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Sachs triangulations and regular maps

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Abstract

A Sachs triangulation of a closed surface S is a triangulation T admitting a vertex-labelling λ in a group G subject to the following conditions: (S1) For any facial triangle t of T with vertices x, y and z, either $\lambda(x)\lambda(y)\lambda(z)=1$ or $\lambda(x)\lambda(z)\lambda(y)=1$. (S2) For any $g, h \in G$, there exists at most one edge in T whose endpoints are labelled g and h.

In this paper we establish various sufficient conditions for a Sachs triangulation to be a regular (symmetrical) map. As an application of these results we construct, for each integer $d \geqslant 2$, a $2d$-valent reflexible symmetrical triangulation of genus $1+d(d-3)/2$.

1. Introduction

In this paper we are interested in constructing highly symmetrical triangulations of closed surfaces. There may be various approaches to this problem. It is hardly surprising, however, that all the known constructions of such triangulations are based on groups. For example, Surowski [6] describes two infinite families of vertex-transitive triangulations of orientable surfaces arising from $PSL(2,p)$, where p is a prime, and from $SL(2,2^r)$; these triangulations are regular maps illustrating the conjugacy map construction of [7]. Another possibility is to try to construct a symmetrical triangulation from a Cayley graph by taking Biggs' method of Cayley maps [1] (see also [4,5]). Here we shall follow the idea suggested by Sachs in [3]: Given a finite group G, take the set of all ordered triples $t=(x,y,z)$ in G such that $xyz=1$ and consider the triples obtained from t by a cyclic permutation of $x,, y, z$ as equal. Then represent each such triple by an oriented topological triangle. Finally, identify each triangle (x,y,z) with (y,x,z') along their xy-sides in the obvious way. This procedure results in a set of triangulated surfaces, which we call *Sachs triangulations*.

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Sachs triangulations have been extensively studied by Voss and Voss in [8–15]. We concentrate on conditions under which a Sachs triangulation attains the highest possible degree of symmetry, i.e., becomes a regular or a reflexible map. These questions seem not to have been considered so far. Also, our approach to Sachs triangulations is different from that in [8–15]. We view a Sachs triangulation as a triangulation endowed with an appropriate labelling of vertices by elements of a group. This approach yields not only new proofs of some basic results about Sachs triangulations obtained in [8–15] but, as a by-product, also a characterization of their label sets.

2. Sachs triangulations

Let T be a triangulation of a closed surface S, i.e., a 2-cell embedding $K \to S$ of a simple graph K in S such that each face is bounded by a 3-cycle of K. Such a 3-cycle will be called a *triangle* of T. Moreover, we shall not use the word triangle to refer to 3-cycles that are not facial.

Let G be a group. A triangulation $T: K \to S$ will be called a *Sachs triangulation* over G if there exists a labelling λ of the vertex set of K by the elements of G satisfying two conditions:

(S1) For each triangle t with vertices u, v and w, either $\lambda(u)\lambda(v)\lambda(w) = 1$ or $\lambda(u)\lambda(w)\lambda(v) = 1$, the unit element of G.

(S2) For any $g, h \in G$, there exists at most one edge in K whose endpoints are labelled g and h.

An example of a Sachs triangulation over the alternating group A_4 is depicted in Fig. 1.

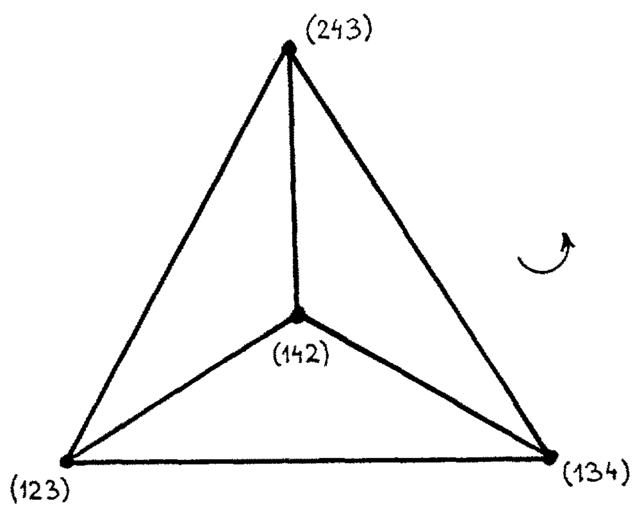

Fig. 1.

Note that if both $\lambda(u)\lambda(v)\lambda(w)=1$ and $\lambda(u)\lambda(w)\lambda(v)=1$, then $\lambda(u)\mu(v)=\lambda(w)^{-1}=\lambda(v)\lambda(u)$, and the converse is also true. In such a case, however, T is the trivial triangulation consisting of a single 3-cycle embedded in the 2-sphere. Indeed, if $t'\neq t$ is a triangle of T sharing the edge uv with t then for the label of the third vertex w' of t' we have $\lambda(w')=\lambda(w)$. But then the edges vw and vw' have endvertices labelled $\lambda(v)$, $\lambda(w)$, and (S2) implies that $w'=w$.

