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The asymptotic numbers of certain kinds of regular toroidal maps

J.M. Szucs

Texas A&M, University at Galveston, P.O. Box 1675, Galveston, TX 77553, USA

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Abstract

The asymptotic number (of isomorphism classes) of toroidal maps of type $(3,6)$ with at most n vertices is found, together with the fraction of those with multiplicity 1. Accurate lower and upper asymptotic estimates are provided for the number of toroidal maps of type $(3,6)$ with a Hamiltonian normal cycle and with at most n vertices. The case of type $(6,3)$ toroidal maps follows by duality. Similar results are obtained for toroidal maps of type $(4,4)$. (Type (p,q)= partition into p-gons, q edges incident to each vertex; normal cycle in a map of type $(3,6)=$ a cycle that leaves, at each of its vertices, exactly two edges on the right; multiplicity of a toroidal map of type $(3,6)=$ the greatest common divisor of the numbers of the three kinds of normal cycles.)

1. Introduction

Toroidal hexagonal graphs (that is, *toroidal polyhexes*) are important in the chemistry of polymers as is attested by the recent papers [4,12] among many other articles. Such graphs are also interesting from a purely mathematical point of view. They were studied in this journal by Altshuler [1,2] and by Kurth [14] under the names of '*regular maps of type* $(6,3)$ *on the torus*' and '*Platonic maps of type* $(6,3)$ *on the torus*'. Negami [15] called the dual graphs '6-regular torus graphs'. Concerning the theory of maps on surfaces, we refer the reader to the book by Coxeter and Moser [6].

A *map* on a surface is a finite graph imbedded on the surface, such that each component of the surface without the graph (that is, each face of the map) is homeomorphic to an open disc. Any vertex of the graph is allowed to be connected with any (not necessarily distinct) vertex by any number of edges. The *dual* map has a vertex inside each face of the original map, and these vertices are connected, without crossings, across common edges of the faces. A map on a surface is said to be *regular* (or Platonic [14]), more precisely, (regular) of *type* (p,q), if it is defined by a cellular

E-mail address: szucsj@tamug.tamu.edu (J.M. Szucs)

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decomposition of the surface into p-gons in such a way that each vertex is incident to exactly q edges (loops are counted twice). On a given surface, types (p,q) and (q,p) are dual to each other. On a torus only the types $(6,3),(3,6)$ and $(4,4)$ exist [2, p. 201]. The type $(3,6)$ is called triangular, and $(4,4)$, quadrangular.

An *isomorphism* between two toroidal maps is a homeomorphism between two tori that sends vertices to vertices, edges to edges, faces to faces, and preserves incidence. Two isomorphic toroidal maps are also isomorphic as graphs and the converse is true [15] for *simple* (= without loops and at most one edge between any two vertices) maps of type $(3,6)$. (For type $(6,3)$, see [11].) In this paper we consider maps up to isomorphisms. In the *asymptotic* formulas it would make no difference if we identified graph-isomorphic rather than isomorphic maps, and in the exact formulas, very little.

Kurth [14] determined the number $N(n)$ (of isomorphism classes) of toroidal maps of type $(3,6)$ with n vertices. By duality, the number of toroidal maps of type $(6,3)$ with k vertices is $N(k/2)$. In chemistry, this is the number of structural (or connectional) isomers for polyhex molecular tori ('bucky tori') with k atoms and is determined for many k empirically in [12]. It follows from Kurth's result that $N(n)$ is a linear combination, with coefficients depending on n in a simple manner, of the sum $\sigma(n)$ of divisors of n, the number $d(n)$ of divisors of n, and the number of divisors of that factor $r_{3,1}(n)$ of n which contains the powers of those prime divisors of n that are congruent to 1 modulo 3. An immediate consequence is that $\sigma(n)/6 \leqslant N(n) \leqslant \sigma(n)/6 + 4d(n)/3$. We obtain from this by classical number theory that the number $\bar{N}(n)$ of toroidal maps of type $(3,6)$ with at most n vertices satisfies the asymptotic equality $\bar{N}(n) = (\pi^2/72)n^2 + \mathrm{O}(n\log n)$. That is, $\bar{N}(n)$ and $\sigma(n)/6$ have the same asymptotics. Note that $\pi^2/72 \approx 0.1371$.

In [2] Altshuler gives the estimate $(n+7)/6 \leqslant N_h(n) \leqslant n/2$ for the number $N_h(n)$ of toroidal maps of type $(3,6)$ with $n, n>3$, vertices and with a Hamiltonian normal cycle. A cycle in a toroidal map of type $(3,6)$ is said to be *normal* if it leaves, at each of its vertices, exactly two edges on the right (left) [1,2]. A *Hamiltonian* cycle is a cycle containing all vertices of the graph. We find $N_h(n)$ and the estimate $0.12489n^2 + \mathrm{O}(n\log n) \leqslant \bar{N}_h(n) \leqslant 0.12491n^2 + \mathrm{O}(n\log n)$ for the number $\bar{N}_h(n)$ of toroidal maps of type $(3,6)$ with at most n vertices and with a Hamiltonian normal cycle. Hence roughly 91% of toroidal maps of type $(3,6)$ have a Hamiltonian normal cycle. We have not found the exact asymptotics of $\bar{N}_h(n)$ (if it exists). However, our lower and upper estimates seem to close in on an exact asymptotic estimate.

Every toroidal map of type $(3,6)$ decomposes into nonintersecting normal cycles in three different ways [1,2]. We define the multiplicity of such a map as the greatest common divisor of the numbers of cycles in these three families. We prove that the number of maps of multiplicity 1 with at most n vertices is $\bar{N}_1(n) = 5n^2/(4\pi^2) + \mathrm{O}(n\log n)$, where $5/(4\pi^2) \approx 0.1267$. It follows that approximately 92% of maps are of multiplicity 1 and almost 99% of the latter have a Hamiltonian normal cycle. This explains some of the empirical findings of Kirby and Pollak [12], who performed a study of $N(n)$ for n up to 5000.

Altshuler [1,2], Negami [15], Kurth [14], Kirby et al. [11], and Kirby and Pollak [12] assign a set of codes to every regular toroidal map of a given type. The conversion between these codings is simple. In any coding, isomorphic maps have the same codes and nonisomorphic maps do not share a code. A generic map of type $(3,6)$ has 6 codes. Every nontrivial automorphism of such a map (that is, one that permutes the three families of nonintersecting normal cycles) induces the coincidence of the codes of the map in pairs or in triples (according as the order of the permutation is 2 or 3) and vice versa. Several results in the papers cited rely on the study of this coincidence.

The next section includes the notation and also the numbers of solutions of certain quadratic residues, needed later. Section 3 provides a coordinate-free treatment of the codes of toroidal maps of type $(3,6)$. In Section 4 we deduce, from Kurth's formula, a simple formula for $N(n)$, the number of isomorphism classes of toroidal maps of type $(3,6)$ with n vertices. The asymptotic expression for $\bar{N}(n)$ follows easily by classical number theory. Section 5 is inspired by Kirby and Pollak [12]. In this section we study the asymptotics of the linear regression of $N(n)$ on n. In Section 6 we discuss the multiplicity of a toroidal map of type $(3,6)$ and determine the asymptotics of $\bar{N}_1(n)$, the number of type $(3,6)$ maps with multiplicity 1 and with at most n vertices. Section 7 is inspired by Altshuler [2]. In this section we determine the number $N_h(n)$ of nonisomorphic toroidal maps of type $(3,6)$ with n vertices and with a Hamiltonian normal cycle. A sequence of upper and lower asymptotic estimates of $\bar{N}_h(n)$ is obtained. These estimates seem to close in on $\bar{N}_h(n)$. Section 8 is devoted to toroidal maps of type $(4,4)$. Their theory is similar to but simpler than that of toroidal maps of type $(3,6)$. The main difference is that an exact asymptotic estimate is found for $\bar{N}_h(n)$, as well.

Familiarity with elementary number theory, including linear and quadratic congruences, is assumed. We also use the infinite product representation of the Riemann zeta function.

