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Nonlinear ∗-Lie derivations on factor von Neumann algebras ☆

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ABSTRACT

In this paper we prove that every nonlinear ∗-Lie derivation from a factor von Neumann algebra into itself is an additive ∗-derivation.

1. Introduction

Let $\mathscr{A}$ be an associative ∗-algebra over the complex field $\mathbb{C}$. A map $\delta : \mathscr{A} \to \mathscr{A}$ is said to be an additive ∗-derivation if it is an additive derivation and satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathscr{A}$. Let $\phi : \mathscr{A} \to \mathscr{A}$ be a map (without the additivity assumption). We say that ϕ is a nonlinear ∗-Lie derivation if $\phi([A, B]_*) = [\phi(A), B]_* + [A, \phi(B)]_*$ for all $A, B \in \mathscr{A}$, where $[A, B]_* = AB - BA^*$. The structure of linear Lie derivations on C^*-algebras has attracted some attention over past years. Johnson [6] proved that every continuous linear Lie derivation from a C^*-algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule $\mathscr{E}$ can be decomposed as $\delta + h$, where $\delta : \mathscr{A} \to \mathscr{E}$ is a derivation and h is a linear mapping from $\mathscr{A}$ into the center of $\mathscr{E}$. Mathieu and Villena [7] proved that every linear Lie derivation on a C^*-algebra can be decomposed into the sum of a derivation and a center-valued trace. In [10], Zhang proved the same result for nest subalgebras of factor von Neumann algebras. Cheung gave in [2] a characterization

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of linear Lie derivations on triangular algebras. Qi and Hou [8] discussed additive ξ-Lie derivations on nest algebras. The most interesting result on additive Lie derivations of prime rings was obtained in [1]. However, the structure of nonlinear Lie derivations or nonlinear $*$-Lie derivations on operator algebras is not clear, it needs to be discussed further. In [3], Cheng and Zhang investigated nonlinear Lie derivations on upper triangular matrix algebras. Yu and Zhang [9] proved that every nonlinear Lie derivations of triangular algebras is the sum of an additive derivation and a map into its center sending commutators to zero. Motivated by these study, we consider nonlinear $*$-Lie derivations on factor von Neumann algebras.

As usual, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field. Let $\mathscr{H}$ be a complex Hilbert space. We denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. Let $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Recall that $\mathscr{M}$ is a factor if its center is $\mathbb{C}I$ where I is the identity of $\mathscr{M}$. Let $\mathscr{M}_{sa}$ be the space of all self-adjoint operators of $\mathscr{M}$.

2. Main result

In this section, we will prove the following theorem.

Theorem 2.1. *Let $\mathscr{M}$ be factor von Neumann algebras acting on a complex Hilbert space $\mathscr{H}$ with* $\dim \mathscr{H} \geqslant 2$. *Suppose that $\phi : \mathscr{M} \to \mathscr{M}$ is a nonlinear $*$-Lie derivation. Then ϕ is an additive $*$-derivation.*

Next we assume that $\mathscr{M}$ is a factor von Neumann algebras acting on a complex Hilbert space $\mathscr{H}$. The following two lemmas can be found in [4].

Lemma 2.1. *Let $A \in \mathscr{M}$. Then $AB = BA^*$ for every $B \in \mathscr{M}$ implies that $A \in \mathbb{R}I$.*

Lemma 2.2. *Let $B \in \mathscr{M}$. Then $AB = BA^*$ for every $A \in \mathscr{M}$ implies that $B = 0$.*

Lemma 2.3. *Let $P \in \mathscr{M}$ be a nontrivial projection and $A \in \mathscr{M}$. Then $AB = BA^*$ for every $B \in P\mathscr{M}(I-P)$ implies that $A = \mu P + \bar{\mu}(I-P)$ for some $\mu \in \mathbb{C}$.*

Proof. It is clear that $(I-P)APT(I-P) = 0$ for all $T \in \mathscr{M}$. Then $(I-P)AP = 0$. Let $X \in P\mathscr{M}P$ and $Y \in (I-P)\mathscr{M}(I-P)$, we have for any $B \in P\mathscr{M}(I-P)$,

$$AXB = XBA^* \text{ and } ABY = BYA^*.$$

On the other hand, we have

$$XAB = XBA^* \text{ and } ABY = BA^*Y.$$

It follows that $(AX - XA)B = B(YA^* - A^*Y) = 0$ for all $B \in P\mathscr{M}(I-P)$. This implies that

$$PAPX - XPAP = (I-P)A^*(I-P)Y - Y(I-P)A^*(I-P) = 0$$

for all $X \in P\mathscr{M}P$ and all $Y \in (I-P)\mathscr{M}(I-P)$. So there exist $\lambda, \mu \in \mathbb{C}$ such that $PAP = \lambda P$ and $(I-P)A^*(I-P) = \mu(I-P)$. Thus,

$$A = \lambda P + PA(I-P) + \bar{\mu}(I-P).$$

Then $\lambda B = BA^*P + \mu B$ for all $B \in P\mathscr{M}(I-P)$. It follows that $\lambda = \mu$ and $(I-P)A^*P = 0$. Hence $A = \mu P + \bar{\mu}(I-P)$ for some $\mu \in \mathbb{C}$. The proof is completed. □

Lemma 2.4. *Let $\phi : \mathscr{M} \to \mathscr{M}$ be a nonlinear $*$-Lie derivation. Then $\phi(0) = 0$, $\phi(\mathbb{C}I) \subseteq \mathbb{C}I$, $\phi(\mathscr{M}_{sa}) \subseteq \mathscr{M}_{sa}$ and $\phi(iA) = i\phi(A)$ for all $A \in \mathscr{M}$.*

Proof. It is easy to verify that $\phi(0) = 0$. Let $T \in \mathscr{M}$, then

$$[\phi(I), T]_* + [I, \phi(T)]_* = \phi([I, T]_*) = \phi(0) = 0.$$

This implies that $\phi(I)T = T\phi(I)^*$ for all $T \in \mathscr{M}$. Hence $\phi(I) = \phi(I)^* \in \mathbb{R}I$, and so we have for $A \in \mathscr{M}_{sa}$,

$$\phi(A) - \phi(A)^* = [\phi(A), I]_* + [A, \phi(I)]_* = \phi([A, I]_*) = \phi(0) = 0.$$

Thus $\phi(\mathscr{M}_{sa}) \subseteq \mathscr{M}_{sa}$. Let $\lambda \in \mathbb{C}$, we have for any $A \in \mathscr{M}_{sa}$,

$$[\phi(A), \lambda I]_* + [A, \phi(\lambda I)]_* = \phi([A, \lambda I]_*) = \phi(0) = 0.$$

It follows from $\phi(\mathscr{M}_{sa}) \subseteq \mathscr{M}_{sa}$ that $A\phi(\lambda I) = \phi(\lambda I)A$ for all $A \in \mathscr{M}_{sa}$. Hence $\phi(\mathbb{C}I) \subseteq \mathbb{C}I$.