From now on we can therefore assume that for each edge uv of K, $\lambda(u)$ does not commute with $\lambda(v)$ (in particular, G is not Abelian). In this case exactly one of the equalities in (S1) holds for t, say $\lambda(u)\lambda(v)\lambda(w)=1$. Then also $\lambda(v)\lambda(w)\lambda(u)=1=\lambda(w)\lambda(u)\lambda(v)$. Thus the cyclic permutation (uvw) gives rise to an unambiguous orientation of t. In other words, each triangle of T is oriented in the sense in which the product of its labels is 1. This leads to the following observation.

Proposition 1. *Every Sachs triangulation is orientable.*

Proof. Suppose T is a Sachs triangulation on a nonorientable surface. Then there exist two adjacent oriented triangles of the form $t=(uvw)$ and $t'=(uvw')$, i.e., the orientations of t and t' are incoherent. Since $\lambda(u)\lambda(v)\lambda(w)=1=\lambda(u)\lambda(v)\lambda(w')$, we have $\lambda(w)=\lambda(w')$ and by (S2), $w=w'$. But then T is trivial and hence orientable, a contradiction. □

Let T be a Sachs triangulation over a group G with labelling λ. By $L(T)$ we denote the *label set* of T, i.e., the set $\{g\in G;\ g=\lambda(v),\ v \text{ a vertex of } T\}$. Take $x, y, z\in L(T)$ such that $xyz=1$. Then by virtue of (S2), there exists at most one triangle t in T whose vertices are labelled x, y, z consistently with the orientation of t. If such a triangle exists we shall denote it by $t=t(x, y, z)$.

There are essentially two ways of constructing Sachs triangulations. The first one consists in taking an orientable triangulation and trying to find a labelling satisfying (S1) and (S2). The other way is based on choosing a triple x, y, z of elements of a group G such that $xyz=1$ and trying to complete the 'initial' triangle $t(x, y, z)$ to a triangulation of a closed surface. We shall now pursue this idea in a greater detail.

Let G be a group. With each triple x, y, z of distinct elements of G such that $xyz=1$ we associate an oriented (topological) triangle $t=t(x, y, z)$ whose vertices are labelled x, y, z consistently with the orientations of t (see Fig. 2).

Let $\Delta(G)$ be the set of all such triangles $t(x, y, z)$. It is easy to see that each unordered pair $\{x, y\}$ of distinct elements of G appears on exactly two triangles, namely $t_1=t(x, y, (xy)^{-1})$ and $t_2=(y, x, (yx)^{-1})$. For any $x\neq y$ we shall now identify (i.e., glue together) the xy-sides of the triangles t_1 and t_2 consistently with their orientation (see Fig. 3). We note that the identification takes place only along 'edges' of triangles, i.e., the vertex corresponding to x in $t(x, y, z)$ is not necessarily the same as the vertex corresponding to x in a different triangle $t(x, y', z')$.

This identification procedure transforms $\Delta(G)$ into a set $\mathscr{S}$ of triangulated orientable surfaces, possibly noncompact. For each pair $\{x, y\}$ of distinct elements of G there

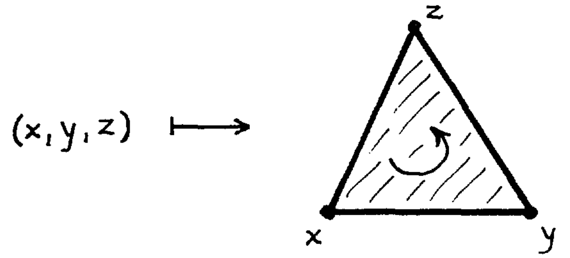

Fig. 2.

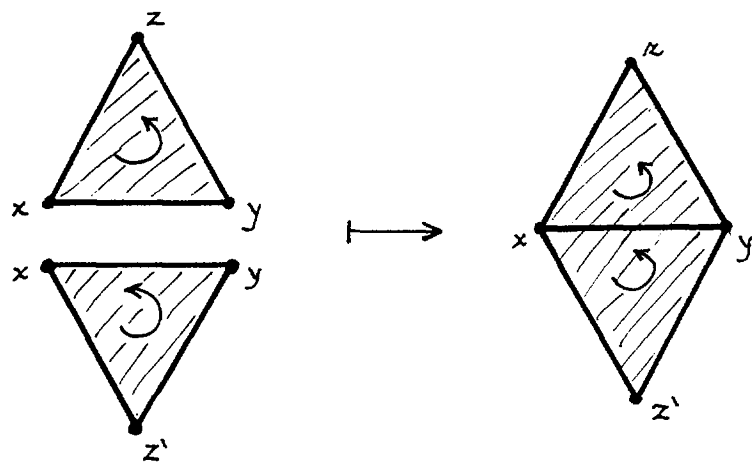

Fig. 3.

exists a unique triangulated surface $T(x, y)$ in $\mathcal{S}$ containing an edge with endpoints labelled x and y.

If the elements x and y commute then the triangles $t(x, y, (xy)^{-1})$ and $t(y, x, (yx)^{-1})$ have the same sides but different orientation. Thus their identification results in the sphere triangulated by a single 3-cycle labelled $x, y, (xy)^{-1} = (yx)^{-1}$, i.e., in the trivial triangulation.

