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On the spectra of some graphs like weighted rooted trees

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Abstract

Let G be a weighted rooted graph of k levels such that, for $j \in \{2, \ldots, k\}$

(1) each vertex at level j is adjacent to one vertex at level $j-1$ and all edges joining a vertex at level j with a vertex at level $j-1$ have the same weight, where the weight is a positive real number;
(2) if two vertices at level j are adjacent then they are adjacent to the same vertex at level $j-1$ and all edges joining two vertices at level j have the same weight;
(3) two vertices at level j have the same degree;
(4) there is not a vertex at level j adjacent to others two vertices at the same level.

We give a complete characterization of the eigenvalues of the Laplacian matrix and adjacency matrix of G. They are the eigenvalues of leading principal submatrices of two nonnegative symmetric tridiagonal matrices of order $k \times k$ and the roots of some polynomials related with the characteristic polynomial of the referred submatrices. By application of the above mentioned results, we derive an upper bound on the largest eigenvalue of a graph defined by a weighted tree and a weighted triangle attached, by one of its vertices, to a pendant vertex of the tree.

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1. Introduction

Let T be an unweighted tree of k levels (the level of a vertex is one more than its distance from the root vertex) with k a positive integer, such that at level j the vertices have degree d_{k-j+1}, and n_{k-j+1} is the number of vertices at level j. In [4], Rojo and Soto characterize the eigenvalues of the adjacency and Laplacian matrix of T. In fact, they stated that the eigenvalues of these matrices are the eigenvalues of principal submatrices of two nonnegative symmetric tridiagonal matrices of order $k \times k$. The codiagonal entries for both matrices are $\sqrt{d_j - 1}$, $2 \leqslant j \leqslant k-1$ and $\sqrt{d_k}$ while the diagonal entries are zero for the adjacency matrix, and d_j, $1 \leqslant j \leqslant k$, in the case of the Laplacian matrix.

Later, Rojo [2] determines the eigenvalues of the adjacency and Laplacian matrix of T where, at level 1, they include two vertices, that is, $n_k = 2$. He also gives results concerning to their multiplicities. It was established that they are the eigenvalues of leading principal submatrices of nonnegative symmetric tridiagonal matrices of order $k \times k$, where the codiagonal entries for these matrices are $\sqrt{d_j - 1}, 2 \leqslant j \leqslant k-1$, while the diagonal entries are $0, \ldots, 0, \pm 1$, in the case of the adjacency matrix and $d_1, d_2, \ldots, d_{k-1}, d_k \pm 1$ in the case of the Laplacian matrix.

Recently, in [3], Rojo and Robbiano determine the eigenvalues of the adjacency and Laplacian matrices of a weighted rooted tree T of k levels such that vertices at the same level have the same degree and the edges joining the vertices at level j with the vertices at level $j+1$ have a weight equal to w_{k-j}, $j \in \{1, \ldots, k-1\}$. They are the eigenvalues of leading principal submatrices of nonnegative symmetric tridiagonal matrices of order $k \times k$. Moreover, they also gave some results concerning their multiplicities. The results in [4] are obtained for $w_{k-j} = 1$, $j \in \{1, \ldots, k-1\}$.

Bearing in mind these results we consider a weighted rooted graph G with n vertices, i.e, G is a weighted graph with a root vertex and k levels such that, for $j \in \{2, \ldots, k\}$,

1. each vertex at level j is adjacent to one vertex at level $j-1$ and all edges joining a vertex at level j with a vertex at level $j-1$ have the same weight, where the weight is a positive real number;
2. if two vertices at level j are adjacent then they are adjacent to the same vertex at level $j-1$ and all edges joining two vertices at level j have the same weight;
3. two vertices at level j have the same degree;
4. there is not a vertex at level j adjacent to others two vertices in the same level.

The techniques used in this paper are similar to those used in [3] and [4].

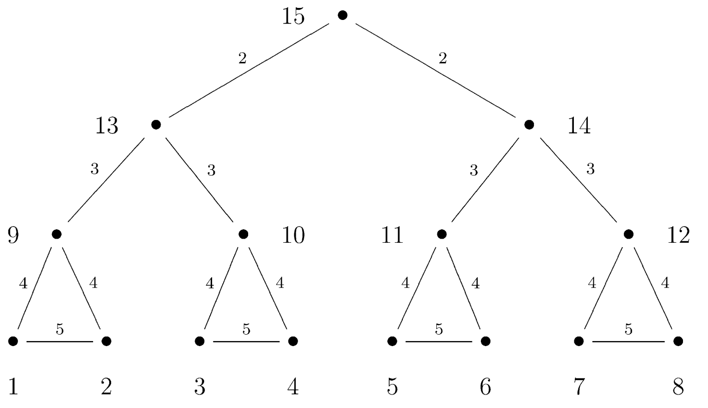
Let Φ_1 be the set of integers $j \in \{1, \ldots, k-1\}$ for which each vertex at the level $k-j+1$ is joined to other vertex at the same level. Observe that if $j \in \Phi_1$ then the number of vertices at the level $k-j+1$ is an even positive integer.

First we study the case $\Phi_1 = \{1\}$.

Using the labels $1, 2, \ldots, n$, in this order, our labeling for the vertices of G is: Label the vertices from the bottom to the root vertex and, in each level, from the left to the right.

Denote by $\{1, 2\}$, $\{3, 4\}$, and so on, the edges at level k. Let u_1 be the weight of these edges. Below we present a graph G in these conditions.

Example 1.1



Note that this graph has four levels with vertex degrees

$$d_1 = 2, \quad d_2 = 3, \quad d_3 = 3, \quad d_4 = 2,$$

the number of vertices in each level is

$$n_1 = 8, \quad n_2 = 4, \quad n_3 = 2, \quad n_4 = 1$$

and the edge weights are

$$u_1 = 5, \quad w_1 = 4, \quad w_2 = 3 \quad \text{and} \quad w_3 = 2.$$

2. Some preliminaries

Let n_{k-j+1} be the number of vertices at level j. Then, for $j \in \{2, 3, \ldots, k-1\}$,

$$n_{k-j} = (d_{k-j+1} - 1)n_{k-j+1},$$
$$n_{k-1} = d_k.$$

Let $m_j = \frac{n_j}{n_{j+1}}$, $j \in \{1, \ldots, k-1\}$ and e_m be the all ones column vector of dimension m.

For each $j \in \{1, 2, \ldots, k-1\}$, we have

$$C_j = \begin{bmatrix} e_{\frac{n_j}{n_{j+1}}} & & 0 \\ & \ddots & \\ 0 & & e_{\frac{n_j}{n_{j+1}}} \end{bmatrix},$$

with n_{j+1} diagonal blocks, where C_j is $n_j \times n_{j+1}$.

The edges joining the vertices at level j with the vertices at level $j+1$ have a weight equal to w_{k-j}. The edges $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ and so on have a weight equal to u_1.

Next we define the adjacency matrix and Laplacian matrix for this type of graphs G with n vertices. Let $e = \{i, j\}$ be an edge of G, we denote by $w(e)$ the weight of the edge e.

Laplacian matrix $L(G) = [l_{ij}]$ is an $n \times n$ matrix defined as

$$l_{ij} = \begin{cases} -w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j, \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ -\sum_{k \neq i} l_{ik} & \text{if } i = j \end{cases}$$

and the adjacency matrix is the $n \times n$ matrix $A(G) = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j, \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ 0 & \text{if } i = j. \end{cases}$$

Recall that $A(G)$ and $L(G)$ are real symmetric matrices.

Here, d_k is the degree of the vertex at level 1 (root vertex), $n_k = 1$ and $n_{j+1}|n_j$, for all $j \in \{1, \ldots, k-1\}$.