2. Notation and the congruences $x^2 + jx + k \equiv 0 \pmod{n}$

Let n be a positive integer. We use the following arithmetical functions:

$\sigma(n)$ is the sum of positive divisors of n;
$d(n)$ is the number of positive divisors of n;
$\phi(n)$ is the number of integers $c, 1 \leqslant c \leqslant n$, relatively prime to n;
$\hat{\phi}(n)$ is the number of integers $c, 1 \leqslant c \leqslant n$, such that neither c nor $c-1$ is relatively prime to n;
$\rho_{jk}(n)$ is the number of residues solving $x^2 + jx + k \equiv 0 \pmod{n}$;
$\gamma(n)$ is equal to 1 if n is a square, otherwise $\gamma(n) = 0$;
$p(n)$ is the number of (positive) prime divisors of n;
$r_{jk}(n)$ is the largest factor of n with prime divisors $\equiv k \pmod{j}$.

The asterisk $*$ denotes Dirichlet convolution: $(f*g)(n)=\sum_{d|n} f(d)g(n/d)$. It is an exercise in the theory of quadratic residues to show that

$$\begin{aligned}
&\rho_{0,-1}(n)=2^{p(n/r_{2,0}(n))+\mathrm{sign}(\log_2 r_{2,0}(n)-2)+1},\\
&\rho_{\pm 1,1}(n)=2^{p(r_{3,1}(n))} \text{ if } n/r_{3,1}(n)=1,3;\ \text{otherwise } \rho_{\pm 1,1}(n)=0,\\
&\rho_{0,1}(n)=2^{p(r_{4,1}(n))} \text{ if } n/r_{4,1}(n)=1,2;\ \text{otherwise } \rho_{0,1}(n)=0.
\end{aligned} \tag{1}$$

The values of $\rho_{1,1}$ and $\rho_{0,1}$ at prime-powers are in [14, Lemma 4.1]. Since the ρ's are multiplicative [3, Theorem 5.28], the values given above are obtained easily. The solutions of $x^2 \pm jx + k \equiv 0 \pmod n$ are transformed into each other by the involution $x \to -x$; consequently, $\rho_{j,k}(n)=\rho_{-j,k}(n)$. We obtain the value of $\rho_{0,-1}$ at a prime-power directly, and then invoke the multiplicativity of $\rho_{0,-1}$ to obtain its value at any number.

Each $\rho(n)$ in (1), not just the first one, can be given by a single formula, using the arithmetical functions above. However, these single formulas are quite bulky. For example, $\rho_{\pm 1,1}(n)=2^{p(r_{3,1}(n))}[1-\mathrm{sign}(r_{3,-1}(n)-1)][1-\mathrm{sign}(|r_{3,0}(n)-2|-1)]$. It is easy to see that

$$\rho_{0,\pm 1}(n),\rho_{\pm 1,1}(n)\leqslant d(n). \tag{2}$$

3. Codes of toroidal maps of type (3, 6)

The isomorphism classes of toroidal maps of type $(3,6)$ and of more general maps have been studied by many authors [1,7,9–12,14,15]. A set of codes is assigned to each isomorphism class in such a way that each code uniquely determines the isomorphism class. The difficulty in counting isomorphism classes lies in the circumstance that the number of distinct codes varies from class to class. In this section we offer a coordinate-free treatment of codes of toroidal maps of type $(3,6)$. Our codes coincide (modulo duality) with those in [11,12]. Codes are not used in the next section.

The notion of a *normal cycle*, introduced by Altshuler [1], is essential in the theory of toroidal maps of type $(3,6)$. A cycle in such a map is called *normal* if it leaves, at each of its vertices, exactly two edges on the right (left). Alshuler [1] proved that a toroidal map of type $(3,6)$ decomposes into nonintersecting normal cycles in three different ways.

Given a toroidal map G of type $(3,6)$, we denote by $\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3$ its families of nonintersecting normal cycles and by l_1,l_2,l_3 (m_1,m_2,m_3) the lengths (numbers) of cycles in $\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3$, respectively. Let v be any vertex of G and let $\Gamma_1,\Gamma_2,\Gamma_3$ denote the cycle in $\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3$, respectively, going through v.

We assign an orientation o_i to each Γ_i in such a 'compatible' way that each Γ_i leaves, at v, exactly one 'outgoing' edge on its right. (See Fig. 1; the opposite sides of each rectangle are identified and each rectangle represents the same torus with different details.) 'Parallel translation' transfers the orientations o_1,o_2,o_3 to all normal cycles. Then o_1,o_2,o_3 are compatible at any other vertex. The only other set of compatible orientations is the reverse one: $-o_1,-o_2,-o_3$. For any vertices v_1,v_2 of G there is a

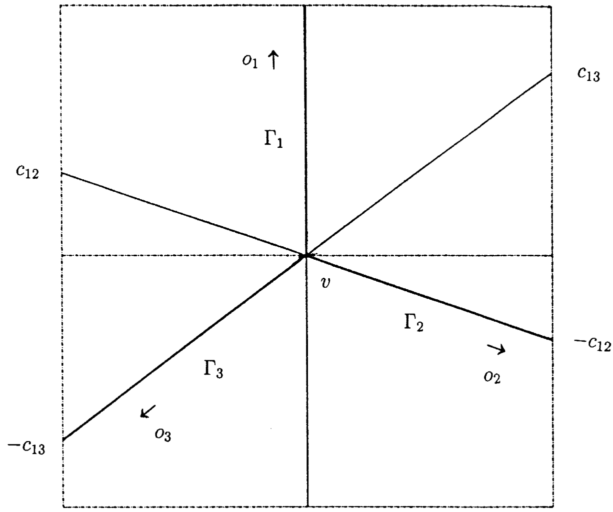

Fig. 1.

unique automorphism $\varphi^{+}_{v_1,v_2}(\varphi^{-}_{v_1,v_2})$ of G that maps v_1 to v_2 and each $\mathfrak{C}_i$ to itself and keeps (reverses) the orientations of normal cycles. We call the automorphisms $\varphi^{\pm}_{v_1,v_2}$ the trivial automorphisms of G. They form a commutative group and $\varphi^{+}_{v,v} = \mathrm{id}$.

For $1 \leqslant i, j \leqslant 3$, $i \neq j$, we denote by c_{ij} the length (in the *graph* G) of the path along Γ_i from v in the direction o_i to the first point of intersection, with Γ_i, of Γ_j, moving from v on Γ_j in the direction $-o_j$ (see Fig. 1). Because of the existence of the trivial automorphisms, the ordered triple (l_i, c_{ij}, m_i), called a *code* of G, depends only on the ordered pair $(\mathfrak{C}_i, \mathfrak{C}_j)$ of families of normal cycles of G. The *set* of 6 (not necessarily distinct) codes of G depends only on its isomorphism class, because it is easy to see that an isomorphism maps each family of nonintersecting normal cycles to such a family and each set of compatible orientations to such a set. The codes belonging to the same map are said to be *equivalent* to each other.

We regard the middle member c of a code (l, c, m) as a residue class mod l. The distinction between numbers and residue classes will be clear from the context, although we usually do not indicate it explicitly. Given (l, c, m), $l, m, c \in \mathbb{Z}$, $l, m > 0$, $0 \leqslant c < l$, there exists a toroidal map G of type $(3,6)$ for which $(l_1, c_{12}, m_1) = (l, c, m)$. The map G can be obtained in this way: In a tiling of the plane by regular triangles we take the coordinate axes X and Y intersecting at an angle of $2\pi/3$ as shown in Fig. 2 and identify the opposite sides of the parallelogram spanned by the vectors $\langle l, 0\rangle$, $\langle c, m\rangle$. This identification produces a torus and the tiling yields G because the tiling is invariant under translation by a vector with integer components. Fig. 2 shows the special case $l = 5$, $c \equiv 4$, $m = 3$. It can be shown that if G' is another map (with its data distinguished from those of G by a prime) such that $(l'_1, c'_{12}, m'_1) = (l, c, m)$, then G and G' are connected with an isomorphism that maps $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$ to $\mathfrak{C}'_1, \mathfrak{C}'_2, \mathfrak{C}'_3$, respectively. We leave the easy details to the reader; see also [14].

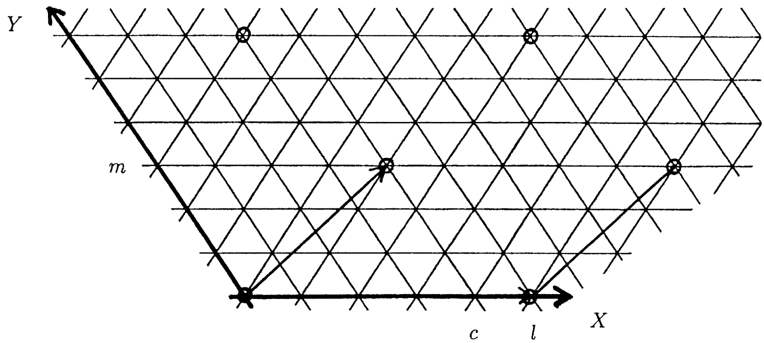

Fig. 2.