Clearly, if the supporting surface of $T = T(x, y)$ is compact, then T is a Sachs triangulation. Therefore, the uniqueness of the triangulation $T(x, y)$ can now be restated as follows:

Proposition 2. *Let T and T' be two Sachs triangulations over a group G, each containing an edge with endpoints labelled g and h, $g, h \in G$. Then T and T' are isomorphic.*

To present an example of the set $\mathcal{S}$ take the group A_4, the alternating group on four elements $1, 2, 3, 4$. Nontrivial elements of A_4 consist of eight 3-cycles and three involutions, each containing two 2-cycles. Two distinct nontrivial elements commute

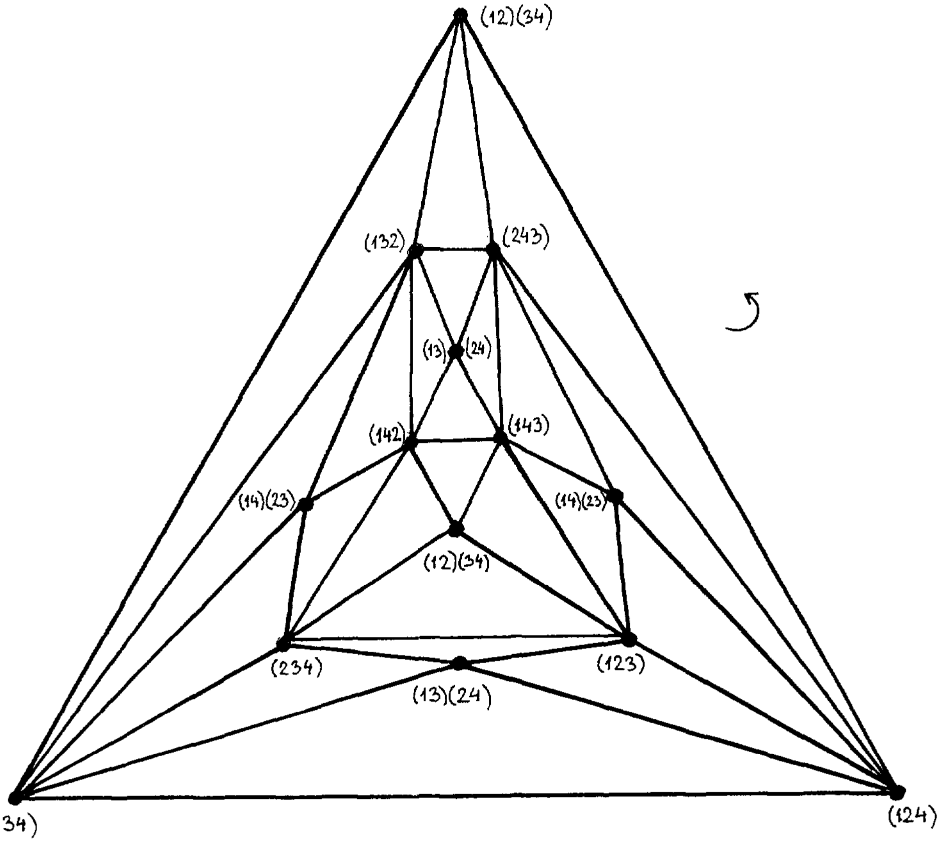

Fig. 4.

if and only if they are both involutions or they are mutually inverse 3-cycles. Since there are $\binom{11}{2}-7=48$ pairs of noncommuting elements in A_4, the triangulations in $\mathscr{S}$ contain 48 edges altogether. A more detailed analysis shows that $\mathscr{S}$ comprises three triangulations of the sphere, two of them being isomorphic to the one in Fig. 1 up to labelling; the remaining triangulation is displayed in Fig. 4.

3. Inner automorphisms and symmetries of Sachs triangulations

The examples presented in the previous section suggest that the group structure of a Sachs triangulation guarantees certain symmetries. Following [2], by an *oriented map* we mean a 2-cell embedding of a graph in an oriented surface. An *automorphism* of a map (not necessarily a triangulation) is an orientation preserving self-homeomorphism of the underlying surface which maps vertices to vertices, edges to edges, faces to faces and preserves the incidence.

In what follows we show that each Sachs triangulation admits map automorphisms induced by inner automorphisms of the group. To see this, let $T(x,y)$ be a Sachs triangulation corresponding to elements x and y of a group G. We know that $T(x,y)$ contains the triangles $t(x,y,z)$ and $t(y,x,z')$ where $z=(xy)^{-1}$ and $z'=(yx)^{-1}$. Now, routine calculations show that the cyclic ordering of labels appearing in the neighbourhood of the vertex of $t(x,y,z)$ labelled x is: $y, z, zyx^{-1}, xzx^{-1}, x^2yx^{-2}, \dots$ (see Fig. 5).

Let v_g denote the inner automorphism of G given by $a \to gag^{-1}$ for each $a \in G$. It is clear from Fig. 5 that v_x induces a rotation of the neighbourhood of the vertex labelled x by two triangles counterclockwise. In the following proposition we show that, in fact, this rotation extends to a map automorphism of $T(x,y)$.