Our labeling for the vertices of G in Example 1.1 yields the block tridiagonal matrices for $L(G)$ and $A(G)$:

$$L(G) = \begin{bmatrix} L_1 \oplus L_2 \oplus L_3 \oplus L_4 & -4C_1 & 0 & 0 \\ -4C_1^{\mathrm{T}} & 11I_4 & -3C_2 & 0 \\ 0 & -3C_2^{\mathrm{T}} & 8I_2 & -2C_3 \\ 0 & 0 & -2C_3^{\mathrm{T}} & 4 \end{bmatrix},$$

where $L_1 = L_2 = L_3 = L_4 = \left[\begin{smallmatrix} 9 & -5 \\ -5 & 9 \end{smallmatrix}\right]$, and

$$A(G) = \begin{bmatrix} A_1 \oplus A_2 \oplus A_3 \oplus A_4 & 4C_1 & 0 & 0 \\ 4C_1^{\mathrm{T}} & 0 & 3C_2 & 0 \\ 0 & 3C_2^{\mathrm{T}} & 0 & 2C_3 \\ 0 & 0 & 2C_3^{\mathrm{T}} & 0 \end{bmatrix},$$

where $A_1 = A_2 = A_3 = A_4 = \left[\begin{smallmatrix} 0 & 5 \\ 5 & 0 \end{smallmatrix}\right]$.

Definition 2.1. Let

$$\delta_1 = u_1 + w_1;$$
$$\delta_j = (d_j - 1)w_{j-1} + w_j, \quad j \in \{2, 3, \ldots, k-1\};$$
$$\delta_k = d_k w_{k-1}.$$

Note that when $w_j = 1$, for all j and $u_1 = 1$, $\delta_j = d_j$ is the vertex degree at level $k - j + 1$.

From the previous definition we can observe that these numbers are the diagonal entries of Laplacian matrix given in Example 1.1. Note also that the upper codiagonal blocks in this matrix is $-w_1C_1$, $-w_2C_2$ and $-w_3C_3$. In general, if $\Phi_1 = \{1\}$, our labelling for the vertices yields the block triadiagonal matrices for $L(G)$ and $A(G)$:

$$L(G) = \begin{bmatrix} L_1 \oplus \cdots \oplus L_{\frac{n_1}{2}} & -w_1C_1 & & & & \\ -w_1C_1^{\mathrm{T}} & \delta_2 I_{n_2} & -w_2C_2 & & & \\ & -w_2C_2^{\mathrm{T}} & \delta_3 I_{n_3} & -w_3C_3 & & \\ & & -w_3C_3^{\mathrm{T}} & \ddots & \ddots & \\ & & & \ddots & \delta_{k-1} I_{n_{k-1}} & -w_{k-1}C_{k-1} \\ & & & & -w_{k-1}C_{k-1}^{\mathrm{T}} & \delta_k \end{bmatrix},$$

where $L_1 = \cdots = L_{\frac{n_1}{2}} = \begin{bmatrix} \delta_1 & -u_1 \\ -u_1 & \delta_1 \end{bmatrix}$ and

$$A(G) = \begin{bmatrix} A_1 \oplus \cdots \oplus A_{\frac{n_1}{2}} & w_1 C_1 & & & & \\ w_1 C_1^{\mathrm{T}} & 0 & w_2 C_2 & & & \\ & w_2 C_2^{\mathrm{T}} & 0 & w_3 C_3 & & \\ & & w_3 C_3^{\mathrm{T}} & \ddots & & \\ & & & & 0 & w_{k-1} C_{k-1} \\ & & & & w_{k-1} C_{k-1}^{\mathrm{T}} & 0 \end{bmatrix},$$

where $A_1 = \cdots = A_{\frac{n_1}{2}} = \begin{bmatrix} 0 & u_1 \\ u_1 & 0 \end{bmatrix}$.

Lemma 2.1 [5]. *The characteristic polynomials, $Q_j(\lambda)$, of the $j \times j$ leading principal submatrices of the $k \times k$ symmetric tridiagonal matrix*

$$Q_k = \begin{bmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{k-1} & b_{k-1} \\ & & & & b_{k-1} & a_k \end{bmatrix}, \tag{2.1}$$

satisfy the three-term recursion formula

$$\begin{aligned} &Q_0(\lambda) = 1, \\ &Q_1(\lambda) = \lambda - a_1, \\ &Q_j(\lambda) = (\lambda - a_j) Q_{j-1}(\lambda) - b_{j-1}^2 Q_{j-2}(\lambda) \end{aligned} \tag{2.2}$$

Lemma 2.2 [1]. *Let A be an $m \times m$ symmetric tridiagonal matrix with nonzero codiagonal entries, where m is a positive integer. Then the eigenvalues of any $(m-1) \times (m-1)$ principal submatrix strictly interlace the eigenvalues of A. In particular, the eigenvalues of A are simple.*

3. The spectrum of $L(G)$ when $\Phi_1 = \{1\}$

Lemma 3.1. *Let M be the block tridiagonal matrix*

$$M = \begin{bmatrix} M_1 \oplus \cdots \oplus M_{\frac{n_1}{2}} & w_1 C_1 & & & & \\ w_1 C_1^{\mathrm{T}} & \alpha_2 I_{n_2} & w_2 C_2 & & & \\ & w_2 C_2^{\mathrm{T}} & \alpha_3 I_{n_3} & w_3 C_3 & & \\ & & w_3 C_3^{\mathrm{T}} & \ddots & & \\ & & & & \alpha_{k-1} I_{n_{k-1}} & w_{k-1} C_{k-1} \\ & & & & w_{k-1} C_{k-1}^{\mathrm{T}} & \alpha_k \end{bmatrix},$$

where $M_1 = \cdots = M_{\frac{n_1}{2}} = \begin{bmatrix} \alpha_1 & u_1 \\ u_1 & \alpha_1 \end{bmatrix}$. Let

$$\begin{aligned}
\beta_1 &= \alpha_1, \\
\beta_2 &= \alpha_2 - \frac{n_1}{n_2}\frac{w_1^2}{(u_1+\alpha_1)}, \quad \alpha_1 \neq -u_1, \\
\beta_j &= \alpha_j - w_{j-1}^2 \frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}}, \quad j \in \{3,\ldots,k\},\ \beta_{j-1} \neq 0.
\end{aligned}$$

If $\beta_1 \neq -u_1, \beta_j \neq 0$, *for* $j \in \{2,\ldots,k-1\}$, *then*

$$\det(M) = ((\beta_1 - u_1)(\beta_1 + u_1))^{\frac{n_1}{2}} \beta_2^{n_2} \cdots \beta_{k-1}^{n_{k-1}} \beta_k.$$

Proof. Suppose that $\beta_1 \neq -u_1$, $\beta_j \neq 0$ for all $j \in \{2,\ldots,k-1\}$. Performing elementary operations without row interchanges to M we obtain the block upper triangular matrix

$$M' = \begin{bmatrix}
M_1 \oplus \cdots \oplus M_{\frac{n_1}{2}} & w_1 C_1 & & & & \\
0 & \beta_2 I_{n_2} & w_2 C_2 & & & \\
 & 0 & \beta_3 I_{n_3} & w_3 C_3 & & \\
 & & 0 & \ddots & & \\
 & & & & & w_{k-1} C_{k-1} \\
0 & & & & 0 & \beta_k
\end{bmatrix}.$$

Then the conclusions follow easily. $\square$

Definition 3.1. Let $P_0(\lambda) = 1$ and

$$\begin{aligned}
P_1(\lambda) &= \lambda - w_1, \\
P_j(\lambda) &= (\lambda - \delta_j) P_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} P_{j-2}(\lambda), \quad j \in \{2,\ldots,k\}.
\end{aligned} \tag{3.1}$$

Definition 3.2. For $j \in \{1,\ldots,k-1\}$, let T_j be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$T_k = \begin{bmatrix}
w_1 & (\sqrt{d_2}-1)w_1 & & & \\
(\sqrt{d_2}-1)w_1 & \delta_2 & & & \\
 & & \ddots & (\sqrt{d_{k-1}}-1)w_{k-2} & \\
 & & (\sqrt{d_{k-1}}-1)w_{k-2} & \delta_{k-1} & (\sqrt{d_k})w_{k-1} \\
 & & & (\sqrt{d_k})w_{k-1} & \delta_k
\end{bmatrix}.$$