Since $\phi(\mathbb{C}I) \subseteq \mathbb{C}I$ and $\phi(\mathscr{M}_{sa}) \subseteq \mathscr{M}_{sa}$, we have $\phi(-\frac{1}{2}I) \in \mathbb{R}I$. It follows from $[\frac{1}{2}iI, \frac{1}{2}iI]_* = -\frac{1}{2}I$ that

$$\begin{aligned} i\phi\left(\frac{1}{2}iI\right) + \frac{1}{2}i\left(\phi\left(\frac{1}{2}iI\right) - \phi\left(\frac{1}{2}iI\right)^*\right) &= \left[\frac{1}{2}iI, \phi\left(\frac{1}{2}iI\right)\right]_* + \left[\phi\left(\frac{1}{2}iI\right), \frac{1}{2}iI\right]_* \\ &= \phi\left(\left[\frac{1}{2}iI, \frac{1}{2}iI\right]_*\right) = \phi\left(-\frac{1}{2}I\right) \in \mathbb{R}I. \end{aligned}$$

Since $\frac{1}{2}i(\phi(\frac{1}{2}iI) - \phi(\frac{1}{2}iI)^*) \in \mathbb{R}I$, we have from above equation that $\phi(\frac{1}{2}iI)^* = -\phi(\frac{1}{2}iI)$. Hence $\phi(-\frac{1}{2}I) = 2i\phi(\frac{1}{2}iI)$. Similarly, we can obtain from the fact $[-\frac{1}{2}iI, -\frac{1}{2}iI]_* = -\frac{1}{2}I$ that $\phi(-\frac{1}{2}iI)^* = -\phi(-\frac{1}{2}iI)$ and $\phi(-\frac{1}{2}I) = -2i\phi(-\frac{1}{2}iI)$. Thus $\phi(\frac{1}{2}iI) = -\phi(-\frac{1}{2}iI)$. Hence

$$\begin{aligned} \phi\left(\frac{1}{2}iI\right) &= \phi\left(\left[-\frac{1}{2}iI, -\frac{1}{2}I\right]_*\right) = \left[\phi\left(-\frac{1}{2}iI\right), -\frac{1}{2}I\right]_* + \left[-\frac{1}{2}iI, \phi\left(-\frac{1}{2}I\right)\right]_* \\ &= -\phi\left(-\frac{1}{2}iI\right) - i\phi\left(-\frac{1}{2}I\right) = \phi\left(\frac{1}{2}iI\right) - i\phi\left(-\frac{1}{2}I\right). \end{aligned}$$

This implies that $\phi(-\frac{1}{2}I) = 0$, and so $\phi(\frac{1}{2}iI) = 0$. For every $A \in \mathscr{M}$, we have

$$\phi(iA) = \phi\left(\left[\frac{1}{2}iI, A\right]_*\right) = \left[\phi\left(\frac{1}{2}iI\right), A\right]_* + \left[\frac{1}{2}iI, \phi(A)\right]_* = i\phi(A).$$

The proof is completed. □

Now we chose a nontrivial projection $P_1 \in \mathscr{M}$ and set $P_2 = I - P_1$. Write $\mathscr{M}_{ij} = P_i\mathscr{M}P_j$, $i, j = 1, 2$. Then we have the following lemma.

Lemma 2.5. *Let $\phi : \mathscr{M} \to \mathscr{M}$ be a nonlinear $*$-Lie derivation and let $U = P_1\phi(P_1)P_2 - P_2\phi(P_1)P_1 \in \mathscr{M}$. Then*

(a) $\phi(A) = AU - UA + P_1\phi(A)P_2$ for all $A \in \mathscr{M}_{12}$;
(b) $\phi(B) = BU - UB + P_2\phi(B)P_1$ for all $B \in \mathscr{M}_{21}$;
(c) $\phi(P_i) = P_iU - UP_i + \lambda_iI$ for some $\lambda_i \in \mathbb{C}$, $i = 1, 2$.

Proof. (a) Since $A = [P_1, A]_*$ for all $A \in \mathscr{M}_{12}$, it follows from $\phi(\mathscr{M}_{sa}) \subseteq \mathscr{M}_{sa}$ that

$$\begin{aligned} \phi(A) &= \phi([P_1, A]_*) = [\phi(P_1), A]_* + [P_1, \phi(A)]_* \\ &= \phi(P_1)A - A\phi(P_1) + P_1\phi(A) - \phi(A)P_1. \end{aligned}$$

This yields

$$P_2\phi(A)P_1 = 0,\ \ P_1\phi(A)P_1 = -A\phi(P_1)P_1,\ \ P_2\phi(A)P_2 = P_2\phi(P_1)A$$

and

$$P_1\phi(P_1)A = A\phi(P_1)P_2. \tag{1}$$

Then

$$\begin{aligned}\phi(A) &= P_2\phi(A)P_2 + P_1\phi(A)P_1 + P_2\phi(A)P_1 + P_1\phi(A)P_2\\ &= P_2\phi(P_1)A - A\phi(P_1)P_1 + P_1\phi(A)P_2\\ &= AU - UA + P_1\phi(A)P_2.\end{aligned}$$

(b) Since $B = [P_2, B]_*$ for all $B \in \mathscr{M}_{21}$, a similar discussion to (a) implies that

$$P_1\phi(B)P_2 = 0,\ \ P_1\phi(B)P_1 = P_1\phi(P_2)B,\ \ P_2\phi(B)P_2 = -B\phi(P_2)P_2$$

and

$$P_2\phi(P_2)B = B\phi(P_2)P_1. \tag{2}$$

It follows from $[P_1, P_2]_* = 0$ and $\phi(\mathscr{M}_{sa}) \subseteq \mathscr{M}_{sa}$ that

$$\phi(P_1)P_2 - P_2\phi(P_1) + P_1\phi(P_2) - \phi(P_2)P_1 = \phi([P_1, P_2]_*) = 0.$$

This yields

$$P_1\phi(P_2)P_2 = -P_1\phi(P_1)P_2,\ \ P_2\phi(P_2)P_1 = -P_2\phi(P_1)P_1. \tag{3}$$

Hence we have from Eq. (3) that

$$\begin{aligned}\phi(B) &= P_1\phi(B)P_1 + P_2\phi(B)P_2 + P_2\phi(B)P_1 + P_1\phi(B)P_2\\ &= P_1\phi(P_2)B - B\phi(P_2)P_2 + P_2\phi(B)P_1\\ &= B\phi(P_1)P_2 - P_1\phi(P_1)B + P_2\phi(B)P_1\\ &= BU - UB + P_2\phi(B)P_1.\end{aligned}$$