Proposition 3. *Let $T=T(x,y)$ be a Sachs triangulation corresponding to elements x and y of a group G. Then the inner automorphism v_x of G induces a map automorphism A_x which maps each triangle $t(a,b,c) \in T$ to $t(v_x(a), v_x(b), v_x(c)) \in T$.*

Proof. First observe that if such a mapping A_x exists, then A_x is necessarily a map automorphism of $T(x,y)$, for it sends triangles sharing a common edge to triangles with the same property. It remains to prove the existence of A_x. However, to do this we only have to verify the following claim:

$$\text{If } t(a,b,c) \in T(x,y) \quad \text{then } t(v_x(a), v_x(b), v_x(c)) \in T(x,y). \qquad (*)$$

This claim is obvious for each triangle incident with the vertex of $t(x,y,z)$ labelled x (see Fig. 5), so we may proceed by induction. Let $t(a,b,c) \in T(x,y)$ be a triangle having a common edge with a triangle for which (*) has already been proved; say,

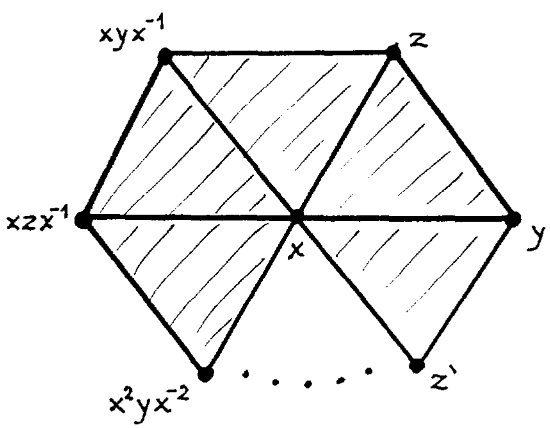

Fig. 5.

$t(b, a, c')$. By induction hypothesis, $t(v_x(b), v_x(a), v_x(c')) \in T(x, y)$. But then, by the definition of $T(x, y)$, also $t(v_x(a), v_x(b), d) \in T(x, y)$ where

$$d = (v_x(a)(v_x(b))^{-1} = (v_x(ab))^{-1} = (v_x(c^{-1}))^{-1} = (v_x(c).$$

This completes the induction step as well as the proof of Proposition 3. □

The above result guarantees the existence of certain symmetries in each Sachs triangulation. As we now show, it has at the same time interesting consequences on the structure of the label set of such a triangulation.

Corollary 4. *Let T be a Sachs triangulation over a group G. Then for each $g \in L(T)$ the inner automorphism v_g of G induces the map automorphism A_g of T described in Proposition* 1.

Proof. It is sufficient to realize that $T = T(g, h)$ for a suitable $h \in L(x, y)$. □

Corollary 5. *Let T be a Sachs triangulation. Then for each $g, h \in L(T)$ it holds that $ghg^{-1} \in L(T)$.*

Proof. By Corollary 4, if $h \in L(T)$ then also $v_g(h) \in L(T)$. □

For the sake of brevity we shall interpret Corollary 5 by saying that the label set of a Sachs triangulation is closed under *conjugation*. This immediately raises the question of characterizing those subsets of G which constitute the label set of a Sachs triangulation. The answer is given in our next theorem. To formulate the result, we define a *triplet* in a group G to be any subset $\{x, y, z\} \subseteq G$ with $xyz = 1$ or $xyz = 1$.

Theorem 6. *Let G be a group. A finite subset M of G forms the label set of a Sachs triangulation over G if and only if the following conditions are satisfied*:

(1) *M is closed under conjugation*;

(2) *M contains a triplet*; *and*

(3) *for any $M' \subseteq M$ such that M' is closed under conjugation and contains a triplet from M, it holds that $M' = M$.*

Proof. *Necessity.* Let T be a Sachs triangulation and let $M = L(T)$. We already know that M satisfies (1) and (2). Now let $M' \subseteq M$ be a set for which (1) and (2) hold. Let $\{a, b, c\}$ be a triplet with $abc = 1$ contained in both M' and M, and let $T' = T(a, b)$ be the corresponding Sachs triangulation. By induction we show that $L(T') \subseteq M'$. This is clear for the elements $a, b, c \in L(T')$. Now let $t(i, j, k)$ be a triangle in T' adjacent to, say, a triangle $t(j, i, k')$ for which it has already been proved that $i, j, k' \in M'$. Since $k = ik'i^{-1}$, the property (2) implies that $k \in M'$. This completes the induction step, i.e., $L(T') \subseteq M'$. But then, using the fact that $t(a, b, c)$ is in both T' and T we have $T' = T$, whence $M = L(T) = L(T') \subseteq M' \subseteq M$. Thus, (3) holds for M as well.

Sufficiency. Let $M \subseteq G$ be a set for which (1)–(3) is true. Take a triplet $\{a,b,c\}$ in M such that $abc=1$ and form the Sachs triangulation $T'=T(a,b)$. Obviously $t(a,b,c)$ is a triangle in T'. Using a part of the above proof of necessity it is clear that $L(T')\subseteq M$. Since $L(T')$ satisfies (1) and (2), the condition (3) applied to M yields $L(T')=M$. The proof is complete. □

In this context it is interesting to mention that the multiplicity of labels in a Sachs triangulation has been considered by Voss in [15]. The result reads as follows: Let $T=T(x,y)$ be a Sachs triangulation over a group G generated by x and y. Let $N_G(x)$ denote the normalizer of $x \in G$ and let k be the smallest positive integer such that x^k belongs to the centre $Z(G)$ of G. Then the number of vertices of T labelled x is $|N_G(x)|/k|Z(G)|$. Moreover, conjugate elements of G have the same multiplicity in $L(G)$.