Lemma 3.2. *If* T_j, $j \in \{1,\ldots,k-1\}$, *is the leading principal submatrix referred in Definition 3.2, we have*

$$\det(\lambda I - T_j) = P_j(\lambda), \quad j \in \{1,\ldots,k\}.$$

Proof. For $Q_k = T_k$ we have, by (2.1),

$$\begin{aligned}
a_1 &= w_1; \\
a_j &= \delta_j, \quad j \in \{2,\ldots,k\}; \\
b_j &= (\sqrt{d_{j+1}} - 1) w_j, \quad j \in \{1,\ldots,k-2\}; \\
b_{k-1} &= (\sqrt{d_k}) w_{k-1}.
\end{aligned}$$

Because $n_{k-j} = (d_{k-j+1} - 1)n_{k-j+1}$, $j \in \{2, \ldots, k-1\}$ and $n_{k-1} = d_k$, we have

$$\left(\sqrt{d_{j+1} - 1}\right) w_j = \left(\sqrt{\frac{n_j}{n_{j+1}}}\right) w_j, \quad j \in \{1, \ldots, k-2\}$$

and

$$\left(\sqrt{d_k}\right) w_{k-1} = \left(\sqrt{n_{k-1}}\right) w_{k-1} = \sqrt{\frac{n_{k-1}}{n_k}} w_{k-1}.$$

So, (2.2) gives the polynomials $P_j(\lambda)$, $j \in \{0, \ldots k\}$. □

Corollary 3.3. *For $j \in \{1, \ldots, k\}$ the zeros of the polynomial $P_j(\lambda)$ are real and simple.*

Proof. Let λ_0 such that $P_j(\lambda_0) = 0$.From Lemma 3.2, λ_0 is an eigenvalue of the matrix T_j. Since T_j is a real symmetric matrix, $\lambda_0 \in \mathbb{R}$. Moreover, from Lemma 2.2, λ_0 is a simple eigenvalue of T_j. Thus, λ_0 is a simple zero of $P_j(\lambda)$. □

Let $\Phi = \{1, \ldots, k-1\}$ and $\Omega = \{j \in \Phi : n_j > n_{j+1}, j \neq 1\} \cup \{j \in \Phi : n_j > 2n_{j+1}, j = 1\}$. The proof of the next theorem is a slight variation of the proof of [3, Theorem 5].

Theorem 3.4. *Let $P_0(\lambda) = 1$ and*

$$P_1(\lambda) = \lambda - w_1,$$
$$P_j(\lambda) = (\lambda - \delta_j)P_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} P_{j-2}(\lambda), \quad j \in \{2, \ldots, k\}.$$

Then

(a) $\det(\lambda I - L(G)) = (P_1(\lambda) - 2u_1)^{\frac{n_1}{2}} P_1(\lambda)^{\frac{n_1}{2} - n_2} \prod_{i \in \Omega \setminus \{1\}} P_i(\lambda)^{n_i - n_{i+1}} P_k(\lambda)$.
(b) $\sigma(L(G)) = (\cup_{j \in \Omega \cup \{k\}} \{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}) \cup \{\lambda \in \mathbb{R} : P_1(\lambda) = 2u_1\}$.

Proof. (a) We first consider $\lambda \in \mathbb{R}$ such that $P_j(\lambda) \neq 0$, for all $j \in \{1, \ldots, k-1\}$. Applying Lemma 3.1 to the matrix $\lambda I - L(G)$, we have

$$\beta_1 = (\lambda - w_1) - u_1 = P_1(\lambda) - u_1,$$
$$\beta_2 = (\lambda - \delta_2) - \frac{n_1}{n_2} \frac{w_1^2}{u_1 + \beta_1} = \frac{(\lambda - \delta_2)P_1(\lambda) - \frac{n_1}{n_2} w_1^2 P_0(\lambda)}{P_1(\lambda)} = \frac{P_2(\lambda)}{P_1(\lambda)}.$$

For $j \in \{3, \ldots, k\}$,

$$\beta_j = (\lambda - \delta_j) - \frac{n_{j-1}}{n_j} \frac{w_{j-1}^2}{\beta_{j-1}} = \frac{(\lambda - \delta_j)P_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} P_{j-2}(\lambda)}{P_{j-1}(\lambda)} = \frac{P_j(\lambda)}{P_{j-1}(\lambda)}.$$

Therefore,

$$\det(\lambda I - L(G)) = (P_1(\lambda) - 2u_1)^{\frac{n_1}{2}} P_1(\lambda)^{\frac{n_1}{2} - n_2} \prod_{j \in \Omega \setminus \{1\}} P_i(\lambda)^{n_i - n_{i+1}} P_k(\lambda)$$

and the result follows.

Consider now $\lambda_0 \in \mathbb{R}$ such that $P_s(\lambda_0) = 0$ for some $s \in \{1, \ldots, k-1\}$.

Since the zeros of any nonzero polynomial are isolated, which is the case for the polynomials P_j, there exists a neighborhood $N(\lambda_0)$ of λ_0 such that

$$P_j(\lambda) \neq 0 \quad \text{for all } \lambda \epsilon N(\lambda_0) - \{\lambda_0\} \text{ and for all } j \in \{1, \ldots, k-1\}.$$

Hence, the equality follows, for all $\lambda \epsilon N(\lambda_0) - \{\lambda_0\}$. By continuity, taking the limit as λ tends to λ_0 we obtain

$$\det(\lambda_0 I - L(G)) = (P_1(\lambda_0) - 2u_1)^{\frac{n_1}{2}} P_1(\lambda_0)^{\frac{n_1}{2} - n_2} \prod_{j \in \Omega \setminus \{1\}} P_i(\lambda_0)^{n_i - n_{i+1}} P_k(\lambda_0).$$

(b) It is an immediate consequence of part (a). □

The next theorem gives a complete characterization of the eigenvalues of $L(G)$ and some results about their multiplicities.

Theorem 3.5

(a) $\sigma(L(G)) = (\cup_{j \in \Omega \cup \{k\}} \sigma(T_j)) \cup \{2u_1 + w_1\}$.
(b) *The multiplicity of each eigenvalue of the matrix T_j, as an eigenvalue of $L(G)$ is at least $n_j - n_{j+1}$ for $j \in \Omega \setminus \{1\}$, $\frac{n_1 - 2n_2}{2}$ for $j = 1$ and 1 for $j = k$.*
(c) *The eigenvalue $\lambda = 2u_1 + w_1$, of $L(G)$, has multiplicity at least $\frac{n_1}{2}$.*

Proof. (a), (b) and (c) are consequences of Theorem 3.4, Lemma 3.2 and Corollary 3.3. □

Example 3.1. Let G be the graph presented in Example 1.1. For this graph,

$$T_4 = \begin{bmatrix} 4 & 4\sqrt{2} & 0 & 0 \\ 4\sqrt{2} & 11 & 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} & 8 & 2\sqrt{2} \\ 0 & 0 & 2\sqrt{2} & 4 \end{bmatrix}.$$

The eigenvalues of $L(G)$ are the eigenvalues of T_2, T_3, T_4 and $14 = 2u_1 + w_1$. To four decimal places these eigenvalues are

T_2 :	0.8479;	14.1521			each one with multiplicity 2
T_3 :	0.2202;	6.8351;	15.9447		each one with multiplicity 1
T_4 :	0;	2.7347;	8.1881;	16.0772	each one with multiplicity 1
	14				with multiplicity 4.

Using Theorem 3.5 and Lemma 2.2 it is easy to prove that

Theorem 3.6. *The spectral radius of $L(G)$ (the largest eigenvalue of $L(G)$) is the*

$$\max\{\text{the spectral radius of } T_k, 2u_1 + w_1\}.$$

4. The spectrum of $A(G)$ when $\Phi_1 = \{1\}$

The proofs of the following lemmas and theorems are similar to the proofs of Section 3.

