X; \mathbb{Z}_p)$. (§3)
(ii) if $X$ is simply connected and has finitely generated homotopy groups, then the spectral sequence converges in the same sense as the Adams spectral sequence [1] to a quotient of $\pi_\ast X$. (§4)

This mod-$p$ spectral sequence seems to be a good candidate for an Unstable Adams spectral sequence since [2], 2.6, it coincides in the stable range (after a minor reindexing) with the Adams spectral sequence.

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§2. THE SPECTRAL SEQUENCE

2.1. The lower $p$-central series

Let $G$ be a group and $p$ a prime. The lower $p$-central series of $G$ is [8] the filtration

$$G = \Gamma_1 G \geq \Gamma_2 G \geq \ldots \geq \Gamma_r G \geq \ldots,$$

where $\Gamma_r G$ is the subgroup generated by all elements

$$[a_1, \ldots, a_k]^{p^r}$$

for which $k \geq 1$, $kp^r \geq r$, and each $a_j \in G$. The symbol $[\ldots, \ldots, \ldots]$ denotes the simple commutator $[\ldots [\ldots, \ldots, \ldots, \ldots]]$.

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2.2. The spectral sequence

If \( X \) is a connected semi-simplicial complex with base point, let \( GX \) be its loop group complex \([6]\). Then \( GX \) is a free group complex with \( \pi_q GX = \pi_{q+1} X \). We now denote by \( \{E'X\} \) the spectral sequence derived from the homotopy exact couple of the filtered group complex \( GX \),

\[
GX = \Gamma_1 GX \supseteq \Gamma_2 GX \supseteq \ldots \supseteq \Gamma_r GX \supseteq \ldots.
\]

2.3. A generalization

As in \([4]\), 1.6, the above spectral sequence can be generalized to the case of homotopy classes of maps of \( S^{q+1} Y \) into \( X \), \( q \geq 1 \). The obvious generalizations of the results of §3 then hold. For convergence one requires that \( Y \) be finite dimensional, that \( X \) be simply connected, and that both \( H_\ast Y \) and \( \pi_\ast X \) be finitely generated for all \( n \).

§3. PROPERTIES OF \( E^1 X \)

Let \( X \) be a connected semisimplicial complex and \( \{E'X\} \) its mod-\( p \) spectral sequence.

**Theorem 3.1.** \( E^1 X \) is a \( Z_p \)-module and depends only on \( H_\ast(X; Z_p) \).

**Proof.** We have \( GX/\Gamma_2 GX \cong Z_p \otimes GX/[GX, GX] \); thus \([6]\),

\[
\pi_q(GX/\Gamma_2 GX) \cong H_{q+1}(X; Z_p).
\]

The group homotopy type of the \( Z_p \)-module complex \( GX/\Gamma_2 GX \) is, therefore, (see \([5]\)) determined by \( H_\ast(X; Z_p) \).

In order to prove, for \( r > 1 \), that \( \pi_q(\Gamma_r GX/\Gamma_{r+1} GX) \) depends only on \( H_\ast(X; Z_p) \), we recall the definition of the free restricted Lie algebra on a \( Z_p \)-module \( M \). Let \( TM \) be the tensor algebra \( TM = \sum_{r \geq 0} M' \), where \( M' = M \otimes \ldots \otimes M \) \( r \)-times. For \( a, b \in TM \), define \( [a, b] = ab - ba \) and \( a^{[p]} = a^p \); then the free restricted Lie algebra \( LM \) on \( M \) is the smallest sub \( Z_p \)-module of \( TM \) containing \( M \) and closed under the operations \([\; , \; ]\) and \( (\; )^{[p]} \). Put \( L_r M = LM \cap M' \) so that \( LM = \sum_{r \geq 1} L_r M \). For each \( r \), \( L_r M \) is a functor of \( M \). A result of Zassenhaus \([8]\), §2, is

**Proposition 3.3.** If \( G \) is a free group, there is for each \( r \) a natural isomorphism

\[
\Gamma_r G/\Gamma_{r+1} G \cong L_r(G/\Gamma_2 G).
\]

Applying this to \( GX \), we have

**Proposition 3.4.** \( E^1 X \cong \pi_q L(GX/\Gamma_2 GX) \).

From 3.2, 3.4, and Dold's lemma \([5]\), Theorem 3.1 now follows immediately.