4. Sachs triangulations and regular maps

Let T be a Sachs triangulation over a group G and let u be a vertex of T with label $x \in G$. As we have seen in Proposition 3, there is a map automorphism A_x of T which rotates the triangles incident with u two steps counterclockwise. Consider one of such triangles, say uvw, and let d_u, d_v and d_w be the valencies of the vertices u, v and w, respectively. The action of A_x on the neighbourhood of u shows that in the neighbourhood of u vertices with valencies d_v and d_w alternate. Since the same argument can be applied to any vertex adjacent to u, by induction it follows that the valency of every vertex in T is one of the numbers d_u, d_v or d_w. Moreover, if some of these values are odd, say d_u, then necessarily $d_v=d_w$ and hence T contains vertices of at most two valencies. An example of a triangulation with two different valencies is given in Fig. 4. Examples of triangulations with three different valencies can be found in [9].

In this section we concentrate on Sachs triangulations having more symmetries than those induced by inner automorphisms of the group G. In particular, we shall seek for conditions which would guarantee that a given Sachs triangulation is a regular map.

Recall that an oriented map M, not necessarily a triangulation, is called a *regular map* if for any two edges e and f of M with arbitrarily assigned directions there exists a map-automorphism of M sending e to f and preserving the assigned directions. As it is well known (e.g. [1,2]), an oriented map M with q edges has at most $2q$ automorphisms, and the equality is attained if and only if M is regular.

Our next theorem yields an extremely simple sufficient condition for a Sachs triangulation to be a regular map.

Theorem 7. *Let T be a Sachs triangulation with all vertices of odd valency. Then T is a regular map.*

Proof. Let uvw be a triangle of T and let x, y and z be the labels of u, v and w, respectively. In order to prove the theorem it is sufficient to exhibit a map automorphism of T which takes the directed edge uv to uw and another automorphism which takes uv to vu. Since the valency of u is odd, say $2d+1$, the automorphism $(A_x)^{d+1}$ sends v to w leaving u fixed. Thus $(A_x)^{d+1}(uv)=uw$, as required. In particulr, v and w have the same valency. Using the same argument with v in place of u, we conclude that all three valencies are equal to $2d+1$. Now it is easy to check that for the automorphism $A=(A_y)^{d+1}(A_z)^{d+1}(A_x)^{d+1}$ we have $A(uv)=vu$. Since the triangle uvw was arbitrary, Theorem 7 follows. □

A similar theorem with all vertices of the same even valency does not hold. An example can be seen in Fig. 4.

Our next aim is to derive a sufficient condition for the regularity of a Sachs triangulation in algebraic terms. To do this we introduce certain bijections on the label set $L(T)$ of a Sachs triangulation T which comply with its group structure.

A bijection $\varphi: L(T) \to L(T)$ will be called *triangular* if for each triangle $t(x,y,z) \in T$ it holds that $\varphi(x)\varphi(y)\varphi(z)=1$. Similarly, a bijection $\varphi: L(T) \to L(T)$ *will be called antitriangular* if for each triangle $t(x,y,z) \in T$ we have $\varphi(x)\varphi(z)\varphi(y)=1$. Note that each automorphism of the group G which fixes $L(T)$ setwise is a triangular bijection of $L(T)$. This obviously need not be the case with antitriangular bijections.

We start with a technical result which, in some cases, enables to extend triangular bijections to map automorphisms.

Proposition 8. *Let T be a Sachs triangulation over a group G and let φ be a triangular bijection on $L(T)$. Assume that T contains a triangle $t(a,b,c)$ such that $t(\varphi(a),\varphi(b),\varphi(c))$ is again a triangle in T. Then there exists a map automorphism A_φ of T which sends a triangle $t(x,y,z) \in T$ to the triangle $t(\varphi(x),\varphi(y),\varphi(z)) \in T$.*

Proof. First we show that for each $t(x,y,z) \in T$ we have $t(\varphi(x),\varphi(y),\varphi(z)) \in T$. This is obvious for the triangle $t(a,b,c)$, so we may proceed by induction. As in the preceding proofs, let $t=t(x,y,z)$ be a triangle of T sharing a common edge, say xy, with a triangle $t(y,x,z')$ for which $t(\varphi(y),\varphi(x),\varphi(z')) \in T$. Since φ is a triangular bijection on $L(T)$, $t_\varphi=t(\varphi(x),\varphi(y),\varphi(z))$ is a triangle in some Sachs triangulation over G. But t_φ has an edge in common with the triangle $t(\varphi(y),\varphi(x),\varphi(z'))$ which belongs to T, so t_φ itself belongs to T. This completes the induction step.

Now it is clear that the mapping A_φ is a bijection on the set of triangles of T preserving incidence and orientation. This means that A_φ is a map-automorphism of T. □

The reader may have noticed that Proposition 3 is a special case of Proposition 8 while the argument used in its proof is not.

The latter result enables to establish the following sufficient condition for a Sachs triangulation to be regular map.