Lemma 4.1. *Let N be the block tridiagonal matrix*

$$N=\begin{bmatrix} N_1\oplus\cdots\oplus N_{\frac{n_1}{2}} & -w_1C_1 & & & & \\ -w_1C_1^{\mathrm{T}} & \alpha_2 I_{n_2} & -w_2C_2 & & & \\ & -w_2C_2^{\mathrm{T}} & \alpha_3 I_{n_3} & -w_3C_3 & & \\ & & -w_3C_3^{\mathrm{T}} & \ddots & & \\ & & & & \alpha_{k-1}I_{n_{k-1}} & -w_{k-1}C_{k-1} \\ & & & & -w_{k-1}C_{k-1}^{\mathrm{T}} & \alpha_k \end{bmatrix},$$

where $N_1=\cdots=N_{\frac{n_1}{2}}=\begin{bmatrix}\alpha_1 & -u_1\\ -u_1 & \alpha_1\end{bmatrix}$.

Let

$$\beta_1=\alpha_1;$$
$$\beta_2=\alpha_2-\frac{n_1}{n_2}\frac{w_1^2}{(-u_1+\alpha_1)},\quad \alpha_1\neq u_1;$$
$$\beta_j=\alpha_j-w_{j-1}^2\frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}},\quad j\in\{3,\ldots,k\},\ \beta_{j-1}\neq 0.$$

If $\beta_1\neq u_1,\beta_j\neq 0$, for $j\in\{2,\ldots,k-1\}$ then

$$\det(N)=((\beta_1-u_1)(\beta_1+u_1))^{\frac{n_1}{2}}\beta_2^{n_2}\cdots\beta_{k-1}^{n_{k-1}}\beta_k.$$

Definition 4.1. Let $S_0(\lambda)=1$ and

$$S_1(\lambda)=\lambda-u_1,$$
$$S_j(\lambda)=\lambda S_{j-1}(\lambda)-w_{j-1}^2\frac{n_{j-1}}{n_j}S_{j-2}(\lambda),\quad j\in\{2,\ldots,k\}. \tag{4.1}$$

Definition 4.2. For $j\in\{1,\ldots,k-1\}$, let R_j be the $j\times j$ leading principal submatrix of the $k\times k$ symmetric tridiagonal matrix

$$R_k=\begin{bmatrix} u_1 & (\sqrt{d_2-1})w_1 & & & \\ (\sqrt{d_2-1})w_1 & 0 & & & \\ & & \ddots & (\sqrt{d_{k-1}-1})w_{k-2} & \\ & & (\sqrt{d_{k-1}-1})w_{k-2} & 0 & (\sqrt{d_k})w_{k-1} \\ & & & (\sqrt{d_k})w_{k-1} & 0 \end{bmatrix}.$$

Lemma 4.2. *If R_j, $j\in\{1,\ldots,k-1\}$, is the leading principal submatrix referred in Definition 4.2, we have*

$$\det(\lambda I-R_j)=S_j(\lambda),\quad j\in\{1,\ldots,k\}.$$

Corollary 4.3. *For $j\in\{1,\ldots,k\}$ the zeros of the polynomial $S_j(\lambda)$ are real and simple.*

Let $\Phi=\{1,\ldots,k-1\}$ and $\Omega=\{j\in\Phi: n_j>n_{j+1}, j\neq 1\}\cup\{j\in\Phi: n_j>2n_{j+1}, j=1\}$.

Theorem 4.4. *Let $S_0(\lambda)=1$ and*

$$S_1(\lambda)=\lambda-u_1,$$
$$S_j(\lambda)=\lambda S_{j-1}(\lambda)-w_{j-1}^2\frac{n_{j-1}}{n_j}S_{j-2}(\lambda),\quad j\in\{2,\ldots,k\}.$$

Then

(a) $\det(\lambda I - A(G)) = (S_1(\lambda) + 2u_1)^{\frac{n_1}{2}} S_1(\lambda)^{\frac{n_1}{2} - n_2} \prod_{i \in \Omega \setminus \{1\}} S_i(\lambda)^{n_i - n_{i+1}} S_k(\lambda)$.
(b) $\sigma(A(G)) = (\cup_{j \in \Omega \cup \{k\}} \{\lambda \in \mathbb{R} : S_j(\lambda) = 0\}) \cup \{\lambda \in \mathbb{R} : S_1(\lambda) = -2u_1\}$.

The next theorem gives a complete characterization of the eigenvalues of $A(G)$ and some results about their multiplicities.

Theorem 4.5. (a) $\sigma(A(G)) = (\cup_{j \in \Omega \cup \{k\}} \sigma(R_j)) \cup \{-u_1\}$.
(b) *The multiplicity of each eigenvalue of the matrix T_j, as an eigenvalue of $A(G)$ is at least $n_j - n_{j+1}$ for $j \in \Omega \setminus \{1\}$, $\frac{n_1 - 2n_2}{2}$ for $j = 1$ and 1 for $j = k$.*
(c) *The eigenvalue $\lambda = -u_1$, of $A(G)$, has multiplicity at least $\frac{n_1}{2}$.*

Proof. (a), (b) and (c) are consequences of Theorem 4.4, Lemma 4.2 and Corollary 4.3. □

Example 4.1. Let G be the graph presented in Example 1.1. For this graph,

$$R_4 = \begin{bmatrix} 5 & 4\sqrt{2} & 0 & 0 \\ 4\sqrt{2} & 0 & 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} & 0 & 2\sqrt{2} \\ 0 & 0 & 2\sqrt{2} & 0 \end{bmatrix}.$$

The eigenvalues of $A(G)$ are the eigenvalues of R_2, R_3, R_4 and -5. To four decimal places these eigenvalues are

R_2 :	-3.6847;	8.6847			each one with multiplicity 2
R_3 :	-5.9486;	1.6222;	9.3264		each one with multiplicity 1
R_4 :	-6.3527;	-1.3114;	3.2716;	9.3926	each one with multiplicity 1
	-5				with multiplicity 4.

Using Theorem 4.5 and Lemma 2.2 it is easy to prove that

Theorem 4.6. *The spectral radius of $A(G)$ (the largest eigenvalue of $A(G)$) is the spectral radius of R_k.*

5. Bounding the largest eigenvalue of some weighted graphs

Let G be a weighted graph. We denote by $\mu(G)$ and $\lambda(G)$ the largest eigenvalue of $L(G)$ and $A(G)$, respectively.

In [3] the following lemmas are proved.

Lemma 5.1. *Let $G = (V, E)$ be a weighted graph. Let w_e be the weight of $e \in E$. Let $\widetilde{G}$ be the weighted graph obtained from G replacing the weight w_e by $\tilde{w}_e$. Then*

$$\lambda(G) \leqslant \lambda(\widetilde{G}) \quad \text{if } w_e \leqslant \tilde{w}_e$$

and

$$\mu(G) \leqslant \mu(\widetilde{G}) \quad \text{if } w_e \leqslant \tilde{w}_e.$$

Lemma 5.2. *Let $G = (V, E)$ be a weighted graph with n vertices. Let $v \in V$. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be the graph with $(n + 1)$ vertices obtained by adding to G a vertex u and an edge $\{u, v\}$ with weight w. Then*

$$\lambda(G) \leqslant \lambda(\widetilde{G})$$

and

$$\mu(G) \leqslant \mu(\widetilde{G}).$$

Now, let $G = (V, E)$ be a graph defined by a weighted tree and a weighted triangle attached, by one of its vertices, to a pendant vertex of the tree.

Let us denote by e_x the excentricity of x (largest distance from x to any other vertex) and by $\operatorname{diam}(G)$ the diameter of G ($\operatorname{diam}(G) = \max\{d(v, z) : z, v \in V\}$ and $d(v, z)$ is the distance from v to z, i.e., the length of the shortest path from v to z).

Suppose, w.l.g., that x_1, x_2, x_3 are the vertices of the triangle and x_1, x_2 have degree two. Let $u \in V$ such that

$$d(x_1, u) = \left\lceil \frac{e_{x_1}}{2} \right\rceil,$$

where $\lceil a \rceil$ is the smallest integer greater than or equal to a.