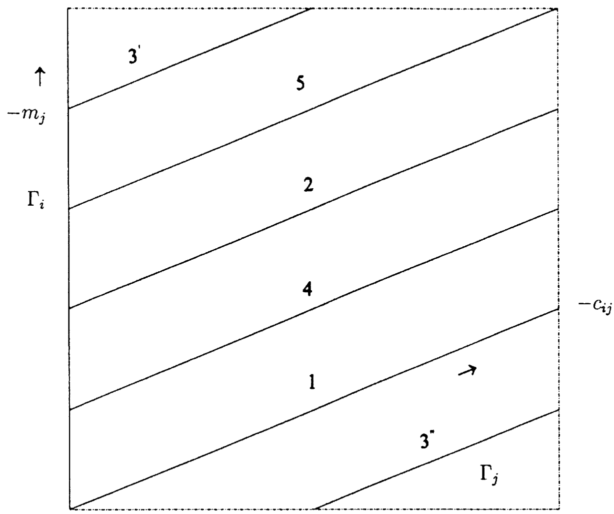

Fig. 3.

It follows from the preceding paragraph that one code determines the other codes of a map. Let i, j, k be a permutation of $1, 2, 3$. It is obvious that $c_{ij} + c_{ik} \equiv m_i \,(\mathrm{mod}\, l_i)$. Furthermore, traveling on Γ_j from v in the direction $-o_j$, we meet Γ_i at its points with the 'coordinates' $0, c_{ij}, 2c_{ij}, 3c_{ij}, \ldots, 0, \ldots,$ where 0 is the coordinate of v. The number of distinct points is the additive order of c_{ij} in $\mathbb{Z} \,\mathrm{mod}\, l_i$, which is $l_i/\mathrm{gcd}(l_i, c_{ij})$. Hence $l_j = m_i l_i/\mathrm{gcd}\,(l_i, c_{ij}) = n/\mathrm{gcd}\,(l_i, c_{ij})$ and $m_j = n/l_j = \mathrm{gcd}\,(l_i, c_{ij})$, where n denotes the number of vertices of G. Moreover, the Γ_i-coordinate of the point with Γ_j-coordinate c_{ji} is $-m_j$. Therefore, c_{ji} is *uniquely determined* by the system of congruences $c_{ji} \equiv 0 \,(\mathrm{mod}\, m_i), -c_{ij}(c_{ji}/m_i) \equiv -m_j \,(\mathrm{mod}\, l_i)$. Fig. 3 shows the special case $l_i/\mathrm{gcd}(l_i, c_{ij}) = 5,\ c_{ji}/m_i \equiv 2 \,(\mathrm{mod}\, l_j/m_i)$. Combination of the results of this paragraph yields the following formulas:

$$\begin{aligned} &n = l_i m_i,\ m_j = \gcd(l_i, c_{ij}),\ l_j = n/\gcd(l_i, c_{ij}),\ c_{ji} \equiv 0 \,(\mathrm{mod}\, m_i),\\ &c_{ij} + c_{ik} \equiv m_i \,(\mathrm{mod}\, l_i),\ c_{ij}c_{ji} \equiv m_i m_j \,(\mathrm{mod}\, n),\\ &(m_i - c_{ij})(m_k - c_{kj}) \equiv m_i m_k \,(\mathrm{mod}\, n),\\ &c_{ij}(m_j - c_{jk}) \equiv m_i m_j \,(\mathrm{mod}\, n),\ (m_i - c_{ij})c_{ki} \equiv m_i m_k \,(\mathrm{mod}\, n). \end{aligned} \tag{3}$$

If two of the 6 codes of G coincide, then G has a nontrivial automorphism. Conversely, a nontrivial automorphism of G determines a transposition $\mathfrak{C}_{i'} \leftrightarrow \mathfrak{C}_{j'}$ or a cycle $\mathfrak{C}_{i'} \to \mathfrak{C}_{j'} \to \mathfrak{C}_{k'}$ by which the codes of G coincide in pairs or triples, respectively. An automorphism φ of the former (latter) kind is called *reflectional* (*rotational*). Let G be represented as a tiling of the plane by regular triangles with appropriate identifications to produce a torus. If it preserves identifications, reflection in a line of the tiling (a third-turn rotation about a vertex) determines a reflectional (rotational) automorphism. Hence the terms ‘reflectional’ and ‘rotational’. (The map in Fig. 2 admits a reflectional automorphism.) It follows from (3) that the maps possessing both reflectional and rotational automorphisms have the (only) code $(m, 0, m)$ or $(3m, 2m, m)$.

Let $\mathrm{aut}\, G$ $(T(G), P(G, v))$ denote the group of restrictions, to the set of vertices of G, of automorphisms of G (of trivial automorphisms of G, of automorphisms of G that leave v and o_1, o_2, o_3 fixed). The elements of $P(G, v)$ different from the neutral element are restrictions of nontrivial automorphisms. The group $\mathrm{aut}\, G$ is the semidirect product (see [13] for a definition) of its subgroup $P(G, v)$ with its normal subgroup $T(G)$. In particular, $P(G, v)$ is isomorphic to $\mathrm{aut}\, G/T(G)$. The group $P(G, v)$ is isomorphic to the two-element or to the three-element (cyclic) group or to the permutation group of $\{1, 2, 3\}$. The observations in this paragraph are due to one of the referees of this paper; see also [14].

4. Asymptotic number of type (3, 6) toroidal maps with at most *n* vertices

Kurth [14, Lemma 3.1(b)] found that the number (of isomorphism classes) of toroidal maps of type $(3, 6)$ with n vertices is

$$N(n) = \frac{1}{6}\left\{\sigma(n) + 3\left(d(n) - d\left(\frac{n}{2}\right) + 2d\left(\frac{n}{4}\right)\right) + 2(\gamma * \rho_{1,1})(n)\right\}. \tag{4}$$

We write the expression for $N(n)$ in a ‘more closed’ form and find an asymptotic formula for $\bar{N}(n) = N(1) + \cdots + N(n)$, the number of type $(3, 6)$ toroidal maps with at most n vertices.

Proposition 1. *For any positive integer n,*

$$(\gamma * \rho_{1,1})(n) = \gamma(r_{3,-1}(n))d(r_{3,1}(n)). \tag{5}$$

Proof. Let $r_{3,1}(n) = p_1^{\alpha_1} \ldots p_t^{\alpha_t} q_1^{\beta_1} \ldots q_u^{\beta_u}$, α_i (>0) even, β_j odd, be the factorization of $r_{3,1}(n)$ into powers of distinct positive primes. First let $n/r_{3,1}(n) = 1$ or 3. We have $m^2 | n$ if and only if $m^2 = p_1^{\alpha'_1} \ldots p_t^{\alpha'_t} q_1^{\beta'_1} \ldots q_u^{\beta'_u}$ with α'_i, β'_j even and such that

$0 \leqslant \alpha_i' \leqslant \alpha_i$, $0 \leqslant \beta_j' \leqslant \beta_j$. We never have $\beta_j' = \beta_j$ for any j and if v denotes the number of α_i' such that $\alpha_i' = \alpha_i$, then $\rho_{1,1}(n/m^2) = 2^{t+u-v}$ by (1). The number of square divisors m^2 of n with a given v is $\prod_i [\alpha_i/2] \cdot \prod_j [(\beta_j+1)/2]$ multiplied by the sum of the vth degree terms in the expansion of $(1+x_1)\dots(1+x_t)$ after the substitution $x_i = 1/(\alpha_i/2)$. Each such term corresponds to a choice of indices i with $\alpha_i' = \alpha_i$. We can transfer 2^{-v} from 2^{t+u-v} to each vth degree term. We obtain that

$$\begin{aligned}
&(\gamma * \rho_{1,1})(n) \\
&\quad = 2^{t+u} \prod_i [\alpha_i/2] \prod_j [(\beta_j+1)/2][1 + (1/2)/(\alpha_1/2)] \cdots [1 + (1/2)/(\alpha_t/2)] \\
&\quad = \prod_i \alpha_i (1 + 1/\alpha_1) \cdots (1 + 1/\alpha_t) \prod_j (\beta_j + 1) \\
&\quad = \prod_i (\alpha_i + 1) \prod_j (\beta_j + 1) = d(n). \qquad (6)
\end{aligned}$$

Consequently, the special case $n/r_{3,1}(n) = 1$ or 3 of Proposition 1 is proved. The general case reduces to this because $\rho_{1,1}(n/m^2) \neq 0$ only if m^2 is a multiple of either $n/r_{3,1}(n)$ or $n/(3r_{3,1}(n))$ according as $\log_3 r_{3,0}(n)$ is even or odd. □

Theorem 2. *We have* $N(n) = \sigma(n)/6 + |2\log_2 r_{2,0}(n) - 1|\ (2\log_2 r_{2,0}(n) + 2)^{-1} d(n) + \gamma(r_{3,-1}(n)) d(r_{3,1}(n))/3$.