(c) Let $X \in \mathscr{M}_{11}$ and $Y \in \mathscr{M}_{22}$, then $XA \in \mathscr{M}_{12}$ and $AY \in \mathscr{M}_{12}$ for all $A \in \mathscr{M}_{12}$. It follows from Eq. (1) that

$$P_1\phi(P_1)XA = XA\phi(P_1)P_2 \ \text{ and } \ P_1\phi(P_1)AY = AY\phi(P_1)P_2. \tag{4}$$

On the other hand, we have from Eq. (1) again

$$XP_1\phi(P_1)A = XA\phi(P_1)P_2 \ \text{ and } \ P_1\phi(P_1)AY = A\phi(P_1)P_2Y. \tag{5}$$

Hence by Eqs. (4) and (5),

$$(P_1\phi(P_1)P_1X - XP_1\phi(P_1)P_1)\mathscr{M}P_2 = \{0\}$$

and

$$P_1\mathscr{M}(P_2\phi(P_1)P_2Y - YP_2\phi(P_1)P_2) = \{0\}.$$

Since $\mathscr{M}$ is a factor von Neumann algebra, we have $P_1\phi(P_1)P_1X = XP_1\phi(P_1)P_1$ for all $X \in \mathscr{M}_{11}$ and $P_2\phi(P_1)P_2Y = YP_2\phi(P_1)P_2$ for all $Y \in \mathscr{M}_{22}$. Then there exist $\lambda_1, \beta_1 \in \mathbb{C}$ such that

$$P_1\phi(P_1)P_1 = \lambda_1 P_1 \text{ and } P_2\phi(P_1)P_2 = \beta_1 P_2.$$

This and Eq. (1) show that $\lambda_1 A = \beta_1 A$ for all $A \in \mathscr{M}_{12}$. Then $\lambda_1 = \beta_1$, and so

$$\begin{aligned}\phi(P_1) &= P_1\phi(P_1)P_1 + P_1\phi(P_1)P_2 + P_2\phi(P_1)P_1 + P_2\phi(P_1)P_2\\ &= P_1\phi(P_1)P_2 + P_2\phi(P_1)P_1 + \lambda_1 I\\ &= P_1U - UP_1 + \lambda_1 I.\end{aligned}$$

From Eq. (2) and a similar discussion to the above, we can show that there exists $\lambda_2 \in \mathbb{C}$ such that

$$P_1\phi(P_2)P_1 = \lambda_2 P_1 \text{ and } P_2\phi(P_2)P_2 = \lambda_2 P_2.$$

Hence by Eq. (3),

$$\begin{aligned}\phi(P_2) &= P_1\phi(P_2)P_1 + P_1\phi(P_2)P_2 + P_2\phi(P_2)P_1 + P_2\phi(P_2)P_2\\ &= -P_1\phi(P_1)P_2 - P_2\phi(P_1)P_1 + \lambda_2 I\\ &= P_2U - UP_2 + \lambda_2 I.\end{aligned}$$

The proof is completed. □

Remark 2.1. Let U be the operator that appears in Lemma 2.5. We define a map $\psi : \mathscr{M} \to \mathscr{M}$ as follows:

$$\psi(X) = \phi(X) - (XU - UX).$$

It is easy to verify that ψ is also a nonlinear $*$-Lie derivation with $\psi(P_i) \in \mathbb{C}I$ for $i = 1, 2$.

Lemma 2.6. *Let ψ be as in Remark 2.1. Then $\psi(\mathscr{M}_{ij}) \subseteq \mathscr{M}_{ij}$ for $i, j = 1, 2$.*

Proof. By the definition of ψ and Lemma 2.5, we see that $\psi(\mathscr{M}_{ij}) \subseteq \mathscr{M}_{ij}$ for $i \neq j$. Since $[P_i, A]_* = 0$ for all $A \in \mathscr{M}_{11}$, it follows from $\psi(P_i) \in \mathbb{C}I$ that

$$0 = \psi([P_i, A]_*) = [\psi(P_i), A]_* + [P_i, \psi(A)]_* = P_i\psi(A) - \psi(A)P_i.$$

Then $P_i\psi(A)P_j = 0$ for $i \neq j$, and so

$$\psi(A) = P_1\psi(A)P_1 + P_2\psi(A)P_2 \tag{6}$$

for all $A \in \mathscr{M}_{11}$. Similarly, we can show that

$$\psi(B) = P_1\psi(B)P_1 + P_2\psi(B)P_2 \tag{7}$$

for all $B \in \mathscr{M}_{22}$. Since $[A, B]_* = [B, A]_* = 0$ for all $A \in \mathscr{M}_{11}$ and $B \in \mathscr{M}_{22}$, we have from Eqs. (6) and (7) that

$$[P_2\psi(A)P_2, B]_* + [A, P_1\psi(B)P_1]_* = [\psi(A), B]_* + [A, \psi(B)]_* = 0$$

and

$$[P_1\psi(B)P_1, A]_* + [B, P_2\psi(A)P_2]_* = [\psi(B), A]_* + [B, \psi(A)]_* = 0.$$

This implies that $[A, P_1\psi(B)P_1]_* = 0$ for all $A \in \mathscr{M}_{11}$ and $[B, P_2\psi(A)P_2]_* = 0$ for all $B \in \mathscr{M}_{22}$. By Lemma 2.2, then

$$P_1\psi(B)P_1 = P_2\psi(A)P_2 = 0.$$

This together with Eqs. (6) and (7) gives us that $\psi(\mathscr{M}_{ii}) \subseteq \mathscr{M}_{ii}$ for $i = 1, 2$. The proof is completed. □

Lemma 2.7. *Let ψ be as in Remark 2.1 and $i, j \in \{1, 2\}$ with $i \neq j$. Then*

(a) $\psi(A_{ii} + A_{ij}) = \psi(A_{ii}) + \psi(A_{ij})$ for all $A_{ii} \in \mathscr{M}_{ii}$ and $A_{ij} \in \mathscr{M}_{ij}$;
(b) $\psi(A_{ii} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ji})$ for all $A_{ii} \in \mathscr{M}_{ii}$ and $A_{ji} \in \mathscr{M}_{ji}$;
(c) $\psi(A_{11} + A_{22}) = \psi(A_{11}) + \psi(A_{22})$ for all $A_{11} \in \mathscr{M}_{11}$ and $A_{22} \in \mathscr{M}_{22}$;
(d) $\psi(A_{12} + A_{21}) = \psi(A_{12}) + \psi(A_{21})$ for all $A_{12} \in \mathscr{M}_{12}$ and $A_{21} \in \mathscr{M}_{21}$.