Let $k = \left\lceil \frac{e_{x_1}}{2} \right\rceil + 1$. For $j = \{1, 2, \ldots, k\}$ let

$$\Delta_{k-j+1} = \max\{d_v : d(v, u) = j - 1\},$$

where d_v is the degree of v. For $j = \{1, 2, \ldots, k - 1\}$ let

$$W_{k-j} = \max\{w_{z,y} : d(z, u) = j - 1,\ d(y, u) = j\},$$

where $w_{z,y}$ is the weight of the edge joining z to y. Let U_1 be the weight of the edge $\{x_1, x_2\}$. Define

$$\begin{aligned}
\delta_1 &= U_1 + W_1; \\
\delta_j &= (\Delta_j - 1)W_{j-1} + W_j, \quad j \in \{2, \ldots, k - 1\}; \\
\delta_k &= \Delta_k W_{k-1}.
\end{aligned}$$

In these conditions, we can prove the following theorem.

Theorem 5.3. *We have*

$$\mu(G) \leqslant \max\left\{ \max_{2 \leqslant j \leqslant k-2} \left\{ \sqrt{\Delta_j - 1}W_{j-1} + \delta_j + \sqrt{\Delta_{j+1} - 1}W_j \right\}, \right.$$
$$\left. \sqrt{\Delta_{k-1} - 1}W_{k-2} + \delta_{k-1} + \sqrt{\Delta_k}W_{k-1}, \sqrt{\Delta_k}W_{k-1} + \delta_k, 2U_1 + W_1 \right\}$$

and

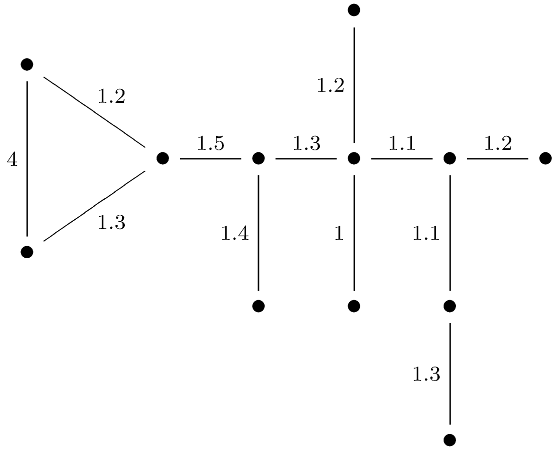
$$\lambda(G) \leqslant \max\left\{ \max_{2 \leqslant j \leqslant k-2} \left\{ \sqrt{\Delta_j - 1}W_{j-1} + \sqrt{\Delta_{j+1} - 1}W_j \right\} \right.$$
$$\left. \sqrt{\Delta_{k-1} - 1}W_{k-2} + \sqrt{\Delta_k}W_{k-1},\ U_1 + \sqrt{\Delta_2 - 1}W_1 \right\}.$$

Proof. Consider the weighted rooted graph with k levels, $\widetilde{G}$, such that

- the vertex u previously consider is the rooted vertex,
- the vertices at level j have degree Δ_{k-j+1}, for $j \in \{1, \ldots, k\}$,
- the edges joining the vertices at level j with the vertices at level $j+1$ have weight equal to W_{k-j}, for $j \in \{1, \ldots, k-1\}$,
- the edges joining two vertices at level k have weight equal to U_1.

Since G is an induced subgraph of $\widetilde{G}$, using the previous lemmas we have $\mu(G) \leqslant \mu(\widetilde{G})$ and $\lambda(G) \leqslant \lambda(\widetilde{G})$. Using the results of Sections 3 and 4, we have that $\mu(\widetilde{G}) = \max\{$the spectral radius of $T_k, 2U_1 + W_1\}$ and $\lambda(\widetilde{G})$ is the spectral radius of R_k. By Grešgorin theorem the result follows. □

Example 5.1. Consider the following graph G



Then, if x_1, x_2, x_3 are the vertices previously defined, $e_{x_1} = 6$ and $k = 4$.
Using the previous notation, we have

$$W_3 = \max\{1.3, 1.2, 1, 1.1\} = 1.3,$$

$$W_2 = \max\{1.5, 1.4, 1.1, 1.2\} = 1.5,$$

$$W_1 = \max\{1.2, 1.3\} = 1.3,$$

$$U_1 = 4,$$

$$\Delta_1 = 2, \quad \Delta_2 = 3, \quad \Delta_3 = 3, \Delta_4 = 4,$$

$$\delta_1 = 5.3, \quad \delta_2 = 2(1.3) + 1.5 = 4.1, \quad \delta_3 = 2(1.5) + 1.3 = 4.3, \quad \delta_4 = 4.(1.3) = 5.2.$$

Therefore

$$T_4 = \begin{bmatrix} 1.3 & 1.3\sqrt{2} & 0 & 0 \\ 1.3\sqrt{2} & 4.1 & 1.5\sqrt{2} & 0 \\ 0 & 1.5\sqrt{2} & 4.3 & 2.6 \\ 0 & 0 & 2.6 & 5.2 \end{bmatrix}.$$

Using Theorem 5.3,

$$\mu(G) \leqslant \max\{1.3\sqrt{2} + 4.1 + 1.5\sqrt{2},\ 1.5\sqrt{2} + 4.3 + 2.6, 2.6 + 5.2, 2(4) + 1.3\} = 9.3.$$

On the other hand the spectral radius of $L(G)$ is 9.2518. Clearly $\mu(G) < \mu(\widetilde{G}) = 9.3$.

For $\lambda(G)$, using Theorem 5.3, we have

$$\lambda(G) \leqslant \max\left\{1.3\sqrt{2} + 1.5\sqrt{2},\ 1.5\sqrt{2} + 2.6,\ 4 + 1.3\sqrt{2}\right\} = 5.8384.$$

On the other hand the spectral radius of $A(G)$ is 4.7489. Clearly $\lambda(G) < \lambda(\widetilde{G}) = 5.8384$.

6. The spectra of $L(G)$ and $A(G)$ in the general case

In this section we are going to generalize the results obtained in sections 3 and 4. The results established in [4] and [3] are corollaries of these results.

Let G be a weighted rooted graph with n vertices and k levels. Let ϕ_1 be the set of integers $j \in \{1, \ldots, k-1\}$ for which each vertex at level $k - j + 1$ is joined to other vertex at the same level and let $\phi_2 = \{1, \ldots, k-1\} \setminus \phi_1$. (If $\phi_1 = \emptyset$ then G is a weighted rooted tree).

Let n_{k-j+1} be the number of vertices at level j, w_{k-j} be the weight of the edges joining the vertices at level j with the vertices at level $j+1$ and for $j \in \phi_1$ let u_j be the weight of the edges at level $k - j + 1$. Then

$$n_{j-1} = \begin{cases} (d_j - 2)n_j & \text{if } j \in \phi_1 \setminus \{1\}, \\ (d_j - 1)n_j & \text{if } j \in \phi_2 \setminus \{1\}. \end{cases}$$
$$n_{k-1} = d_k.$$

Let $m_j = \frac{n_j}{n_{j+1}}$, $j \in \{1, \ldots, k-1\}$.

Definition 6.1. Let

$$\delta_1 = \begin{cases} w_1 & \text{if } 1 \in \phi_2, \\ u_1 + w_1 & \text{if } 1 \in \phi_1, \end{cases}$$
$$\delta_j = \begin{cases} (d_j - 1)w_{j-1} + w_j & \text{if } j \in \phi_2 \setminus \{1\}, \\ (d_j - 2)w_{j-1} + w_j + u_j & \text{if } j \in \phi_1 \setminus \{1\}. \end{cases}$$
$$\delta_k = d_k w_{k-1}.$$

Let

$$V_{n_j} = \begin{cases} I_{n_j} & \text{if } j \in \phi_2, \\ L_1 \oplus \ldots \oplus L_{\frac{n_j}{2}} & \text{if } j \in \phi_1, \end{cases}$$

where $L_1 = \cdots = L_{\frac{n_j}{2}} = \begin{bmatrix} 1 & -\frac{u_j}{\delta_j} \\ -\frac{u_j}{\delta_j} & 1 \end{bmatrix}$ and

$$U_{n_j} = \begin{cases} 0_{n_j} & \text{if } j \in \phi_2, \\ A_1 \oplus \ldots \oplus A_{\frac{n_j}{2}} & \text{if } j \in \phi_1, \end{cases}$$

where $A_1 = \cdots = A_{\frac{n_j}{2}} = \begin{bmatrix} 0 & u_j \\ u_j & 0 \end{bmatrix}$.