Theorem 9. *Let T be a Sachs triangulation over a group G. Suppose that T contains a triangle $t(a,b,c)$ such that the cyclic permutation $a \to b \to c \to a$ and the transposition $a \to b \to a$ both extend to triangular bijections on $L(T)$. Then T is a regular map.*

Proof. Let φ be a triangular bijection on $L(T)$ such that $\varphi(a)=b$, $\varphi(b)=c$, and $\varphi(c)=a$ and let ψ be a triangular bijection for which $\psi(a)=b$ and $\psi(b)=a$. Since $t(\varphi(a),\varphi(b),\varphi(c))=t(a,b,c)$ from Proposition 8 it follows that φ extends to a map automorphism A_φ of T which rotates the triangle $t=t(a,b,c)$ by $2\pi/3$ around its centre consistently with the orientation of the surface. On the other hand, $t(\psi(a),\psi(b),\psi(c))=t(\psi(a),\psi(b),\psi(ab)^{-1})$. Since ψ is a triangular bijection, we have $\psi(a)\psi(b)\psi(ab)^{-1})=1=ba\psi((ab)^{-1})$, whence $\psi((ab)^{-1})=(ba)^{-1}$. Thus $t(\psi(a),\psi(b),\psi(c))=t(b,a,(ba)^{-1})\in T$ and by the same proposition ψ extends to a map automorphism A_φ of T which rotates the ab-side of $t(a,b,c)$ by π around its centre.

As it is well known in the theory of regular maps [1,2], to prove that T is a regular map it is sufficient to exhibit an automorphism Q sending the directed edge ab of this particular triangle abc to ac, and another automorphism R sending ab to ba. Clearly, it is sufficient to take $Q=(A_\varphi)^{-1}A_\psi$ and $R=A_\psi$. This completes the proof. □

It is not clear whether the converse to Theorem 9 holds true. Our next result yields at least a partial converse.

Theorem 10. *Let T be a Sachs triangulation without repeated labels. Then each map-automorphism A of T induces a triangular-bijection α on $L(T)$ by $\alpha(\lambda(u))=\lambda A(u)$ for every vertex u of T.*

Proof. By our assumption the labelling λ of T is a bijection from the vertex-set of T into $L(T)$. Thus the mapping $\alpha: L(T)\to L(T)$ given by $\alpha(\lambda(u))=\lambda A(u)$, u a vertex of T, is a well defined bijection. The triangular property of α immediately follows from the fact that A is a map automorphism. □

Combining the two preceding results we obtain the following corollary.

Corollary 11. *Let T be a Sachs triangulation without repeated labels. Then T is a regular map if and only if there exist triangular bijections φ and ψ and a triangle $t(a,b,c)$ of T such that φ: $a \to b \to c \to a$ and ψ: $a \to b \to a$.*

A *reflection* of a map (not necessarily a triangulation) is a self-homeomorphism of the underlying surface which maps vertices to vertices, edges to edges, faces to faces and preserves the incidence but reverses the orientation. A regular map is *reflexible* if it admits a least one reflection. As the reader may expect, the relationship between reflections of a Sachs triangulation and antitriangular bijections of its label set is similar to the relationship between map-automorphisms and triangular bijections.

We only state the corresponding results; their proofs are similar to those of Proposition 8 and Theorems 9 and 10.

Proposition 12. *Let T be a Sachs triangulation and let σ be an antitriangular bijection $L(T) \to L(T)$. If T contains a triangle $t(a,b,c)$ such that $t(\sigma(a),\sigma(b),\sigma(c)) \in T$ then there exists a reflection B_σ of T which sends $t(a,b,c)$ to $t(\sigma(a),\sigma(c),\sigma(b))$.*

Theorem 13. *Let a Sachs triangulation $T = T(a,b)$ be a regular map. If the transposition $a \to b \to a$ extends to an antitriangular bijection on $L(T)$ then T is reflexible. Conversely, if T is regular and reflexible and does not contain repeated labels such an antitriangular bijection exists.*

5. Regular maps from nilpotent groups

As an application of the above general results we first prove the following theorem:

Theorem 14. *Let $T = T(a,b)$ be a Sachs triangulation over a group G such that the commutator $[a,b]$ belongs to the centre $Z(G)$ of G. Then T is a reflexible regular map. Moreover, T is uniquely determined by the order of $[a,b]$.*

Proof. Take the triangle $t(a,b,c) \in T$, where $c = (ab)^{-1}$. According to Theorems 9 and 13 it is sufficient to show that the permutation $a \to b \to c \to a$ extends to a triangular bijection and that the transposition $a \to b \to a$ extends both to a triangular and an antitriangular bijection on $L(T)$. Before doing this we shall investigate the structure of the set $L(T)$.

Let $h = [a,b]$. Since $h \in Z(G)$, we have

$$[b,c] = b(b^{-1}a^{-1})b^{-1}(ab) = a^{-1}b^{-1}ab = a^{-1}b^{-1}ab(a^{-1}b^{-1}ba)$$
$$= a^{-1}b^{-1}[a,b]ba = a^{-1}b^{-1}hba = h \cdot a^{-1}b^{-1}ba = h.$$

Similarly, $[c,a] = h$. This symmetry property will be useful in subsequent considerations.

We shall prove that the vertices of each triangle of T have labels ah^k, bh^l and ch^m for suitable integers k, l, and m. This is obvious for the triangle $t(a,b,c)$. Let $t(x,y,z)$ be a triangle such that for the adjacent triangle $t(y,x,z')$ we already know that, say, $x = ah^k$, $y = bh^l$, and $z' = ch^m$. But then, recalling that $h \in Z(G)$, we have

$$z = (xy)^{-1} = (ah^k bh^l)^{-1} = (ab)^{-1} h^{-k-l} = ch^{-k-l}.$$

By induction, the labels of each triangle in T have the required form. Thus, $L(T) = \{ah^k, bh^1, ch^m;\ k,l,m \text{ integers}\}$.