Proof. It is a simple exercise to show that $d(n) - d(n/2) + 2d(n/4) = |2\log_2 r_{2,0}(n) - 1|(\log_2 r_{2,0}(n) + 1)^{-1} d(n)$. Theorem 2 follows from this, from Proposition 1, and from (4). □

By the definitions, $d(r_{3,1}(n)) \leqslant d(n)$; furthermore, the coefficient of $d(n)$ in Theorem 2 is clearly $\leqslant 1$. It is obvious that $\sigma(n) \geqslant n+1$. It is classical that $\lim_{n\to\infty} [d(n)/n^{\epsilon}] = 0$ for any positive ϵ [8, Theorem 315, p. 260]. Therefore, by Theorem 2, $N(n)$ behaves much like $\sigma(n)/6$ for n large. However, the number of codes assigned to all type $(3,6)$ toroidal maps with n vertices is easily seen to be $\sigma(n)$. Therefore, these graphs 'almost surely' have exactly 6 distinct codes.

The maximum asymptotic order of $N(n)$ is provided by $\limsup_{n\to\infty} [\sigma(n)/(n \log\log n)] = e^{\gamma}$, where γ is Euler's constant [8, Theorem 323, p. 266]. The minimum asymptotic order is $n/6$, provided by the sequence of primes. It is also classical that $\sigma(1) + \cdots + \sigma(n) = (\pi^2/12)n^2 + O(n\log n)$ [8, Theorem 324, p. 266] and $d(1) + \cdots + d(n) = n\log n + n(2\gamma - 1) + O(\sqrt{n})$ [8, Theorem 320, p. 264]. Therefore, by Theorem 2, $\bar{N}(n) = (\pi^2/72)n^2 + O(n\log n)$. The average order of $N(n)$ is $(\pi^2/36)n$ in the spirit of [8, end of p. 266]. We also have $\sigma(n)/6 \leqslant N(n) \leqslant \sigma(n)/6 + (4/3)d(n)$. The main result is stated as follows:

Theorem 3. *For* $n > 1$ *we have* $\bar{N}(n) = (\pi^2/72)n^2 + O(n\log n)$, *where* $\pi^2/72 \approx 0.1371$.

5. The linear regression of $N(n)$ on n

The asymptotics of $N(n)$ explains the phenomena observed by Kirby and Pollak empirically in [12]: $N(n)$ depends on n nearly linearly and the relation becomes 'more' linear when the number of prime factors of n is taken into account. Kirby and Pollak actually study the number $H(k)$ (our notation) of hexagonal rather than triangular toroidal maps with k vertices. They exclude the maps whose duals have the codes $(k/2,0,1)\sim(k/2,1,1)(\sim(1,0,k/2))$ (these are the duals with a loop) and for every *even* m they identify the maps whose duals have the inequivalent codes $(2,0,m)$ and $(2,1,m)$ because these type $(6,3)$ maps are graph-isomorphic. Then $H(k)=N(k/2)-2$ if $8|k$ and $H(k)=N(k/2)-1$ otherwise. The asymptotics of $H(k)$ is therefore the same as that of $N(k/2)$.

Kirby and Pollak obtain the linear regression line $y=0.1406x+0.7075$ for the points $x=k,\ y=H(k),\ k=6,\dots,250$. We obtain from the asymptotics of $\bar{N}(n)$ that the slope m of the regression line $y=mx+b$ for $x=k, y=N(k), k=1,\dots,n$ is $m=\pi^2/36+\mathrm{O}(\log n/n)\approx 0.2742+\mathrm{O}(\log n/n)$ and the y-intercept b is $\mathrm{O}(\log n)$:

$$\sum_1^n k^2 = n^3/3+\mathrm{O}(n^2),$$

$$\begin{aligned}\sum_1^n kN(k) &= n\sum_1^n N(k)-\sum_1^{n-1}N(k)-\cdots-\sum_1^2 N(k)-\sum_1^1 N(k)\\ &= n\bar{N}(n)-\bar{N}(n-1)-\cdots-\bar{N}(2)-\bar{N}(1)\\ &= \pi^2 n^3/72-(\pi^2/72)[(n-1)^2+\cdots+1^2]+\mathrm{O}(n^2\log n)\\ &= (\pi^2/72)(2/3)n^3+\mathrm{O}(n^2\log n),\end{aligned}$$

$$\begin{aligned}m &= \frac{(\pi^2/72)(2/3)n^3+\mathrm{O}(n^2\log n)-(\pi^2n^2/72+\mathrm{O}(n\log n))(n+1)/2}{n^3/3+\mathrm{O}(n^2)-n(n+1)^2/4}\\ &= \pi^2/36+\mathrm{O}(\log n/n),\end{aligned}$$

$$\begin{aligned}b &= \pi^2 n/72+\mathrm{O}(\log n)-[(n+1)/2][\pi^2/36+\mathrm{O}(\log n/n)]\\ &= \mathrm{O}(\log n).\end{aligned}$$

Since 0.2743 is close to twice 0.1406, the slope of the empirical linear regression line of Kirby and Pollak is close to the slope of the asymtotic linear regression line. Careful examination of the estimate of b shows that it is asymptotically equal to one-nth of the sum of half of the excess of $\bar{N}(n)$ over its estimate and the excess of $(\bar{N}(1)+\cdots+\bar{N}(n))/n$ over its estimate.

The number of codes of a toroidal map of type $(3,6)$ is $6,3,2$, or 1 (see Section 3). However, for every even m, Kirby and Pollak identify the maps whose duals have the inequivalent codes $(2,1,m)$ and $(2,0,m)$. It can be verified by using (3) that if m is even, then $(2,1,m)\sim(2m,m,1)\sim(2m,m+1,1)$ and if m is even and $m\geqslant 4$, then

$(2,0,m) \sim (m,0,2) \sim (m,2,2)$, are the complete sets of codes belonging to the maps in question. Therefore, Table 1 in [12] cannot have more than one line with 4 codes, even if it is continued infinitely.

6. Multiplicity of a toroidal map of type (3,6). Asymptotic number of maps with multiplicity 1

At the beginning of this paper we defined the *multiplicity* of a toroidal map of type $(3,6)$ as the gcd of the numbers of the three kinds of normal cycles. This multiplicity can be defined in the following more revealing manner: The code $(\mu l, \mu c, \mu m)$, $\mu = 2,3,\ldots$, belongs to the type $(3,6)$ map obtained from the one associated with the code (l,c,m) by dividing every triangle into μ^2 little triangles. Therefore, the gcd of the three members of a code of a map does not change when we pass to another code of the map. This can also be seen from (3). In the next paragraph we show that this gcd is equal to the multiplicity of the map.

Let G be a toroidal map of type $(3,6)$, let l_i, m_i, $i=1,2,3$, denote the length and the number of cycles in each of the three families of normal cycles in G, and let μ stand for the gcd of the 3 members of a code of G. It follows from (3) that $\mu = \gcd(m_i, m_j)$ for any $i \neq j$. Therefore, μ is equal to the multiplicity of G. Let (i,j,k) be any permutation of $1,2,3$. Let $\lambda_i = l_j/m_k$. This definition is unambiguous because $l_j m_j = l_k m_k = n$, the number of vertices. Let $g_{ij} = \gcd(\lambda_i, \lambda_j)$. Suppose that $\mu = 1$. Then $g_{ij} = l_k/(m_i m_j)$, and thus $g_{ij} = (\mu l_k)/(m_i m_j)$ in the general case because the λ_i do not change in 'subdividing' a map. Therefore, $g_{ij} = \mu n/(m_i m_j m_k)$ and thus g_{ij} is independent of i,j and is denoted by g. Hence $\mu^2 \lambda_1 \lambda_2 \lambda_3 / g^2 = (l_1/m_2)\,(l_2/m_3)(l_3/m_1)/[(l_1/(m_2 m_3))(l_2/(m_1 m_3))] = l_3 m_3 = n$. On the other hand, it follows from $g = (\mu l_k)/(m_i m_j)$ that $m_i = (\mu/g)(l_k/m_j) = (\mu/g)\lambda_i$. Therefore, $l_i = n/m_i = (\mu/g)\lambda_j \lambda_k$ and $\gcd(l_i, m_i) = \mu \gcd(g, \lambda_i/g)$.