So, Laplacian matrix is

$$L(G)=\begin{bmatrix} \delta_1 V_{n_1} & -w_1C_1 & & & & \\ -w_1C_1^{\mathrm{T}} & \delta_2 V_{n_2} & -w_2C_2 & & & \\ & -w_2C_2^{\mathrm{T}} & \delta_3 V_{n_3} & -w_3C_3 & & \\ & & -w_3C_3^{\mathrm{T}} & \ddots & \ddots & \\ & & & \ddots & \delta_{k-1}V_{n_{k-1}} & -w_{k-1}C_{k-1} \\ & & & & -w_{k-1}C_{k-1}^{\mathrm{T}} & \delta_k \end{bmatrix}$$

and the adjacency matrix is

$$A(G)=\begin{bmatrix} U_{n_1} & w_1C_1 & & & & \\ w_1C_1^{\mathrm{T}} & U_{n_2} & w_2C_2 & & & \\ & w_2C_2^{\mathrm{T}} & U_{n_3} & w_3C_3 & & \\ & & w_3C_3^{\mathrm{T}} & \ddots & & \\ & & & & U_{n_{k-1}} & w_{k-1}C_{k-1} \\ & & & & w_{k-1}C_{k-1}^{\mathrm{T}} & 0 \end{bmatrix}.$$

Lemma 6.1. *Let M be the block tridiagonal matrix*

$$M=\begin{bmatrix} \alpha_1 U_{n_1}^* & w_1C_1 & & & & \\ w_1C_1^{\mathrm{T}} & \alpha_2 U_{n_2}^* & w_2C_2 & & & \\ & w_2C_2^{\mathrm{T}} & \alpha_3 U_{n_3}^* & w_3C_3 & & \\ & & w_3C_3^{\mathrm{T}} & \ddots & & \\ & & & & \alpha_{k-1}U_{n_{k-1}}^* & w_{k-1}C_{k-1} \\ & & & & w_{k-1}C_{k-1}^{\mathrm{T}} & \alpha_k \end{bmatrix},$$

where $U_{n_j}^* = I_{n_j} + \frac{1}{\alpha_j}U_{n_j}$, $j \in \phi_1$. Let

$$\beta_1=\alpha_1,$$
$$\beta_j=\begin{cases} \alpha_j - w_{j-1}^2 \frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}} & if\ \ j-1\in\phi_2,\ \beta_{j-1}\neq 0, \\ \alpha_j - w_{j-1}^2 \frac{n_{j-1}}{n_j}\frac{1}{u_{j-1}+\beta_{j-1}} & if\ \ j-1\in\phi_1,\ \beta_{j-1}\neq -u_{j-1}. \end{cases} \tag{6.1}$$

If $\beta_j \neq -u_j$ for $j\in\phi_1$ and $\beta_j\neq 0$ for $j\in\phi_2$ then

$$\det(M)=\prod_{j\in\phi_1}\big((\beta_j-u_j)(\beta_j+u_j)\big)^{\frac{n_j}{2}}\prod_{j\in\phi_2}\beta_j^{n_j}\beta_k.$$

Proof. Suppose that $\beta_j \neq -u_j$ for $j\in\phi_1$ and $\beta_j\neq 0$ for $j\in\phi_2$. Performing elementary operations without row interchanges to M we obtain the block upper triangular matrix

$$M'=\begin{bmatrix} \beta_1 U_{n_1}^* & w_1C_1 & & & & \\ 0 & \beta_2 U_{n_2}^* & w_2C_2 & & & \\ & 0 & \beta_3 U_{n_3}^* & w_3C_3 & & \\ & & 0 & \ddots & & \\ & & & \ddots & \beta_{k-1}U_{n_{k-1}}^* & w_{k-1}C_{k-1} \\ & & & & 0 & \beta_k \end{bmatrix}.$$

Then the conclusion follows easily. □

Definition 6.2. Let $P_0(\lambda) = 1$, $P_1(\lambda) = \lambda - w_1$ and

$$P_j(\lambda) = \begin{cases} (\lambda - \delta_j)P_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} P_{j-2}(\lambda) & \text{if } j \in \phi_2 \backslash \{1\} \cup \{k\}, \\ (\lambda - \delta_j + u_j)P_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} P_{j-2}(\lambda) & \text{if } j \in \phi_1 \backslash \{1\}. \end{cases}$$

Let

$$r_j = \begin{cases} \sqrt{d_j - 1} & \text{if } j \in \phi_2, \\ \sqrt{d_j - 2} & \text{if } j \in \phi_1 \end{cases}$$

and

$$s_j = \begin{cases} \delta_j & \text{if } j \in \phi_2, \\ \delta_j - u_j & \text{if } j \in \phi_1. \end{cases}$$

Definition 6.3. For $j \in \{1, \ldots, k-1\}$, let T_j be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$T_k = \begin{bmatrix} s_1 & r_2 w_1 & & & \\ r_2 w_1 & s_2 & & & \\ & & \ddots & r_{k-1} w_{k-2} & \\ & & r_{k-1} w_{k-2} & s_{k-1} & (\sqrt{d_k}) w_{k-1} \\ & & & (\sqrt{d_k}) w_{k-1} & \delta_k \end{bmatrix}.$$

Lemma 6.2. *If T_j, $j \in \{1, \ldots, k-1\}$, is the leading principal submatrix referred in Definition 6.3, we have*

$$\det(\lambda I - T_j) = P_j(\lambda), \quad j \in \{1, \ldots, k\}.$$

The proof of the previous lemma is similar to the proof of Lemma 3.2.

Corollary 6.3. *For $j \in \{1, \ldots, k\}$ the zeros of the polynomial $P_j(\lambda)$ referred in Definition 6.2 are real and simple.*

The proof of the previous corollary is similar to the proof of Corollary 3.3.

Let $\Omega = \{j \in \phi_2 : n_j > n_{j+1}\} \cup \{j \in \phi_1 : n_j > 2n_{j+1}\}$.

Theorem 6.4. *Let $P_0(\lambda) = 1$, $P_1(\lambda) = \lambda - w_1$ and*

$$P_j(\lambda) = \begin{cases} (\lambda - \delta_j)P_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} P_{j-2}(\lambda) & \text{if } j \in \phi_2 \backslash \{1\} \cup \{k\}, \\ (\lambda - \delta_j + u_j)P_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} P_{j-2}(\lambda) & \text{if } j \in \phi_1 \backslash \{1\}. \end{cases} \tag{6.2}$$

Then

(a) $\det(\lambda I - L(G)) = \prod_{j \in \phi_1} (P_j(\lambda) - 2u_j P_{j-1}(\lambda))^{\frac{n_j}{2}} P_j(\lambda)^{\frac{n_j}{2} - n_{j+1}} \prod_{j \in \phi_2} P_j(\lambda)^{n_j - n_{j+1}} P_k(\lambda)$.

(b) $\sigma(L(G)) = (\cup_{j \in \Omega \cup \{k\}} \{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}) \cup (\cup_{j \in \phi_1} \{\lambda \in \mathbb{R} : P_j(\lambda) = 2u_j P_{j-1}(\lambda)\})$.