3.5. Presentation of \( E^1 X \) in terms of \( H_\ast(X; Z_p) \)

It turns out that \( E^1 X \) is simpler than the corresponding term in Curtis's spectral sequence. There follows a presentation of \( E^1 X \) for \( p = 2 \) (A. K. Bousfield, unpublished). A similar but more complicated presentation exists for \( p \) odd.
Proposition. Let $X$ be simply connected, $p = 2$; then there is a natural isomorphism

$$(E^1X)_{j+1} \approx \sum_{i \geq 0} (L^j(S^{-1}H_*(X; Z_2)))_{i+1} \otimes \pi_jL(AS_i),$$

where

1. $S^{-1}H_*(X; Z_2)$ is $H_*(X; Z_2)$ with gradation reduced by 1.
2. $L^j$ is the free restricted graded Lie algebra functor [7], §6.
3. the groups $\pi_jL(AS_i)$ are as in [2], 5.4.

§4. Convergence of the Spectral Sequence

Denote by $\pi_*(X; p)$ the quotient of $\pi_*(X)$ by the subgroup of elements of finite order prime to $p$.

Theorem 4.1. If $X$ is simply connected and has finitely generated homotopy groups; then $\{E^rX\}$ is weakly convergent [3, XV. 2], and $E^\infty X$ is the graded group associated with a filtration of $\pi_*(X; p)$.

Proof. It suffices to show that, for each $r$,

$$(4.2.) \quad u \in \text{Im}[\pi_*(\Gamma_rGX) \to \pi_*(\Gamma_rGX)] \text{ for all } s \geq r \text{ if and only if } u \text{ is of finite order prime to } p.$$ 

Since each $\pi_*(\Gamma_rGX) / \Gamma_{r+1}GX$ is a $Z_p$-module, the "if" part of 4.2 is obvious. Now, in view of 3.1 and the assumptions on $X$, the groups $\pi_q\Gamma_rGX$ are all finitely generated. Therefore, an element of $\pi_*(\Gamma_rGX)$ is of finite order prime to $p$ if it is infinitely divisible by $p$. By [8], 11, there is a semisimplicial map $\xi(r) : \Gamma_rGX \to \Gamma_{pr}GX$ sending $a \to a^p$. Now 4.2 follows easily from

Lemma (4.3). For each $q$ there is an $n_q$ such that $pr \geq n_q$ implies

$$\xi_q(r) : \pi_q\Gamma_rGX \to \pi_q\Gamma_{pr}GX$$

is an isomorphism.

Proof. Filter $\Gamma_rGX$, for each $s$, by

$$\Gamma_sGX = \Gamma_{s,1}GX \supseteq \Gamma_{s,2}GX \supseteq \ldots \supseteq \Gamma_{s,m}GX \supseteq \ldots,$$

where, for any group $G$, $\Gamma_{s,m}G$ is the subgroup generated by all elements

$$[a_1, \ldots, a_k]^p$$

for which $k \geq m$, $kp^t \geq s$, and each $a_j \in G$. If $m \geq s$, $\Gamma_{s,m}G$ is the $m$-th term in the lower central series of $G$. Now by [8], 15, $\xi(r)$ induces isomorphisms

$$\Gamma_{r,m}GX / \Gamma_{r,m+1}GX \cong \Gamma_{pr,m}GX / \Gamma_{pr,m+1}GX$$

for $m < pr$. Furthermore, by a theorem of Curtis [4], 1.3, for each $q$ there is an $N$ such that $m \geq N$ implies $\Gamma_{m,m}GX$ is $q$-connected. Put $n_q = N$; then $\Gamma_{r,pr}GX = \Gamma_{pr,pr}GX$ is $q$-connected for $pr \geq n_q$. Iterated application of the five-lemma now demonstrates 4.3.

References


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