Kirby and Pollak define the following three cases: (I) $\gcd(l_i, m_i) | c_{ij}$ for some i,j; (II) $\mu = 1, \gcd(l_i, m_i) \dagger c_{ij}, i,j = 1,\ldots,3,\ i \neq j$; (III) $\mu \neq 1, \gcd(l_i, m_i) \dagger c_{ij}, i,j = 1,\ldots,3$, $i \neq j$. Since $\mu = \gcd(l_i, m_i, c_{ij})$, the condition $\gcd(l_i, m_i) | c_{ij}$ is equivalent to $\mu = \gcd(l_i, m_i)$, which in turn is equivalent to $\gcd(g, \lambda_i/g) = 1$ by the formula at the end of the preceding paragraph. Therefore, the equality $n = l_i m_i = \mu^2 g(\lambda_1/g)(\lambda_2/g)(\lambda_3/g)$, where the factors in parantheses are mutually relatively prime, shows that the smallest n for which case II occurs is obtained when $\mu = 1$, $g = 2\cdot 3\cdot 5$, $\{\lambda_1, \lambda_2, \lambda_3\} = \{2g, 3g, 5g\}$, if such a map exists. The map with code $(450,5,2)$, found empirically by Kirby and Pollak, actually has these parameters. A smallest case III map is obviously obtained by 'doubling' a smallest case II map, that is, considering the map with code (900,10,4) in accordance with [12]. Every case III map can be reduced to case II by dividing all of its parameters by its multiplicity μ. The theorem below, combined with Theorem 3, shows that the 'probability' that a random map is of multiplicity 1 is $90/\pi^4 \approx 0.9239$.

The several functions O used throughout the proof of Theorem 4 have a universal constant, which can be taken to equal the constant of the function O in Theorem 3 if the order $n \log n$ there is replaced with $n\log(n+1)(\sim n \log n)$ *and* the cases $n = 0,1$

are included (with $\bar{N}(0)=0$). Working with the order $n\log(n+1)$ allows us to deal with all terms of all series in (7) uniformly, including the terms $\bar{N}(\mathrm{int}(n/p^2))$, $\bar{N}(\mathrm{int}(n/(pq)^2)),\ldots$, when int (n/p^2) or int $(n/(pq)^2),\ldots$ is 1 or 0. ('int' denotes integer part.) For these terms Theorem 3 does not hold in its original form. For clarity, we also use the further functions O′ of order n with universal constant = 4.

Theorem 4. *The number of toroidal maps of type* (3,6) *of multiplicity* 1 *and with at most* $n(>1)$ *vertices is* $\bar{N}_1(n)=(5/(4\pi^2))n^2+\mathrm{O}(n\log n)$, *where* $5/(4\pi^2)\approx 0.1267$.

Proof. We determine the number $\bar{N}_{>1}(n)$ of toroidal maps of type (3,6) with multiplicity larger than 1 and with at most n vertices ($n=0,1,\ldots$). A map with at most n vertices falls in this category if and only if its multiplicity is divisible by one of the primes, that is, by 2, 3, 5, etc. The number of maps with at most n vertices whose multiplicity is divisible by a given number v is $\bar{N}(\mathrm{int}(n/v^2))$ (recall that $\bar{N}(0)=0$). Of course, the multiplicity of a map can be divisible by several primes. Consequently,

$$\bar{N}_{>1}(n)=\sum_p \bar{N}(\mathrm{int}\,(n/p^2))-\sum_{p<q}\bar{N}(\mathrm{int}\,(n/(pq)^2))$$
$$+\sum_{p<q<r}\bar{N}(\mathrm{int}\,(n/(pqr)^2))-\cdots, \tag{7}$$

where $p, q, r,\ldots$ are primes. In each series all terms are 0 beginning from a certain term and all terms of all series are 0 beginning with a certain series. It follows from (7) combined with Theorem 3 that

$$\bar{N}_{>1}(n)/(\pi^2/72)$$
$$=\sum_p(\mathrm{int}^2(n/p^2)+\mathrm{O}\,(\mathrm{int}\,(n/p^2)\log(\mathrm{int}\,(n/p^2)+1)))$$
$$-\sum_{p<q}(\mathrm{int}^2(n/(pq)^2)+\mathrm{O}\,(\mathrm{int}\,(n/(pq)^2)\log(\mathrm{int}\,(n/(pq)^2)+1)))$$
$$+\sum_{p<q<r}(\mathrm{int}^2(n/(pqr)^2)+\mathrm{O}\,(\mathrm{int}\,(n/(pqr)^2)\log(\mathrm{int}\,(n/(pqr)^2)+1)))-\cdots$$
$$=\mathrm{O}'(n)+\sum_p((n/p^2)^2+\mathrm{O}((n/p^2)\log(n/p^2+1)))$$
$$-\sum_{p<q}((n/(pq)^2)^2+\mathrm{O}((n/(pq)^2)\log(n/(pq)^2+1)))$$
$$+\sum_{p<q<r}((n/(pqr)^2)^2+\mathrm{O}((n/(pqr)^2)\log(n/(pqr)^2+1)))-\cdots, \tag{8}$$

where the second equality holds because $\mathrm{int}(n/p^2)\log(\mathrm{int}(n/p^2)+1)\leqslant(n/p^2)\log(n/p^2+1)$ and similar inequalities hold for the expressions in the other O's and because

$$
\begin{aligned}
0 &\leqslant \sum_p ((n/p^2)^2 - \mathrm{int}^2(n/p^2)) + \sum_{p<q} ((n/(pq)^2)^2 - \mathrm{int}^2(n/(pq)^2)) \\
&\quad + \sum_{p<q<r} ((n/(pqr)^2)^2 - \mathrm{int}^2(n/(pqr)^2)) + \cdots \\
&\leqslant \sum_p 2n/p^2 + \sum_{p<q} 2n/(pq)^2 + \sum_{p<q<r} 2n/(pqr)^2 + \cdots \\
&= 2n\left(\sum_p 1/p^2 + \sum_{p<q} 1/(pq)^2 + \sum_{p<q<r} 1/(pqr)^2 + \cdots\right) \\
&\leqslant 2n.
\end{aligned}
$$

The last line here holds because

$$
\begin{aligned}
&\sum_p 1/p^2 + \sum_{p<q} 1/(pq)^2 + \sum_{p<q<r} 1/(pqr)^2 + \cdots \\
&\quad < \sum_{k=1}^{\infty} k^{-2} - 1 = \pi^2/6 - 1 < 1.
\end{aligned}
\tag{9}
$$

It follows from (8) and (9) that

$$
\begin{aligned}
&\bar{N}_{>1}(n)/(\pi^2/72) \\
&\quad = \mathrm{O}'(n) + n^2\left(\sum_p (1/p^4) - \sum_{p<q} (1/(pq)^4) + \sum_{p<q<r} (1/(pqr)^4) - \cdots\right) \\
&\qquad + \sum_p \mathrm{O}((n/p^2)\log(n/p^2+1)) + \sum_{p<q} \mathrm{O}((n/(pq)^2)\log(n/(pq)^2+1)) \\
&\qquad + \sum_{p<q<r} \mathrm{O}(n/(pqr)^2\log(n/(pqr)^2+1)) + \cdots \\
&\quad = \mathrm{O}'(n) + n^2(1 - 1/\zeta(4)) \\
&\qquad + \mathrm{O}\left(\sum_p [n\log(n+1)/p^2]\right) + \mathrm{O}\left(\sum_{p<q} [n\log(n+1)/(pq)^2]\right) \\
&\qquad + \mathrm{O}\left(\sum_{p<q<r} [n\log(n+1)/(pqr)^2]\right) + \cdots \\
&\quad = \mathrm{O}'(n) + n^2(1 - 1/\zeta(4)) \\
&\qquad + \mathrm{O}\left(n\log(n+1)\left[\sum_p (1/p^2) + \sum_{p<q} (1/(pq)^2) + \sum_{p<q<r} (1/(pqr)^2) + \cdots\right]\right) \\
&\quad = \mathrm{O}'(n) + n^2(1 - 1/\zeta(4)) + \mathrm{O}(n\log(n+1)),
\end{aligned}
$$

because

$$\sum_{p}(1/p^4)-\sum_{p<q}(1/(pq)^4)+\sum_{p<q<r}(1/(pqr)^4)-\cdots=1-1/\zeta(4),$$

where ζ denotes the Riemann zeta function [8, a consequence of Theorem 280, p. 246]. Since $n\log(n+1)\sim n\log n$, the proof is completed by substituting $\zeta(4)=\pi^4/90$ [8, p. 245] into the expression just obtained for $\bar{N}_{>1}(n)$ and then using the asymptotic estimate of $\bar{N}(n)$ from Theorem 3 in the obvious equality $\bar{N}_1(n)=\bar{N}(n)-\bar{N}_{>1}(n)$. $\square$