Proof. We first consider $\lambda \in \mathbb{R}$ such that $P_j(\lambda) \neq 0$, for all $j \in \{1, \ldots, k-1\}$. Applying Lemma 6.1 to the matrix $\lambda I - L(G)$, we have

$$\beta_1 = \lambda - \delta_1 = P_1(\lambda) - u_1 \quad \text{if } 1 \in \phi_1$$

and

$$\beta_1 = \lambda - \delta_1 = P_1(\lambda) \quad \text{if } 1 \in \phi_2.$$

This gives

$$\beta_1 = \frac{P_1(\lambda) - u_1 P_0(\lambda)}{P_0(\lambda)} \quad \text{if } 1 \in \phi_1$$

and

$$\beta_1 = \frac{P_1(\lambda)}{P_0(\lambda)} \quad \text{if } 1 \in \phi_2.$$

Suppose now that $j \geqslant 2$ and

$$\beta_{j-1} = \frac{P_{j-1}(\lambda)}{P_{j-2}(\lambda)} - u_{j-1} \quad \text{if } j-1 \in \phi_1 \tag{6.3}$$

and

$$\beta_{j-1} = \frac{P_{j-1}(\lambda)}{P_{j-2}(\lambda)} \quad \text{if } j-1 \in \phi_2. \tag{6.4}$$

If $j-1 \in \phi_2$, by (6.1)

$$\beta_j = (\lambda - \delta_j) - \frac{n_{j-1}}{n_j}\frac{w_{j-1}^2}{\beta_{j-1}}.$$

By (6.4), we have

$$\beta_j = (\lambda - \delta_j) - \frac{n_{j-1}}{n_j} w_{j-1}^2 \frac{P_{j-2}(\lambda)}{P_{j-1}(\lambda)},$$

that is,

$$\beta_j = \frac{(\lambda - \delta_j)P_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} w_{j-1}^2 P_{j-2}(\lambda)}{P_{j-1}(\lambda)}.$$

If $j \in \phi_1$, using (6.2), we have

$$\beta_j = \frac{(\lambda - \delta_j)P_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} w_{j-1}^2 P_{j-2}(\lambda)}{P_{j-1}(\lambda)} = \frac{P_j(\lambda) - u_j P_{j-1}(\lambda)}{P_{j-1}(\lambda)}$$

and if $j \in \phi_2 \cup \{k\}$, using (6.2), we have

$$\beta_j = \frac{P_j(\lambda)}{P_{j-1}(\lambda)}.$$

Suppose now that $j-1 \in \phi_1$. By (6.1) we have,

$$\beta_j = (\lambda - \delta_j) - w_{j-1}^2 \frac{n_{j-1}}{n_j} \frac{1}{u_{j-1} + \beta_{j-1}}.$$

By (6.3),

$$\beta_j = (\lambda - \delta_j) - w_{j-1}^2 \frac{n_{j-1}}{n_j} \frac{1}{u_{j-1} + \left(\frac{P_{j-1}(\lambda)}{P_{j-2}(\lambda)} - u_{j-1}\right)}.$$

Therefore

$$\beta_j = \frac{(\lambda - \delta_j)P_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} P_{j-2}(\lambda)}{P_{j-1}(\lambda)}.$$

Again, if $j \in \phi_1$, using (6.2), we have

$$\beta_j = \frac{(\lambda - \delta_j)P_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} w_{j-1}^2 P_{j-2}(\lambda)}{P_{j-1}(\lambda)} = \frac{P_j(\lambda) - u_j P_{j-1}(\lambda)}{P_{j-1}(\lambda)}$$

and if $j \in \phi_2 \cup \{k\}$, using (6.2), we have

$$\beta_j = \frac{P_j(\lambda)}{P_{j-1}(\lambda)}.$$

Therefore,

$$\det(\lambda I - L(G)) = \prod_{j \in \phi_1} (P_j(\lambda) - 2u_j P_{j-1}(\lambda))^{\frac{n_j}{2}} P_j(\lambda)^{\frac{n_j}{2}} P_{j-1}(\lambda)^{-n_j} \times \prod_{j \in \phi_2 \cup \{k\}} P_j(\lambda)^{n_j} P_{j-1}(\lambda)^{-n_j},$$

$$\det(\lambda I - L(G)) = \prod_{j \in \phi_1} (P_j(\lambda) - 2u_j P_{j-1}(\lambda))^{\frac{n_j}{2}} P_j(\lambda)^{\frac{n_j}{2}} \times \prod_{j \in \phi_2 \cup \{k\}} P_j(\lambda)^{n_j} \prod_{j \in \phi_1 \cup \phi_2} P_j(\lambda)^{-n_{j+1}},$$

$$\det(\lambda I - L(G)) = \prod_{j \in \phi_1} (P_j(\lambda) - 2u_j P_{j-1}(\lambda))^{\frac{n_j}{2}} P_j(\lambda)^{\frac{n_j}{2} - n_{j+1}} \prod_{j \in \phi_2} P_j(\lambda)^{n_j - n_{j+1}} P_k(\lambda).$$

and the result follows.

Consider now $\lambda_0 \in \mathbb{R}$ such that $P_s(\lambda_0) = 0$ for some $s \in \{1, \ldots, k-1\}$.

Since the zeros of any nonzero polynomial are isolated, which is the case for the polynomials $P_j(\lambda)$, there exists a neighborhood $N(\lambda_0)$ of λ_0 such that

$$P_j(\lambda) \neq 0 \quad \text{for all } \lambda \epsilon N(\lambda_0) - \{\lambda_0\} \text{ and for all } j \in \{1, \ldots, k-1\}.$$

Hence, the equality follows, for all $\lambda \epsilon N(\lambda_0) - \{\lambda_0\}$. By continuity, taking the limit as λ tends to λ_0 we obtain

$$\det(\lambda_0 I - L(G)) = \prod_{j \in \phi_1} (P_j(\lambda_0) - 2u_j P_{j-1}(\lambda_0))^{\frac{n_j}{2}} P_j(\lambda_0)^{\frac{n_j}{2} - n_{j+1}} \times \prod_{j \in \phi_2} P_j(\lambda_0)^{n_j - n_{j+1}} P_k(\lambda_0).$$

(b) It is an immediate consequence of part (a). □

The next theorem gives a complete characterization of the eigenvalues of Laplacian matrix. In fact, they are the eigenvalues of leading principal submatrices of T_k and the roots of some polynomials related with these submatrices.

Theorem 6.5

(a) $\sigma(L(G)) = (\cup_{j\in\Omega\cup\{k\}}\sigma(T_j)) \cup (\cup_{j\in\phi_1}\{\lambda \in \mathbb{R} : \det(\lambda I - T_j) = 2u_j \det(\lambda I - T_{j-1})\})$.
(b) *The multiplicity of each eigenvalue of the matrix T_j, as an eigenvalue of $L(G)$ is at least $n_j - n_{j+1}$ for $j \in \phi_2$, $\frac{n_j-2n_{j+1}}{2}$ for $j \in \phi_1$ and 1 for $j = k$.*
(c) *For $j \in \phi_1$, each root of the polynomial*

$$\det(\lambda I - T_j) = 2u_j \det(\lambda I - T_{j-1})$$

is an eigenvalue of $L(G)$ with multiplicity at least $\frac{n_j}{2}$.

Proof. (a), (b) and (c) are consequences of Theorem 6.4, Lemma 6.2 and Corollary 6.3. □

For $A(G)$ we have similar results.

Lemma 6.6. *Let N be the block tridiagonal matrix*

$$N = \begin{bmatrix} \alpha_1 V_{n_1} & -w_1 C_1 & & & & \\ -w_1 C_1^{\mathrm{T}} & \alpha_2 V_{n_2} & -w_2 C_2 & & & \\ & -w_2 C_2^{\mathrm{T}} & \alpha_3 V_{n_3} & -w_3 C_3 & & \\ & & -w_3 C_3^{\mathrm{T}} & \ddots & & \\ & & & & \alpha_{k-1} V_{n_{k-1}} & -w_{k-1} C_{k-1} \\ & & & & -w_{k-1} C_{k-1}^{\mathrm{T}} & \alpha_k \end{bmatrix}.$$

Let

$$\beta_1 = \alpha_1,$$
$$\beta_j = \begin{cases} \alpha_j - w_{j-1}^2 \frac{n_{j-1}}{n_j} \frac{1}{\beta_{j-1}} & if\ j-1 \in \phi_2, \beta_{j-1} \neq 0, \\ \alpha_j - w_{j-1}^2 \frac{n_{j-1}}{n_j} \frac{1}{-u_{j-1}+\beta_{j-1}} & if\ j-1 \in \phi_1, \beta_{j-1} \neq u_{j-1}. \end{cases} \tag{6.5}$$