Next we prove that for any integers k and l we have $ah^k \neq bh^l$. If this is not the case, then $b=ah^{k-l}$, which together with the fact that $h \in Z(G)$ implies that a and b commute. But then $h=1$ and $x=y$, a contradiction. By the symmetry mentioned above, $bh^l \neq ch^m$ and $ch^m \neq ah^k$ for any k, l and m.

To show that T is a regular map, we employ Theorem 9. Define the mapping φ on $L(T)$ as follows: $\varphi(a)=b$, $\varphi(b)=c$, $\varphi(c)=a$ and $\varphi(xh^k)=\varphi(x)h^k$ for $x \in \{a,b,c\}$ and any integer k. Using the preceding facts it is clear that φ is a bijection $L(T) \to L(T)$. To show that φ is triangular take a triangle in T, say $t(ah^k, bh^l, ch^m)$. Since $abc = ah^k bh^l ch^m$, we obtain $h^{k+l+m}=1$. Consequently,

$$\varphi(ah^k)\varphi(bh^l)\varphi(ch^m) = bh^k ch^l ah^m = bcah^{k+l+m} = bca = 1,$$

and φ is triangular.

To define ψ on $L(T)$ put $\psi(ah^k)=bh^{-k}$, $\psi(bh^l)=ah^{-l}$, and $\psi(ch^m)=\psi((ab)^{-1}h^m)=ch^{1-m}$. As above, ψ is a bijection $L(T) \to L(T)$. Let t be a triangle of T, this time say $t=t(bh^l, ah^k, c'h^m)$. Note that

$$c' = (ba)^{-1} = a^{-1}b^{-1} = (b^{-1}a^{-1}ab)a^{-1}b^{-1} = b^{-1}a^{-1}[a,b] = ch.$$

Now,

$$\begin{aligned}\varphi(bh^l)\varphi(ah^k)\varphi(c'h^m) &= \varphi(bh^l)\varphi(ah^k)\varphi(ch^{m+1})\\ &= ah^{-l}b^{-k}hch^{-m} = abch^{-(k+l+m)} = 1.1 = 1.\end{aligned}$$

Thus ψ is a triangular bijection, as well.

Now, it is readily seen that $t(\varphi(a), \varphi(b), \varphi(c)) = t(a,b,c) \in T$ and $t(\psi(a), \psi(b), \psi(c)) = t(b,a,ch) = t(b,a,c') \in T$. By virtue of Theorem 9, the triangulation T is a regular map.

We further show that T is reflexible. Define the mapping σ on $L(T)$ by setting $\sigma(ah^k)=ah^k$, $\sigma(bh^l)=ch^l$ and $\sigma(ch^m)=bh^m$. Obviously, σ is a bijection $L(T) \to L(T)$. If, say, $t(ah^k, bh^l, ch^m)$ is a triangle in T then

$$\sigma(ah^k)\sigma(ch^m)\sigma(bh^l) = ah^k bh^m ch^l = ah^k bh^l ch^m = 1.$$

Therefore σ is antitriangular. Moreover, since $t(\sigma(a), \sigma(b), \sigma(c)) = t(a,b,c)$ belongs to T, Proposition 12 and Theorem 13 imply that T is reflexible.

To finish the proof it remains to show that if $T=T(a,b)$ and $T'=T(a',b')$ are two Sachs triangulations over a group G and G', respectively, such that $h=[a,b] \in Z(G)$, $h'=[a',b'] \in Z(G)$ and both h and h' have the same order, then T and T' are isomorphic maps. But this can easily be proved by the inductive method used throughout. □

Routine calculations show that in a Sachs triangulation satisfying the assumptions of Theorem 14 the vertices in the neighbourhood of the vertex labelled a in the triangle $t(a,b,c)$ are successively labelled $b, c, hb, h^{-1}c, h^2b, h^{-2}c, \ldots,$. Combining this with the facts observed in the course of the previous proof we deduce that T is a $2d$-valent map, where d is the order of $h=[a,b]$ in G. The number of triangles in T is twice the number of ordered pairs (ah^k, bh^l), where $0 \leqslant k, l \leqslant d-1$, i.e., $2d^2$. Thus T has $3d$ vertices and $3d^2$ edges. Using the Euler formula we obtain that the genus of T is equal to $1+d(d-3)/2$.

The condition imposed on the group $G=\langle a,b\rangle$ in Theorem 14 is obviously very restrictive. It is therefore natural to ask whether it can be replaced by some weaker or more natural group-theoretical condition. In fact, a very simple condition of this type is at hand. If G is a nilpotent group of class 2 then the whole commutator subgroup $[G,G]$ is contained in $Z(G)$. In particular, $[a,b]\in Z(G)$, which implies the following result:

Theorem 15. *Let $G=\langle a,b\rangle$ be a finite nilpotent group of class 2. Then $T=T(a,b)$ is a regular and reflexible map.*

For nilpotent groups of higher classes, however, the theorem does not hold. It is sufficient to take the dihedral group $D(2^m)=\langle a,b;\, a^{2^m}=b^2=(ab)^2+1\rangle$, $m\geqslant 2$, which is nilpotent of class m. The underlying graph K of $T(a,b)$ consists of a 2^m-cycle with 4-valent vertices labelled $b, ab, a^2b, \ldots, a^{2^m-1}b$, and two additional nonadjacent vertices labelled a and a^{-1} connected with every vertex on the cycle. For $m\geqslant 3$ the graph K is not regular, so $T(a,b)$ is not a regular map unless $m=2$.