7. Number of type (3, 6) toroidal maps with a Hamiltonian normal cycle

In [2] Altshuler gave upper and lower estimates for the number $N_h(n)$ of non-isomorphic toroidal maps of type $(3,6)$ with n vertices and with a Hamiltonian normal cycle. In this section we find an exact expression for $N_h(n)$ and give lower and upper asymptotic estimates for the number $\bar{N}_h(n)$ of nonisomorphic maps with a Hamiltonian normal cycle and with at most n vertices. These upper and lower asymptotic estimates seem to close in on an asymptotic estimate.

Given the code (l_1,c_{12},m_1) of a toroidal map of type $(3,6)$, we write $\varphi_{ij}(l_1,c_{12},m_1)$ for the code (l_i,c_{ij},m_i) of the same map (for this notation, see Section 3). Let the number of vertices n be one and the same given number until Theorem 5. If $m_1,m_i=1$, then c_{ij} in $(n,c_{ij},1)=\varphi_{ij}(n,c_{12},1)$ depends only on c_{12}. We write $\varphi_{ij}(c_{12})$ for c_{ij}. This meaning of φ_{ij} is compatible with the other one and does not cause confusion.

Let $C_h(n)$ be the set of codes belonging to toroidal maps of type $(3,6)$ with a Hamiltonian normal cycle of length n. The elements of $C_h(n)$ are the codes obtained from those of the form $(n,c,1)$ by application of the six φ_{ij}. First, we find $\operatorname{card} C_h(n)$, the cardinality of $C_h(n)$. The φ_{ij} form a permutation group of $C_h(n)$ and $N_h(n)$ is the number of orbits in $C_h(n)$ of this group, which number is equal to the average number of invariant elements in $C_h(n)$ under the φ_{ij} by Burnside's lemma (see [5]).

It follows from (3) that each $\varphi_{ij}(c)$ is defined if and only if the corresponding congruence below is solvable $\bmod\, n$, and then $\varphi_{ij}(c)$ is the only solution.

$$\begin{aligned}
&\varphi_{12}(c)-c\equiv 0, \qquad \varphi_{13}(c)+c\equiv 1,\\
&\varphi_{21}(c)c\equiv 1, \qquad (1-\varphi_{32}(c))(1-c)\equiv 1, \qquad\qquad (10)\\
&(1-\varphi_{23}(c))c\equiv 1, \qquad \varphi_{31}(c)(1-c)\equiv 1.
\end{aligned}$$

$\varphi_{12}(c)=c$ and $\varphi_{13}(c)=1-c$ are always defined and $\varphi_{13}^2=\mathrm{id}$, $\varphi_{13}\varphi_{21}=\varphi_{23}$, $\varphi_{13}\varphi_{32}=\varphi_{31}$. Furthermore, $\varphi_{13}(c)=c$ exactly when n is odd and $c=(n+1)/2 \pmod n$, in which case the codes equivalent to $(n,c,1)$ are $(n,2,1),(n,-1,1)$. If n is odd, then these are the codes of the only type $(3,6)$ map with a reflectional automorphism with respect to a Hamiltonian normal cycle. We call this map and its codes *exceptional*. If n is even, then such a map does not exist. (If $n>3$ is odd, then the codes $(n,(n+1)/2,1),(n,2,1),(n,-1,1)$ are distinct. If $n=1$ or $n=3$, then they are equal and provide all type $(3,6)$ maps of multiplicity 1 with reflectional *and* rotational symmetry; see also Section 3.)

Congruences (10) show that a code $(n,c,1)$ is equivalent to codes of the form (l,C,m) with $m \neq 1$ if and only if at least one of c or $c-1$ is not (mutiplicatively) invertible mod n.

Let S_1 (S_2) be the set of residue classes $c \bmod n$ such that one of $c, \varphi_{13}(c)$ $(=1-c)$ is invertible and the other is not (neither c nor $\varphi_{13}(c)$ is invertible). The mapping $c \to \varphi_{13}(c)$ maps S_1 (S_2) onto itself and $\varphi_{13}^{-1} = \varphi_{13}$. Note that if c is invertible and $1-c$ is not, then φ_{13} reverses this: $\varphi_{13}(c)$ is *not* invertible and $1-\varphi_{13}(c)$ *is*. A similar observation holds when c is not invertible and $1-c$ is. Furthermore, $\varphi_{13}(c) \neq c$ if $c \in S_1 \cup S_2$. Therefore, $(1/2)\operatorname{card} S_1$ is equal to the number of residue classes $c \bmod n$ such that c is not invertible and $1-c$ is.

Let $c \in S_2$. It follows from (10) that $(n, \varphi_{13}(c), 1)$ is the only code of the form $(n, C, 1)$ that is equivalent to and distinct from $(n,c,1)$. The 4 further codes equivalent to $(n,c,1)$ are all distinct, since otherwise the corresponding map would have a nontrivial automorphism, which would necessarily be reflectional (since in the presence of a rotational automorphism all normal cycles have the same length) with respect to the *unique* Hamiltonian normal cycle, and thus $(n,c,1)$ would be exceptional. However, all normal cycles of all exceptional maps are of multiplicity 1. Therefore, the number of codes (l,C,m) with $m \neq 1$ that are equivalent to a code $(n,c,1)$ with $c \in S_2$, is $2 \operatorname{card} S_2$ because c and $\varphi_{13}(c)$ are distinct residue classes mod n when $c \in S_2$.

Let $c \in S_1$. For the sake of definiteness we assume that c is not invertible and $1-c$ is. Then (10) informs us that $\varphi_{32}(c)$, $\varphi_{31}(c)$ exist but $\varphi_{21}(c), \varphi_{23}(c)$ do not. It follows easily from (10) that $\varphi_{31}(c)$ is invertible and $1-\varphi_{31}(c)$ is not. Therefore, $\varphi_{31}(c) \in S_1$ and thus $\varphi_{32}(c) = \varphi_{13}(\varphi_{31}(c)) \in S_1$ and $\varphi_{32}(c) \neq \varphi_{31}(c)$. Hence each of the unordered pairs $(c, \varphi_{13}(c))$ and $(\varphi_{32}(c), \varphi_{31}(c))$ contains distinct residue classes and the pairs are either disjoint or equal. Therefore, 4 *or* 2 *distinct codes of the form* $(n,C,1)$ *are equivalent to* $(n,c,1)$. *Accordingly, the number of distinct code(s) equivalent to* $(n,c,1)$ *and of the form* (l,C,m), $m \neq 1$, *is* 2 *or* 1 because the number of distinct codes of a map is 1,2,3, or 6. The italicized statements also hold when c is invertible and $1-c$ is not, and thus for all $c \in S_1$. Consequently, the total number of codes (l,C,m) with $m \neq 1$ that are equivalent to a code $(n,c,1)$ with $c \in S_1$ is $(1/2) \operatorname{card} S_1$.