Proof. (a) Let $X_{jj} \in \mathscr{M}_{jj}$. It follows from $[X_{jj}, A_{ij}]_* = [X_{jj}, A_{ii} + A_{ij}]_*$ and Lemma 2.6 that

$$\begin{aligned}[\psi(X_{jj}), A_{ij}]_* + [X_{jj}, \psi(A_{ij})]_* &= [\psi(X_{jj}), A_{ii} + A_{ij}]_* + [X_{jj}, \psi(A_{ii} + A_{ij})]_* \\ &= [\psi(X_{jj}), A_{ij}]_* + [X_{jj}, \psi(A_{ii} + A_{ij})]_*.\end{aligned}$$

Hence

$$X_{jj}(\psi(A_{ii} + A_{ij}) - \psi(A_{ij})) = (\psi(A_{ii} + A_{ij}) - \psi(A_{ij}))X_{jj}^* \tag{8}$$

for all $X_{jj} \in \mathscr{M}_{jj}$. Taking $X_{jj} = P_j$ in Eq. (8), we have from the fact $\psi(A_{ij}) \in \mathscr{M}_{ij}$ that

$$P_j\psi(A_{ii} + A_{ij})P_i = P_j(\psi(A_{ii} + A_{ij}) - \psi(A_{ij}))P_i = 0. \tag{9}$$

Also, we have from Eq. (8) and Lemma 2.2 that

$$P_j\psi(A_{ii} + A_{ij})P_j = P_j(\psi(A_{ii} + A_{ij}) - \psi(A_{ij}))P_j = 0. \tag{10}$$

Clearly, it follows from Eq. (8) that $P_i(\psi(A_{ii} + A_{ij}) - \psi(A_{ij}))X_{jj}^* = 0$ for all $X_{jj} \in \mathscr{M}_{jj}$. This implies that

$$P_i\psi(A_{ii} + A_{ij})P_j = \psi(A_{ij}). \tag{11}$$

On the other hand, we have from Lemma 2.6 and the fact $[A_{ii}, X_{ii}]_* = [A_{ii}+A_{ij}, X_{ii}]_*$ for all $X_{ii} \in \mathscr{M}_{ii}$ that

$$\begin{aligned}[\psi(A_{ii}), X_{ii}]_* + [A_{ii}, \psi(X_{ii})]_* &= [\psi(A_{ii} + A_{ij}), X_{ii},]_* + [A_{ii} + A_{ij}, \psi(X_{ii})]_* \\ &= [\psi(A_{ii} + A_{ij}), X_{ii},]_* + [A_{ii}, \psi(X_{ii})]_*.\end{aligned}$$

Hence

$$(\psi(A_{ii} + A_{ij}) - \psi(A_{ii}))X_{ii} = X_{ii}(\psi(A_{ii} + A_{ij}) - \psi(A_{ii}))^*.$$

By Lemmas 2.1 and 2.6, there exists a scalar $\lambda \in \mathbb{R}$ such that

$$P_i\psi(A_{ii} + A_{ij})P_i = \psi(A_{ii}) + \lambda P_i. \tag{12}$$

Combining Eqs. (9)–(12), we obtain that

$$\psi(A_{ii}+A_{ij})=\psi(A_{ii})+\psi(A_{ij})+\lambda P_i. \tag{13}$$

For each $X_{ij}\in\mathscr{M}_{ij}$, we have from Eq. (13) that there exists a scalar $\alpha\in\mathbb{R}$ such that

$$\begin{aligned}\psi(-X_{ij}A_{ij}^*)+\psi(A_{ii}X_{ij})+\alpha P_i &\\ &=\psi(-X_{ij}A_{ij}^*+A_{ii}X_{ij})=\psi([A_{ii}+A_{ij},X_{ij}]_*)\\ &=[\psi(A_{ii}+A_{ij}),X_{ij}]_*+[A_{ii}+A_{ij},\psi(X_{ij})]_*\\ &=[\psi(A_{ii})+\psi(A_{ij})+\lambda P_i,X_{ij}]_*+[A_{ii}+A_{ij},\psi(X_{ij})]_*\\ &=\psi([A_{ij},X_{ij}]_*)+\psi([A_{ii},X_{ij}]_*)+\lambda X_{ij}\\ &=\psi(-X_{ij}A_{ij}^*)+\psi(A_{ii}X_{ij})+\lambda X_{ij}.\end{aligned}$$

Then $\lambda X_{ij}=\alpha P_i$ for each $X_{ij}\in\mathscr{M}_{ij}$. This implies that $\lambda=0$, and so by Eq. (13) we have $\psi(A_{ii}+A_{ij})=\psi(A_{ii})+\psi(A_{ij})$.

(b) Let $X_{ji}\in\mathscr{M}_{ji}$. Then

$$\psi([A_{ii}+A_{ji},X_{ji}]_*)=[\psi(A_{ii}+A_{ji}),X_{ji}]_*+[A_{ii}+A_{ji},\psi(X_{ji})]_*. \tag{14}$$

On the other hand, it follows from (a) that

$$\begin{aligned}\psi([A_{ii}+A_{ji},X_{ji}]_*)&=\psi(-X_{ji}A_{ji}^*-X_{ji}A_{ii}^*)=\psi(-X_{ji}A_{ji}^*)+\psi(-X_{ji}A_{ii}^*)\\ &=\psi([A_{ji},X_{ji}]_*)+\psi([A_{ii},X_{ji}]_*)\\ &=[\psi(A_{ji}),X_{ji}]_*+[A_{ji},\psi(X_{ji})]_*\\ &\quad+[\psi(A_{ii}),X_{ji}]_*+[A_{ii},\psi(X_{ji})]_*\\ &=[\psi(A_{ii})+\psi(A_{ji}),X_{ji}]_*+[A_{ii}+A_{ji},\psi(X_{ji})]_*.\end{aligned}$$

Hence by Eq. (14),

$$[\psi(A_{ii}+A_{ji}),X_{ji}]_*=[\psi(A_{ii})+\psi(A_{ji}),X_{ji}]_*$$

for all $X_{ji}\in\mathscr{M}_{ji}$. By Lemma 2.3,

$$\psi(A_{ii}+A_{ji})-\psi(A_{ii})-\psi(A_{ji})=\mu P_j+\bar{\mu}P_i \tag{15}$$

for some $\mu\in\mathbb{C}$. Since $[X_{jj},A_{ji}]_*=[X_{jj},A_{ii}+A_{ji}]_*$ for all $X_{jj}\in\mathscr{M}_{jj}$, we have from Lemma 2.6 and Eq. (15) that

$$\begin{aligned}[\psi(X_{jj}),A_{ji}]_*+[X_{jj},\psi(A_{ji})]_*&=[\psi(X_{jj}),A_{ii}+A_{ji}]_*+[X_{jj},\psi(A_{ii}+A_{ji})]_*\\ &=[\psi(X_{jj}),A_{ji}]_*+[X_{jj},\psi(A_{ii}+A_{ji})]_*\\ &=[\psi(X_{jj}),A_{ji}]_*+[X_{jj},\psi(A_{ji})+\mu P_j]_*.\end{aligned}$$

Then $\mu X_{jj}=\mu X_{jj}^*$ for all $X_{jj}\in\mathscr{M}_{jj}$, and so $\mu=0$. By Eq. (15), hence $\psi(A_{ii}+A_{ji})=\psi(A_{ii})+\psi(A_{ji})$.