There is still another way of modifying Theorem 14. An analysis of its proof yields that the result only depends on the fact that for the triangle $t(a,b,c)$, where $c=(ab)^{-1}$, the commutators $[a,b]$, $[b,c]$ and $[c,a]$ are all equal. Thus, by replacing the assumption that $[a,b]\in Z(G)$ by the condition that $[a,b]=[b,c]=[c,a]$ we obtain another version of Theorem 14. It may be somewhat surprising that all the three versions considered are in fact equivalent. This follows from the next group-theoretical result.

Theorem 16. *Let $G=\langle a,b\rangle$ be a finite group. Then, the following three statements are equivalent:*

(a) *G is nilpotent of class 2.*

(b) $[a,b]\in Z(G)$.

(c) *If $abc=1$, then $[a,b]=[b,c]=[c,a]$.*

Proof. ((a)⇔(b)) Using the lower central series of G it is easy to see that a group G is nilpotent of class two if and only if $[G,G]\subseteq Z(G)$. Thus, the implication (a)⇒(b) is immediate. For the converse, it is sufficient to show that if $G=\langle a,b\rangle$ and $[a,b]\in Z(G)$, then $[G,G]\subseteq Z(G)$. In doing this, two well-known commutator identities will be helpful:

$$[xy,z]=[y,z]^x[x,z], \tag{1}$$

$$[x,yz]=[x,y][x,z]^y. \tag{2}$$

(As usual, x^a stands for axa^{-1}.) We proceed in two steps. The first one will be to establish the following claim:

$$\text{If } x\in\{a,b\} \text{ and } w\in G, \quad \text{then } [w,x]\in Z(G). \tag{3}$$

We employ induction on the length $|w|$ of w expressed as a word over the alphabet $\{a,b\}$. (This is possible since a and b have finite order.) The statement (3) is obviously true if $|w|=1$. Assume that (3) holds for all words of length $\leqslant m$ and let w be a word of length $m+1$. Then $w=va$ or $w=vb$ for some word v with $|v|=m$. Let $x=a$. The identity (1) and the assumption (b) then imply that

$$[va,a]=[a,a]^v[v,a]=[v,a]$$

and

$$[vb,a]=[b,a]^v[v,a]=([a,b]^{-1})^v[v,a]=[a,b]^{-1}[v,a].$$

Since $[v,a]\in Z(G)$ by the induction hypothesis, and $[a,b]\in Z(G)$ by the assumption (b), in both cases we obtain that $[w,a]\in Z(G)$. The proof that $[w,b]\in Z(G)$ can be obtained by simply interchanging a and b in the above considerations. Thus (3) is proved.

To finish the proof of the implication (b)$\Rightarrow$(a) it remains to prove the following:

If $u,w\in G$, *then* $[u,w]\in Z(G)$. (4)

Again, we employ induction on $|w|$. If $|w|=1$, then (4) reduces to (3). So we assume that (4) holds for all words of length $\leqslant m$ and let w be a word of length $m+1$. As above, $w=va$ or $w=vb$. The identity (2) now implies that

$$[u,va]=[u,v]\,[u,a]^v.$$

Since $[u,a]\in Z(G)$ by (3), we have $[u,a]^v=[u,a]\in Z(G)$. Moreover, $[u,v]\in Z(G)$ by the induction hypothesis. Hence, $[u,va]\in Z(G)$, and similarly $[u,vb]\in Z(G)$. This concludes the induction step as well as the proof of (b)$\Rightarrow$(a).

((b)$\Leftrightarrow$(c)) First assume that (b) holds. We show that $[b,c]=[a,b]=[c,a]$. Substituting $(ab)^{-1}$ for c in $[b,c]$ and $[c,a]$ we obtain that $[b,c]=[a^{-1},b^{-1}]$ and $[c,a]=[b^{-1},a]$. Since $[a,b]\in Z(G)$ it follows that

$$[c,a]=[b^{-1},a]=[b^{-1},a]b^{-1}b=[a,b]^{b^{-1}}=[a,b]$$

and

$$[b\,c]=[a^{-1},b^{-1}]=[a^{-1},b^{-1}]a^{-1}a=[b^{-1},a]^{a^{-1}}=[c,a]^{a^{-1}}=[a,b]^{a^{-1}}=[a,b].$$

Thus $[c,a]=[a,b]=[b,c]$, as required.

Conversely, assume that (c) holds. From our assumption it follows that $[a,b]=[a^{-1},b^{-1}]=[b^{-1},a]$. Hence,

$$a[a,b]=a[a^{-1},b^{-1}]=b^{-1}ab=b^{-1}aba^{-1}a=[b^{-1},a]a=[a,b]a$$

and

$$b[a,b]=b[b^{-1},a]=aba^{-1}=aba^{-1}b^{-1}b=[a,b]b.$$

Thus $[a,b]$ commutes with both a and b and consequently with every word over $\{a,b\}$. Since $G=\langle a,b\rangle$ is finite, it commutes with every element of G, i.e., $[a,b]\in Z(G)$. This completes the proof. □

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