Codes $(n,c,1)$ and $(n,c',1)$ with $c \in S_1$ and $c' \in S_2$ cannot be equivalent to each other because according to the preceding two paragraphs, the first kind is equivalent to 1 or 2 codes of the form (l,C,m), $m \neq 1$, and the second kind, to 4 such codes. If both c and $1-c$ are invertible, then by (10), the code $(n,c,1)$ is equivalent to no code of the form $(l,C,m), m \neq 1$. Furthermore, the number of codes of the form $(n,c,1)$ is n. Consequently, the preceding two paragraphs imply that

$$\operatorname{card} C_h(n) = n + 2 \operatorname{card} S_2 + (1/2)\operatorname{card} S_1. \tag{11}$$

Every residue class $c \bmod n$ falls in one of three mutually disjoint categories: (1) both c and $1-c$ are noninvertible; (2) c is noninvertible and $1-c$ is invertible; (3) c is invertible. The cardinalities of these classes are: $\hat{\phi}(n)$ (definition); $(1/2) \operatorname{card} S_1$

(shown 3 paragraphs ago); $\phi(n)$ (definition). Therefore, $n=\hat{\phi}(n)+(1/2)\,\mathrm{card}\,S_1+\phi(n)$. Combining this with (11) and with the obvious equality $\hat{\phi}(n)=\mathrm{card}\,S_2$ gives that

$$\mathrm{card}\,C_h(n)=2n-\phi(n)+\hat{\phi}(n). \tag{12}$$

This is also the number of codes invariant under φ_{12}. Since $\varphi_{32}\varphi_{21}\varphi_{32}=\varphi_{13}$, $\varphi_{13}\varphi_{21}\varphi_{13}=\varphi_{32}$, the group elements $\varphi_{13},\varphi_{21},\varphi_{32}$ have the *same number* of invariant codes in $C_h(n)$. We find the number of φ_{21}-invariant codes in $C_h(n)$. Suppose that some element (l_1,c_{12},m_1) of $C_h(n)$ with $m_1\neq 1$ is φ_{21}-invariant. Since we have shown 4 paragraphs ago that the codes equivalent to the codes $(n,C,1), C\in S_2$, are never multiple, $(l_1,c_{12},m_1)\sim(n,C,1)$ for some $C\in S_1$. Then our arguments 3 paragraphs ago imply that $m_2=m_3=1$, which is impossible because $m_2=m_1\neq 1$ by the definition of φ_{21}. Consequently, all φ_{21}-invariant codes in $C_h(n)$ are of the form $(n,c,1)$. By (10), $\varphi_{21}(n,c,1)=(n,c,1)$ exactly when $c^2\equiv 1\,(\mathrm{mod}\,n)$.

φ_{23} and φ_{31} are inverses of each other, and thus they have the *same* invariant codes in $C_h(n)$. If (l,c,m) is a code of *any* toroidal map of type $(3,6)$, then the multiplicities of the three kinds of normal cycles are the third members of $(l,c,m),\varphi_{23}(l,c,m)$, $\varphi_{31}(l,c,m)$. Therefore, if (l,c,m) *belongs to* $C_h(n)$ and is invariant under φ_{23}, then $m=1$. Furthermore, it follows from (10) that $\varphi_{23}(n,c,1)=(n,c,1)$ if and only if $c^2-c+1\equiv 0\,(\mathrm{mod}\,n)$.

Theorem 5. *For any given positive integer n we have*

$$N_h(n)=(1/6)[2n-\phi(n)+\hat{\phi}(n)+3\rho_{0,-1}(n)+2\rho_{-1,1}(n)]. \tag{13}$$

Proof. This follows from (12) and from the two paragraphs after (12) by Burnside's lemma and by Section 2. □

Corollary 6. (Altshuler [2, Theorem 4.1, p. 216]). *We have* $(n+7)/6\leqslant N_h(n)\leqslant n/2$ for $n>3$.

Proof. For the upper estimate we only use the properties of φ_{13}. Given n, the number of codes of the form $(n,c,1)$, where c is a residue class mod n, is n. The mapping $(n,c,1)\to(n,1-c,1)(=\varphi_{13}(n,c,1))$ is involutional and transforms each code to an equivalent one. Its only fixed point is $(n,(n+1)/2,1)$, and then n is odd and $(n,(n+1)/2,1)$ has the equivalent codes $(n,2,1),(n,-1,1)$, which are different from $(n,(n+1)/2,1)$ and from each other if $n>3$. Therefore, if $n(>3)$ is odd, then $N_h(n)\leqslant(n-1)/2$. If n is even, then by similar arguments, $N_h(n)\leqslant n/2$.

By Theorem 5 and Section 2, $N_h(n)=(n+7)/6$ when $n>3$ is prime and $n\equiv-1\,(\mathrm{mod}\,3)$, and $N_h(n)=(n+11)/6$ when n is prime and $n\equiv 1\,(\mathrm{mod}\,3)$. Let $n, n>3$, be composite. Then $\phi(n)\leqslant n-2$ and $\mathrm{sign}(\log_2 r_{2,0}(n)-2)+p(n/r_{2,0}(n))\geqslant 0$, and thus Theorem 5 combined with Section 2 shows that $N_h(n)\geqslant(n+8)/6$. □

Corollary 6 is due to Altshuler [2, Theorem 4.1, p. 216]. (We note that in his Theorem 4.2 after Theorem 4.1 Altshuler states that for a given number of vertices, each set of three distinct normal cycle lengths arises from no more than one toroidal map of type $(3,6)$ up to isomorphism. However, codes $(45,6,1)$ and $(45,21,1)$ belong to two nonisomorphic maps, both with normal cycle lengths 45, 15, and 9.)

Theorem 5 can be used to study the asymptotics of $\bar{N}_h(n)$, the number of nonisomorphic toroidal maps of type $(3,6)$ with a Hamiltonian normal cycle and with at most n vertices, by studying the asymptotics of $\hat{\phi}^{-}(n) = \hat{\phi}(1) + \cdots + \hat{\phi}(n)$. We could only find lower and upper asymptotic estimates of $\hat{\phi}^{-}(n)$. These lower and upper estimates of $\hat{\phi}^{-}(n)$ seem to close in on an exact asymptotic estimate. In the proposition below $p_1 < p_2 < \cdots$ denotes the sequence of all primes. Note that $\omega_k^{(i)} \neq 0$ if and only if $k/2 \leqslant i \leqslant k^2/4$.

Proposition 7. *We have* $\chi_{2t}n^2 + \mathrm{O}(n) \leqslant \hat{\phi}^{-}(n) \leqslant \chi_{2t-1}n^2 + \mathrm{O}(n)$ *for* $t = 1, 2, \ldots,$ *where*

$$\chi_s = (1/2)\sum_{i=1}^{s} \sum_{2\sqrt{i}\leqslant k\leqslant 2i} (-1)^{i-1}\omega_k^{(i)} \sum_{\lambda_1<\cdots<\lambda_k} (p_{\lambda_1}\ldots p_{\lambda_k})^{-2} \tag{14}$$

and $\omega_k^{(i)}$ *is the number of sets of* i *(distinct) pairs* $(l_1, m_1), \ldots, (l_i, m_i)$ *of integers such that* $l_\iota \neq m_\kappa, 1 \leqslant \iota, \kappa \leqslant i$ *and* $\{l_\iota\colon 1 \leqslant \iota \leqslant i\} \cup \{m_\kappa\colon 1 \leqslant \kappa \leqslant i\} = \{1, 2, \ldots, k\}$.

Proof. The proof is somewhat similar to that of Theorem 4. Given integers a, c, the numbers c and $c-1$ are both not relatively prime to a if and only if two distinct primes, say p and q, exist such that $pq|a, p|c, q|c-1$. By the Chinese Remainder Theorem [3, Theorem 5.26, p. 117], among any pq consecutive integers there is exactly one c such that $p|c, q|(c-1)$. Since the number of $a, 1 \leqslant a \leqslant n$, such that $pq|a$ is obviously $\mathrm{int}(n/(pq))$, it follows that for a given positive integer n and a given pair of distinct primes p, q, the cardinality of the set $S(n; p, q)$ of ordered pairs (a, c) such that $1 \leqslant c \leqslant a \leqslant n, pq|a, p|c, q|(c-1)$ is $1 + \cdots + \mathrm{int}(n/(pq)) = \mathrm{int}(n/(pq))(\mathrm{int}(n/(pq)) + 1)/2$. This is the 'portion' of $\hat{\phi}^{-}(n)$ due to the distinct primes p and q, in that order. The entire $\hat{\phi}^{-}(n)$ is obtained if we consider all ordered pairs p, q of distinct primes. (Of course, pairs (p, q) such that $pq > n$ do not contribute; we then have $\mathrm{int}(n/(pq)) = 0$.) However, a given pair (a, c) may be included many times due to many pairs (p, q).