(c) Let $X_{11}\in\mathscr{M}_{11}$. It follows from $[X_{11},A_{11}]_*=[X_{11},A_{11}+A_{22}]_*$ and Lemma 2.6 that

$$\begin{aligned}[\psi(X_{11}),A_{11}]_*+[X_{11},\psi(A_{11})]_*&=[\psi(X_{11}),A_{11}+A_{22}]_*+[X_{11},\psi(A_{11}+A_{22})]_*\\ &=[\psi(X_{11}),A_{11}]_*+[X_{11},\psi(A_{11}+A_{22})]_*.\end{aligned}$$

Then

$$X_{11}(\psi(A_{11}+A_{22})-\psi(A_{11}))=(\psi(A_{11}+A_{22})-\psi(A_{11}))X_{11}^*$$

for every $X_{11}\in\mathscr{M}_{11}$. Applying the same argument as in (a), we can show that

$$P_1\psi(A_{11}+A_{22})P_2=P_2\psi(A_{11}+A_{22})P_1=0 \tag{16}$$

and

$$P_1\psi(A_{11}+A_{22})P_1=\psi(A_{11}). \tag{17}$$

From the fact $[X_{22},A_{22}]_*=[X_{22},A_{11}+A_{22}]_*$ for all $X_{22}\in\mathscr{M}_{22}$, similarly, we can obtain that

$$P_2\psi(A_{11}+A_{22})P_2=\psi(A_{22}). \tag{18}$$

Combining Eqs. (16)–(18), we see that $\psi(A_{11}+A_{22})=\psi(A_{11})+\psi(A_{22})$.

(d) Let $X_{12}\in\mathscr{M}_{12}$. It is clear that

$$\psi([A_{12}+A_{21},X_{12}]_*)=[\psi(A_{12}+A_{21}),X_{12}]_*+[A_{12}+A_{21},\psi(X_{12})]_*. \tag{19}$$

On the other hand, we have from (c) that

$$\begin{aligned}\psi([A_{12}+A_{21},X_{12}]_*)&=\psi(A_{21}X_{12}-X_{12}A_{12}^*)=\psi(A_{21}X_{12})+\psi(-X_{12}A_{12}^*)\\&=\psi([A_{21},X_{12}]_*)+\psi([A_{12},X_{12}]_*)\\&=[\psi(A_{21}),X_{12}]_*+[A_{21},\psi(X_{12})]_*\\&\quad+[\psi(A_{12}),X_{12}]_*+[A_{12},\psi(X_{12})]_*\\&=[\psi(A_{12})+\psi(A_{21}),X_{12}]_*+[A_{12}+A_{21},\psi(X_{12})]_*.\end{aligned}$$

This and Eq. (19) show that

$$[\psi(A_{12}+A_{21}),X_{12}]_*=[\psi(A_{12})+\psi(A_{21}),X_{12}]_*$$

for all $X_{12}\in\mathscr{M}_{12}$. By Lemma 2.3, there exists a scalar $\mu\in\mathbb{C}$ such that

$$\psi(A_{12}+A_{21})=\psi(A_{12})+\psi(A_{21})+\mu P_1+\bar{\mu}P_2. \tag{20}$$

We see from Eq. (20) that for each $X_{11}\in\mathscr{M}_{11}$ there exists a scalar $\alpha\in\mathbb{C}$ such that

$$\begin{aligned}\psi([X_{11},A_{12}+A_{21}]_*)&=\psi(X_{11}A_{12}-A_{21}X_{11}^*)\\&=\psi(X_{11}A_{12})+\psi(-A_{21}X_{11}^*)+\alpha P_1+\bar{\alpha}P_2.\end{aligned}$$

On the other hand, it follows from Eq. (20) again

$$\begin{aligned}\psi([X_{11},A_{12}+A_{21}]_*)&=[\psi(X_{11}),A_{12}+A_{21}]_*+[X_{11},\psi(A_{12}+A_{21})]_*\\&=[\psi(X_{11}),A_{12}]_*+[\psi(X_{11}),A_{21}]_*+[X_{11},\psi(A_{12})]_*\\&\quad+[X_{11},\psi(A_{21})]_*+[X_{11},\mu P_1+\bar{\mu}P_2]_*\\&=\psi([X_{11},A_{12}]_*)+\psi([X_{11},A_{21}]_*)+[X_{11},\mu P_1+\bar{\mu}P_2]_*\\&=\psi(X_{11}A_{12})+\psi(-A_{21}X_{11}^*)+\mu(X_{11}-X_{11}^*).\end{aligned}$$

Hence $\mu(X_{11}-X_{11}^*)=\alpha P_1+\bar{\alpha}P_2$. This implies that $\bar{\alpha}=0$, then $\mu(X_{11}-X_{11}^*)=0$ for all $X_{11}\in\mathscr{M}_{11}$, and so $\mu=0$. Therefore, we have from Eq. (20) that $\psi(A_{12}+A_{21})=\psi(A_{12})+\psi(A_{21})$. The proof is completed. □

Lemma 2.8. *Let ψ be as in Remark 2.1. Then $\psi(\sum_{i,j=1}^2 A_{ij})=\sum_{i,j=1}^2\psi(A_{ij})$ for all $A_{ij}\in\mathscr{M}_{ij}$.*