If $\beta_j \neq u_j$ for $j \in \phi_1$ and $\beta_j \neq 0$ for $j \in \phi_2$ then

$$\det(N) = \prod_{j\in\phi_1} \left((\beta_j - u_j)(\beta_j + u_j)\right)^{\frac{n_j}{2}} \prod_{j\in\phi_2} \beta_j^{n_j} \beta_k.$$

Proof. Suppose that $\beta_j \neq u_j$ for $j \in \phi_1$ and $\beta_j \neq 0$ for $j \in \phi_2$. Performing elementary operations without row interchanges to N we obtain the block upper triangular matrix

$$N' = \begin{bmatrix} \beta_1 V_{n_1} & -w_1 C_1 & & & & \\ 0 & \beta_2 V_{n_2} & -w_2 C_2 & & & \\ & 0 & \beta_3 V_{n_3} & -w_3 C_3 & & \\ & & 0 & \ddots & & \\ & & & \ddots & \beta_{k-1} V_{n_{k-1}} & -w_{k-1} C_{k-1} \\ & & & & 0 & \beta_k \end{bmatrix}.$$

Then the conclusion follows easily. □

Definition 6.4. Let $S_0(\lambda) = 1$, $S_1(\lambda) = \lambda - u_1$ and

$$S_j(\lambda) = \begin{cases} \lambda S_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} S_{j-2}(\lambda) & \text{if } j \in \phi_2 \backslash \{1\} \cup \{k\}, \\ (\lambda - u_j) S_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} S_{j-2}(\lambda) & \text{if } j \in \phi_1 \backslash \{1\}. \end{cases}$$

Let

$$r_j = \begin{cases} \sqrt{d_j - 1} & \text{if } j \in \phi_2, \\ \sqrt{d_j - 2} & \text{if } j \in \phi_1 \end{cases}$$

and

$$s_j^* = \begin{cases} 0 & \text{if } j \in \phi_2, \\ u_j & \text{if } j \in \phi_1. \end{cases}$$

Definition 6.5. For $j \in \{1, \ldots, k-1\}$, let R_j be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$R_k = \begin{bmatrix} s_1^* & r_2 w_1 & & & \\ r_2 w_1 & s_2^* & & & \\ & & \ddots & r_{k-1} w_{k-2} & \\ & & r_{k-1} w_{k-2} & s_{k-1}^* & (\sqrt{d_k}) w_{k-1} \\ & & & (\sqrt{d_k}) w_{k-1} & 0 \end{bmatrix}.$$

Lemma 6.7. *If* R_j, $j \in \{1, \ldots, k-1\}$, *is the leading principal submatrix referred in Definition* 6.5, *we have*

$$\det(\lambda I - R_j) = S_j(\lambda), \quad j \in \{1, \ldots, k\}.$$

Corollary 6.8. *For* $j \in \{1, \ldots, k\}$ *the zeros of the polynomial* $S_j(\lambda)$ *referred in Definition* 6.4 *are real and simple*.

Let $\Omega = \{j \in \phi_2 : n_j > n_{j+1}\} \cup \{j \in \phi_1 : n_j > 2n_{j+1}\}$.

Theorem 6.9. *Let* $S_0(\lambda) = 1$, $S_1(\lambda) = \lambda - u_1$ *and*

$$S_j(\lambda) = \begin{cases} \lambda S_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} S_{j-2}(\lambda) & \text{if } j \in \phi_2 \backslash \{1\} \cup \{k\}, \\ (\lambda - u_j) S_{j-1}(\lambda) - w_{j-1}^2 \frac{n_{j-1}}{n_j} S_{j-2}(\lambda) & \text{if } j \in \phi_1 \backslash \{1\}. \end{cases} \tag{6.6}$$

Then

(a) $\det(\lambda I - A(G)) = \prod_{j \in \phi_1} (S_j(\lambda) + 2u_j S_{j-1}(\lambda))^{\frac{n_j}{2}} S_j(\lambda)^{\frac{n_j}{2} - n_{j+1}} \prod_{j \in \phi_2} S_j(\lambda)^{n_j - n_{j+1}} S_k(\lambda)$.
(b) $\sigma(A(G)) = (\cup_{j \in \Omega \cup \{k\}} \{\lambda \in \mathbb{R} : S_j(\lambda) = 0\}) \cup (\cup_{j \in \phi_1} \{\lambda \in \mathbb{R} : S_j(\lambda) = -2u_j S_{j-1}(\lambda)\})$.

The proof of this theorem is similar to the proof of Theorem 6.4.

The next theorem gives a complete characterization of the eigenvalues of the adjacency matrix. In fact, they are the eigenvalues of leading principal submatrices of R_k and the roots of some polynomials related with these submatrices.

Theorem 6.10

(a) $\sigma(A(G)) = (\cup_{j \in \Omega \cup \{k\}} \sigma(R_j)) \cup (\cup_{j \in \phi_1} \{\lambda \in \mathbb{R} : \det(\lambda I - R_j) = -2u_j \det(\lambda I - R_{j-1})\})$.

(b) *The multiplicity of each eigenvalue of the matrix* R_j, *as an eigenvalue of* $A(G)$ *is at least* $n_j - n_{j+1}$ *for* $j \in \phi_2$, $\frac{n_j - 2n_{j+1}}{2}$ *for* $j \in \phi_1$ *and* 1 *for* $j = k$.

(c) *For* $j \in \phi_1$, *each root of the polynomial*

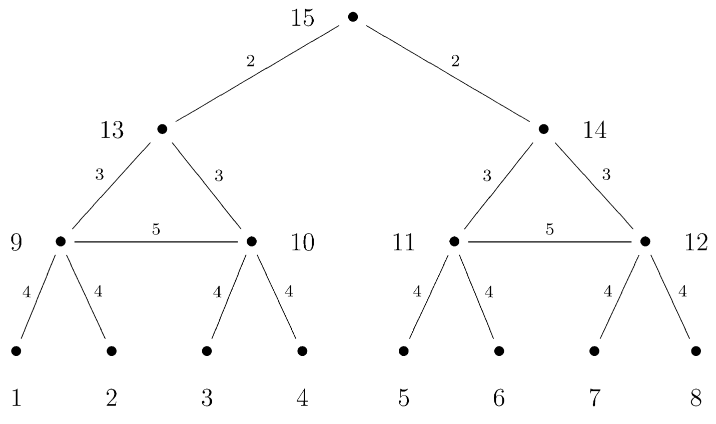
$$\det(\lambda I - R_j) = -2u_j \det(\lambda I - R_{j-1})$$

is an eigenvalue of $A(G)$ *with multiplicity at least* $\frac{n_j}{2}$.

Proof. (a), (b) and (c) are consequences of Theorem 6.9, Lemma 6.7 and Corollary 6.8. □

Next, we present an example for $\Phi_1 = \{2\}$.

Example 6.1



This graph has four levels with vertex degrees

$$d_1 = 1, \quad d_2 = 4, \quad d_3 = 3, \quad d_4 = 2,$$

the number of vertices in each level is

$$n_1 = 8, \quad n_2 = 4, \quad n_3 = 2, \quad n_4 = 1$$

and the edge weights are

$$u_2 = 5, \quad w_1 = 4, \quad w_2 = 3 \quad \text{and} \quad w_3 = 2.$$

Let $\delta_1 = w_1 = 4, \delta_2 = u_2 + (d_2 - 2)w_1 + w_2 = 16, \delta_3 = (d_3 - 1)w_2 + w_3 = 8$ and $\delta_4 = d_4 w_3 = 4$.

For this graph let

$$T_4 = \begin{bmatrix} 4 & 4\sqrt{2} & 0 & 0 \\ 4\sqrt{2} & 11 & 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} & 8 & 2\sqrt{2} \\ 0 & 0 & 2\sqrt{2} & 4 \end{bmatrix}.$$

The eigenvalues of $L(G)$ are the eigenvalues of T_1, T_3, T_4 and the roots of the polynomial $\lambda^2 - 25\lambda + 52$. To four decimal places these eigenvalues are

T_1 :	4				with multiplicity 4
T_3 :	0.2202	6.8351	15.9445		each one with multiplicity 1
T_4 :	0	2.7347	8.1881	16.0772	each one with multiplicity 1
2.2897	22.7103				each one with multiplicity 2.

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