Proof. Let $i,j\in\{1,2\}$ with $i\neq j$. Then

$$[A_{ii}+A_{ji},T_{ii}]_*=[A_{ii}+A_{ij}+A_{ji},T_{ii}]_*$$

for all $T_{ii}\in\mathscr{M}_{ii}$, and so by Lemma 2.6,

$$[\psi(A_{ii}+A_{ij}+A_{ji})-\psi(A_{ii}+A_{ji}),T_{ii}]_*=0 \tag{21}$$

for all $T_{ii}\in\mathscr{M}_{ii}$. It follows from Lemmas 2.1, 2.6 and 2.7(b) that

$$P_i\psi(A_{ii}+A_{ij}+A_{ji})P_i=\psi(A_{ii})+\lambda_1P_i \tag{22}$$

for some $\lambda_1\in\mathbb{R}$. By Eq. (21), we have

$$P_j(\psi(A_{ii}+A_{ij}+A_{ji})-\psi(A_{ii}+A_{ji}))T_{ii}=0$$

for all $T_{ii}\in\mathscr{M}_{ii}$. It follows from Lemmas 2.6 and 2.7(b) that

$$P_j\psi(A_{ii}+A_{ij}+A_{ji})P_i=\psi(A_{ji}). \tag{23}$$

On the other hand, we have from Lemma 2.6 and the fact $[A_{ij},T_{jj}]_*=[A_{ii}+A_{ij}+A_{ji},T_{jj}]_*$ for all $T_{jj}\in\mathscr{M}_{jj}$ that

$$[\psi(A_{ii}+A_{ij}+A_{ij})-\psi(A_{ij}),T_{jj}]_*=0$$

for all $T_{jj}\in\mathscr{M}_{jj}$. Then by Lemmas 2.1 and 2.6, there is a $\lambda_2\in\mathbb{R}$ such that

$$P_j\psi(A_{ii}+A_{ij}+A_{ji})P_j=\lambda_2P_j. \tag{24}$$

Also, we have $P_i(\psi(A_{ii}+A_{ij}+A_{ij})-\psi(A_{ij}))T_{jj}=0$ for all $T_{jj}\in\mathscr{M}_{jj}$. This implies that

$$P_i\psi(A_{ii}+A_{ij}+A_{ji})P_j=\psi(A_{ij}). \tag{25}$$

Combining Eqs. (22)–(25), we obtain that

$$\psi(A_{ii}+A_{ij}+A_{ji})=\psi(A_{ii})+\psi(A_{ij})+\psi(A_{ji})+\lambda_1P_i+\lambda_2P_j. \tag{26}$$

It follows from Eq. (26) that for each $T_{ii}\in\mathscr{M}_{ii}$ there exist $\alpha_1,\alpha_2\in\mathbb{R}$ such that

$$\begin{aligned}
&\psi(T_{ii}A_{ii}-A_{ii}T_{ii}^*)+\psi(T_{ii}A_{ij})+\psi(-A_{ji}T_{ii}^*)+\alpha_1P_i+\alpha_2P_j\\
&=\psi(T_{ii}A_{ii}+T_{ii}A_{ij}-A_{ii}T_{ii}^*-A_{ji}T_{ii}^*)=\psi([T_{ii},A_{ii}+A_{ij}+A_{ji},]_*)\\
&=[\psi(T_{ii}),A_{ii}+A_{ij}+A_{ji}]_*+[T_{ii},\psi(A_{ii}+A_{ij}+A_{ji})]_*\\
&=[\psi(T_{ii}),A_{ii}+A_{ij}+A_{ji}]_*+[T_{ii},\psi(A_{ii})+\psi(A_{ij})+\psi(A_{ji})+\lambda_1P_i+\lambda_2P_j]_*\\
&=\psi([T_{ii},A_{ii}]_*)+\psi([T_{ii},A_{ij}]_*)+\psi([T_{ii},A_{ji}]_*)+\lambda_1(T_{ii}-T_{ii}^*)\\
&=\psi(T_{ii}A_{ii}-A_{ii}T_{ii}^*)+\psi(T_{ii}A_{ij})+\psi(-A_{ji}T_{ii}^*)+\lambda_1(T_{ii}-T_{ii}^*).
\end{aligned}$$

This implies that $\lambda_1 = \alpha_1 = \alpha_2 = 0$. On the other hand, by Eq. (26) and Lemma 2.7(d), we have for any $T_{jj} \in \mathscr{M}_{jj}$,

$$\begin{aligned}\psi(T_{jj}A_{ji}) + \psi(-A_{ij}T_{jj}^*) &= \psi(T_{jj}A_{ji} - A_{ij}T_{jj}^*) = \psi([T_{jj}, A_{ii} + A_{ij} + A_{ji},]_*) \\ &= [\psi(T_{jj}), A_{ii} + A_{ij} + A_{ji}]_* \\ &\quad + [T_{jj}, \psi(A_{ii} + A_{ij} + A_{ji})]_* \\ &= [\psi(T_{jj}), A_{ii} + A_{ij} + A_{ji}]_* \\ &\quad + [T_{jj}, \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}) + \lambda_1 P_i + \lambda_2 P_j]_* \\ &= \psi([T_{jj}, A_{ii}]_*) + \psi([T_{jj}, A_{ij}]_*) \\ &\quad + \psi([T_{jj}, A_{ji}]_*) + \lambda_2(T_{jj} - T_{jj}^*) \\ &= \psi(-A_{ij}T_{jj}^*) + \psi(T_{jj}A_{ji}) + \lambda_2(T_{jj} - T_{jj}^*).\end{aligned}$$

This implies that $\lambda_2 = 0$. Hence by Eq. (26), we have for $i \neq j$,

$$\psi(A_{ii} + A_{ij} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}). \tag{27}$$

Since $[T_{11}, \sum_{i,j=1}^2 A_{ij}]_* = [T_{11}, A_{11} + A_{12} + A_{21}]_*$ for all $T_{11} \in \mathscr{M}_{11}$, it follows from the fact $\psi(T_{11}) \in \mathscr{M}_{11}$ that

$$\left[T_{11}, \psi\left(\sum_{i,j=1}^{2} A_{ij}\right) - \psi(A_{11} + A_{12} + A_{21})\right]_* = 0.$$

By Lemmas 2.2, 2.6 and Eq. (27), then

$$P_1\psi\left(\sum_{i,j=1}^{2} A_{ij}\right)P_1 = \psi(A_{11}). \tag{28}$$

Also, we can show that

$$P_1\psi\left(\sum_{i,j=1}^{2} A_{ij}\right)P_2 = \psi(A_{12}), \quad P_2\psi\left(\sum_{i,j=1}^{2} A_{ij}\right)P_1 = \psi(A_{21}). \tag{29}$$

Since $[T_{22}, \sum_{i,j=1}^2 A_{ij}]_* = [T_{22}, A_{22} + A_{12} + A_{21}]_*$ for all $T_{22} \in \mathscr{M}_{22}$, it follows from the fact $\psi(T_{22}) \in \mathscr{M}_{22}$ that

$$\left[T_{22}, \psi\left(\sum_{i,j=1}^{2} A_{ij}\right) - \psi(A_{22} + A_{12} + A_{21})\right]_* = 0.$$

By Lemmas 2.2, 2.6 and Eq. (27), then

$$P_2\psi\left(\sum_{i,j=1}^{2} A_{ij}\right)P_2 = \psi(A_{22}). \tag{30}$$

Combining Eqs. (28)–(30), we have $\psi(\sum_{i,j=1}^2 A_{ij}) = \sum_{i,j=1}^2 \psi(A_{ij})$. The proof is completed. $\square$

Lemma 2.9. *Let ψ be as in Remark 2.1. Then $\psi(A_{ij} + B_{ij}) = \psi(A_{ij}) + \psi(B_{ij})$ for all $A_{ij}, B_{ij} \in \mathscr{M}_{ij}$, $i, j = 1, 2$.*

Proof. Let $i, j \in \{1, 2\}$ with $i \neq j$. Then $[T_{ij}, P_j]_* = T_{ij} - T_{ij}^*$ for all $T_{ij} \in \mathscr{M}_{ij}$, and so by Lemmas 2.6 and 2.7(d),

$$\begin{aligned}\psi(T_{ij}) + \psi(-T_{ij}^*) &= \psi(T_{ij} - T_{ij}^*) = [\psi(T_{ij}), P_j]_* + [T_{ij}, \psi(P_j)]_* \\ &= \psi(T_{ij}) - \psi(T_{ij})^* + T_{ij}\psi(P_j) - \psi(P_j)T_{ij}^*.\end{aligned}$$

Since $\psi(P_j) \in \mathbb{C}I$ and $\psi(-T_{ij}^*), \psi(T_{ij})^* \in \mathscr{M}_{ji}$, we have from above equation that $T_{ij}\psi(P_j) = 0$ for all $T_{ij} \in \mathscr{M}_{ij}$. Hence $\psi(P_j) = 0$ for $j = 1, 2$.

Let $A_{ij}, B_{ij} \in \mathscr{M}_{ij}$ $(i \neq j)$. Then

$$[P_i + A_{ij}, P_j + B_{ij}]_* = A_{ij} + B_{ij} - A_{ij}^* - B_{ij}A_{ij}^*,$$

and so by Lemmas 2.8, 2.7 and 2.6,

$$\begin{aligned}&\psi(A_{ij} + B_{ij}) + \psi(-A_{ij}^*) + \psi(-B_{ij}A_{ij}^*) \\ &\quad= \psi(A_{ij} + B_{ij} - A_{ij}^* - B_{ij}A_{ij}^*) = \psi([P_i + A_{ij}, P_j + B_{ij}]_*) \\ &\quad= [\psi(P_i + A_{ij}), P_j + B_{ij}]_* + [P_i + A_{ij}, \psi(P_j + B_{ij})]_* \\ &\quad= [\psi(P_i) + \psi(A_{ij}), P_j + B_{ij}]_* + [P_i + A_{ij}, \psi(P_j) + \psi(B_{ij})]_* \\ &\quad= [\psi(A_{ij}), P_j + B_{ij}]_* + [P_i + A_{ij}, \psi(B_{ij})]_* \\ &\quad= \psi(A_{ij}) + \psi(B_{ij}) - \psi(A_{ij})^* - B_{ij}\psi(A_{ij})^* - \psi(B_{ij})A_{ij}^*.\end{aligned}$$

This implies that

$$\psi(A_{ij} + B_{ij}) = \psi(A_{ij}) + \psi(B_{ij}) \tag{31}$$

for all $A_{ij}, B_{ij} \in \mathscr{M}_{ij}$ $(i \neq j)$. Let $A_{ii}, B_{ii} \in \mathscr{M}_{ii}$ and $T_{ij} \in \mathscr{M}_{ij}$ $(i \neq j)$. It follows from Eq. (31) and Lemma 2.6 that

$$\begin{aligned}\psi([A_{ii} + B_{ii}, T_{ij}]_*) &= \psi(A_{ii}T_{ij} + B_{ii}T_{ij}) = \psi(A_{ii}T_{ij}) + \psi(B_{ii}T_{ij}) \\ &= \psi([A_{ii}, T_{ij}]_*) + \psi([B_{ii}, T_{ij}]_*) \\ &= [\psi(A_{ii}), T_{ij}]_* + [A_{ii}, \psi(T_{ij})]_* \\ &\quad + [\psi(B_{ii}), T_{ij}]_* + [B_{ii}, \psi(T_{ij})]_*. \\ &= \psi(A_{ii})T_{ij} + A_{ii}\psi(T_{ij}) + \psi(B_{ii})T_{ij} + B_{ii}\psi(T_{ij}).\end{aligned}$$

On the other hand, we have from Lemma 2.6 that

$$\begin{aligned}\psi([A_{ii} + B_{ii}, T_{ij}]_*) &= [\psi(A_{ii} + B_{ii}), T_{ij}]_* + [A_{ii} + B_{ii}, \psi(T_{ij})]_* \\ &= \psi(A_{ii} + B_{ii})T_{ij} + A_{ii}\psi(T_{ij}) + B_{ii}\psi(T_{ij}).\end{aligned}$$

Hence $(\psi(A_{ii} + B_{ii}) - \psi(A_{ii}) - \psi(B_{ii}))T_{ij} = 0$ for all $T_{ij} \in \mathscr{M}_{ij}$. This implies that $\psi(A_{ii} + B_{ii}) = \psi(A_{ii}) + \psi(B_{ii})$ for all $A_{ii}, B_{ii} \in \mathscr{M}_{ii}$. The proof is completed. □

Lemma 2.10. *Let ψ be as in Remark 2.1 and let $A_{ii}, B_{ii} \in \mathscr{M}_{ii}$ and $A_{ij}, B_{ij} \in \mathscr{M}_{ij}$ $(i \neq j)$. Then*

(a) $\psi(A_{ii}B_{ii}) = \psi(A_{ii})B_{ii} + A_{ii}\psi(B_{ii})$, $\psi(A_{ij}B_{ji}) = \psi(A_{ij})B_{ji} + A_{ij}\psi(B_{ji})$;
(b) $\psi(A_{ii}B_{ij}) = \psi(A_{ii})B_{ij} + A_{ii}\psi(B_{ij})$, $\psi(A_{ij}B_{jj}) = \psi(A_{ij})B_{jj} + A_{ij}\psi(B_{jj})$.

Proof. (a) Let $X_{ij} \in \mathscr{M}_{ij}$. Then $A_{ii}X_{ij} = [A_{ii}, X_{ij}]_*$, and so by Lemma 2.6, we have

$$\psi(A_{ii}X_{ij}) = [\psi(A_{ii}), X_{ij}]_* + [A_{ii}, \psi(X_{ij})]_* = \psi(A_{ii})X_{ij} + A_{ii}\psi(X_{ij}).$$

This yields that

$$\begin{aligned}\psi(A_{ii}B_{ii})X_{ij} + A_{ii}B_{ii}\psi(X_{ij}) = \psi(A_{ii}B_{ii}X_{ij}) &= \psi(A_{ii})B_{ii}X_{ij} + A_{ii}\psi(B_{ii}X_{ij})\\ &= \psi(A_{ii})B_{ii}X_{ij} + A_{ii}\psi(B_{ii})X_{ij} + A_{ii}B_{ii}\psi(X_{ij}).\end{aligned}$$

Then $(\psi(A_{ii}B_{ii}) - \psi(A_{ii})B_{ii} - A_{ii}\psi(B_{ii}))X_{ij} = 0$ for all $X_{ij} \in \mathscr{M}_{ij}$. It follows that $\psi(A_{ii}B_{ii}) = \psi(A_{ii})B_{ii} + A_{ii}\psi(B_{ii})$. Since $A_{ij}B_{ji} = [A_{ij}, B_{ji}]_*$, we have from Lemma 2.6 that

$$\psi(A_{ij}B_{ji}) = [\psi(A_{ij}), B_{ji}]_* + [A_{ij}, \psi(B_{ji})]_* = \psi(A_{ij})B_{ji} + A_{ij}\psi(B_{ji}). \tag{32}$$

(b) Let $T_{ji} \in \mathscr{M}_{ji}$ $(i \neq j)$. It follows from Eq. (32) that

$$\begin{aligned}\psi(A_{ii}B_{ij})T_{ji} + A_{ii}B_{ij}\psi(T_{ji}) = \psi(A_{ii}B_{ij}T_{ji}) &= \psi(A_{ii})B_{ij}T_{ji} + A_{ii}\psi(B_{ij}T_{ji})\\ &= \psi(A_{ii})B_{ij}T_{ji} + A_{ii}\psi(B_{ij})T_{ji} + A_{ii}B_{ij}\psi(T_{ji}).\end{aligned}$$

Then $(\psi(A_{ii}B_{ij}) - \psi(A_{ii})B_{ij} - A_{ii}\psi(B_{ij}))T_{ji} = 0$ for all $T_{ji} \in \mathscr{M}_{ji}$. This implies that $\psi(A_{ii}B_{ij}) = \psi(A_{ii})B_{ij} + A_{ii}\psi(B_{ij})$. Similarly, we have

$$\begin{aligned}T_{ji}\psi(A_{ij}B_{jj}) + \psi(T_{ji})A_{ij}B_{jj} = \psi(T_{ji}A_{ij}B_{jj}) &= \psi(T_{ji}A_{ij})B_{jj} + T_{ji}A_{ij}\psi(B_{jj})\\ &= \psi(T_{ji})A_{ij}B_{jj} + T_{ji}\psi(A_{ij})B_{jj} + T_{ji}A_{ij}\psi(B_{jj}),\end{aligned}$$

and so $\psi(A_{ij}B_{jj}) = \psi(A_{ij})B_{jj} + A_{ij}\psi(B_{jj})$. The proof is completed. □

Now we are in a position to prove our main theorem.

Proof of Theorem 2.1. Let $A, B \in \mathscr{M}$. Then $A = \sum_{i,j=1}^{2} A_{ij}$ and $B = \sum_{i,j=1}^{2} B_{ij}$ for some $A_{ij}, B_{ij} \in \mathscr{M}_{ij}$. Let ψ be as in Remark 2.1. It follows from Lemmas 2.8 and 2.9 that

$$\psi(A + B) = \sum_{i,j=1}^{2} \psi(A_{ij} + B_{ij}) = \sum_{i,j=1}^{2} (\psi(A_{ij}) + \psi(B_{ij})) = \psi(A) + \psi(B).$$

By Lemmas 2.10 and 2.6,

$$\begin{aligned}\psi(AB) &= \psi(A_{11}B_{11}) + \psi(A_{11}B_{12}) + \psi(A_{12}B_{21}) + \psi(A_{12}B_{22})\\ &\quad + \psi(A_{21}B_{11}) + \psi(A_{21}B_{12}) + \psi(A_{22}B_{21}) + \psi(A_{22}B_{22})\\ &= \psi(A_{11})B_{11} + A_{11}\psi(B_{11}) + \psi(A_{11})B_{12} + A_{11}\psi(B_{12})\\ &\quad + \psi(A_{12})B_{21} + A_{12}\psi(B_{21}) + \psi(A_{12})B_{22} + A_{12}\psi(B_{22})\\ &\quad + \psi(A_{21})B_{11} + A_{21}\psi(B_{11}) + \psi(A_{21})B_{12} + A_{21}\psi(B_{12})\\ &\quad + \psi(A_{22})B_{21} + A_{22}\psi(B_{21}) + \psi(A_{22})B_{22} + A_{22}\psi(B_{22})\\ &= \psi(A_{11})B + A_{11}\psi(B) + \psi(A_{12})B + A_{12}\psi(B)\\ &\quad + \psi(A_{21})B + A_{21}\psi(B) + \psi(A_{22})B + A_{22}\psi(B)\\ &= \psi(A)B + A\psi(B).\end{aligned}$$

It follows that ψ is also an additive derivation. Let $A = B + iC$ where $B, C \in \mathscr{M}_{sa}$. By Lemma 2.4, then

$$\psi(A^*) = \psi(B) - \psi(iC) = \psi(B) - i\psi(C) = \psi(A)^*$$

for all $A \in \mathscr{M}$. Hence ψ is an additive $*$-derivation. The proof is completed. □

By Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let $\mathscr{H}$ be an infinite dimensional complex Hilbert space and $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a nonlinear $*$-Lie derivation. Then there exists $T \in \mathscr{B}(\mathscr{H})$ satisfying $T + T^* = 0$ such that $\phi(A) = AT - TA$ for all $A \in \mathscr{B}(\mathscr{H})$.*

Proof. It follows from Theorem 2.1 that ϕ is an additive $*$-derivation. By the result of [5], then ϕ is linear, and so it is inner. Thus there exists $S \in \mathscr{B}(\mathscr{H})$ such that $\phi(A) = AS - SA$ for all $A \in \mathscr{B}(\mathscr{H})$. Hence

$$A^*S - SA^* = \phi(A^*) = \phi(A)^* = S^*A^* - A^*S^*$$

for all $A \in \mathscr{B}(\mathscr{H})$. This implies that $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Set $T = S - \frac{1}{2}\lambda I$, then $T + T^* = 0$ and $\phi(A) = AT - TA$ for all $A \in \mathscr{B}(\mathscr{H})$. The proof is completed. □

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