Let $(p_{\mu_1}, p_{\nu_1}), \ldots, (p_{\mu_i}, p_{\nu_i})$ be a set of i pairs of primes, let $\lambda_1 < \cdots < \lambda_k$ be the list of all distinct μ_ι, ν_κ, let $\Psi : \iota \to \lambda_\iota : \{1, \ldots, k\} \to \{\lambda_1, \ldots, \lambda_k\}$, let $P = p_{\lambda_1} \cdot \cdots \cdot p_{\lambda_k}$, and let $P \leqslant n$. A pair $(a, c), 1 \leqslant c \leqslant a \leqslant n$, is in $S(n; p_{\mu_1}, p_{\nu_1}) \cap \cdots \cap S(n; p_{\mu_i}, p_{\nu_i})$ if and only if $\mu_\iota \neq \nu_\kappa, 1 \leqslant \iota, \kappa \leqslant i$, and $p_{\mu_\iota}|c, p_{\nu_\kappa}|(c-1), P|a, 1 \leqslant \iota, \kappa \leqslant i$. Therefore, it follows from the Chinese Remainder Theorem just like before that the cardinality of $S(n; p_{\mu_1}, p_{\nu_1}) \cap \cdots \cap S(n; p_{\mu_i}, p_{\nu_i})$ is $\mathrm{int}(n/P)(\mathrm{int}(n/P) + 1)/2$ if the set of i pairs $(\Psi^{-1}(\mu_1), \Psi^{-1}(\nu_1)), \ldots, (\Psi^{-1}(\mu_i), \Psi^{-1}(\nu_i))$ is one like in the statement of Proposition 7; otherwise $S(n; p_{\mu_1}, p_{\nu_1}) \cap \cdots \cap S(n; p_{\mu_i}, p_{\nu_i}) = \emptyset$. Consequently, the sum of the cardinalities of $S(n; p_{\mu_1}, p_{\nu_1}) \cap \cdots \cap S(n; p_{\mu_i}, p_{\nu_i})$ such that i and $P, P \leqslant n$, are given (or equivalently, i and k and $\lambda_1, \ldots, \lambda_k$ are given) is $\omega_k^{(i)}\mathrm{int}(n/P)(\mathrm{int}(n/P) + 1)/2$. Note that the latter is equal to 0 if $P > n$, in which case $S(n; p_{\mu_1}, p_{\nu_1}) \cap \cdots \cap S(n; p_{\mu_i}, p_{\nu_i}) = \emptyset$.

Therefore (see the comment right before Proposition 7 concerning the range of k),

$$(1/2)\sum_{i=1}^{s}\sum_{2\sqrt{i}\leqslant k\leqslant 2i}(-1)^{i-1}\omega_k^{(i)}\sum_{\lambda_1<\cdots<\lambda_k}\operatorname{int}(n/(p_{\lambda_1}\cdots p_{\lambda_k}))(\operatorname{int}(n/(p_{\lambda_1}\cdots p_{\lambda_k}))+1) \tag{15}$$

is the number of pairs (c,a) in $\bigcup_{p\neq q} S(n;p,q)$ if each pair belonging to exactly m sets $S(n;p,q)$ is counted $\sum_{i=1}^{s}(-1)^{i-1}\binom{m}{i}$ times. (Here we use the convention that $\binom{m}{i}=0$ for $i>m$. It is obvious that if $s\geqslant m$, then the pair is counted once.) Since for every m, the multiplicity of counting is $\geqslant 1$ or $\leqslant 1$ according as s is odd or even, the expression in (15) is $\geqslant\hat{\phi}^{-}(n)$ or $\leqslant\hat{\phi}^{-}(n)$ according as s is odd or even. Replacing $\operatorname{int}(n/(p_{\lambda_1}\cdots p_{\lambda_k}))(\operatorname{int}(n/(p_{\lambda_1}\cdots p_{\lambda_k}))+1)=\operatorname{int}^2(n/(p_{\lambda_1}\cdots p_{\lambda_k}))+\operatorname{int}(n/(p_{\lambda_1}\cdots p_{\lambda_k}))$ with $n^2/(p_{\lambda_1}\cdots p_{\lambda_k})^2$ in (15) produces $\chi_s n^2$. The error caused by this is not larger than

$$K(s)\sum_{2\leqslant k\leqslant 2s}\sum_{\lambda_1<\cdots<\lambda_k}[(n/(p_{\lambda_1}\cdots p_{\lambda_k}))^2-\operatorname{int}^2(n/(p_{\lambda_1}\cdots p_{\lambda_k}))+n/(p_{\lambda_1}\cdots p_{\lambda_k})]$$

$$\leqslant K(s)n\sum_{2\leqslant k\leqslant 2s}\sum_{\lambda_1<\cdots<\lambda_k}3/(p_{\lambda_1}\cdots p_{\lambda_k}), \tag{16}$$

where $K(s)=(1/2)\sum_{i=1}^{s}\sum_{2\sqrt{i}\leqslant k\leqslant 2i}\omega_k^{(i)}$ depends on s only. Since the infinite series appearing in (16) are all convergent and there are only finitely many of them, the error is $\mathrm{O}(n)$, and thus Proposition 7 is proved. □

Although we have not proved it rigorously, the χ_s seem to close in on a definite value. The first five χ are approximately $0.06377, 0.05205, 0.05383, 0.05330$ and 0.05342. Recurrence relations can be found for the $\omega_k^{(i)}$ and the infinite series in (14) can be calculated recursively, using $\sum_p 1/p^{2k}, k=1,2,\ldots$. The $\omega_k^{(i)}$ grow fast. These calculations will be published in another paper. Proposition 7 combined with Theorem 5 leads to lower and upper estimates for $\bar{N}_h(n)$:

Theorem 8. *For every $t, t=1,2,\ldots,$*

$$(1/6-1/(2\pi^2)+\chi_{2t}/6)n^2+\mathrm{O}(n\log n)$$
$$\leqslant\bar{N}_h(n)\leqslant(1/6-1/(2\pi^2)+\chi_{2t-1}/6)n^2+\mathrm{O}(n\log n).$$

Proof. This follows directly from Theorem 5, Proposition 7, and Section 2 because $\phi(1)+\cdots+\phi(n)=3n^2/\pi^2+\mathrm{O}(n\log n)$ [8, Theorem 330, p. 268] and $d(1)+\cdots+d(n)=n\log n+n(2\gamma-1)+\mathrm{O}(\sqrt{n})$, where γ is Euler's constant [8, Theorem 320, p. 264]. □

Using approximations of χ_4 and χ_5, we obtain that

$$0.12489n^2+\mathrm{O}(n\log n)\leqslant\bar{N}_h(n)\leqslant 0.12491n^2+\mathrm{O}(n\log n).$$

Therefore, asymptotically, 'most' toroidal maps of type $(3,6)$ of multiplicity 1 (almost 99% of them) have a Hamiltonian normal cycle.

8. Toroidal maps of type $(4,4)$

The theory of toroidal maps of type $(4,4)$ is similar to that of triangular maps, and is actually simpler. A *normal* cycle leaves, at each of its vertices, exactly one edge on the right (left). A map G decomposes into two families of nonintersecting normal cycles: $\mathfrak{C}_1$ and $\mathfrak{C}_2$. In contrast to the $(3,6)$ case, any choice of orientations for $\mathfrak{C}_1, \mathfrak{C}_2$ is just as good as any other. We denote one by o_1, o_2 for $\mathfrak{C}_1, \mathfrak{C}_2$, respectively, choose a vertex v of G, and denote by Γ_i the cycle in $\mathfrak{C}_i$ that goes through v. If now G' is another map with orientations o_1', o_2' on its families of normal cycles $\mathfrak{C}_1', \mathfrak{C}_2'$, then an isomorphism from G to G' can map o_1, o_2 to any of the four combinations of $\pm o_1'$ with $\pm o_2'$. Given vertices v_1 and v_2 of G, a unique automorphism exists that maps v_1 to v_2, $\mathfrak{C}_i$ to $\mathfrak{C}_i$ $(i=1,2)$ and keeps (reverses) orientation on normal cycles. These automorphisms are called *trivial*. In contrast to the $(3,6)$ case, not every automorphism mapping each $\mathfrak{C}_i$ to itself is trivial. Correspondingly, a nontrivial automorphism of G 'modulo the trivial automorphisms' is described by a permutation of $\{1,2\}$ (which can be the identical one) *and* the *sign* of the automorphism, which is $+1$ (-1) if it maps o_1, o_2 to o_1, o_2 or to $-o_1, -o_2$ (to $-o_1, o_2$ or to $o_1, -o_2$).

Two codes belong to each of $\mathfrak{C}_1, \mathfrak{C}_2$. The codes belonging to $\mathfrak{C}_i$ are (l_i, c_{ij}, m_i), $(l_i, -c_{ij}, m_i)$, where l_i (m_i) is the length (number) of normal cycles in $\mathfrak{C}_i$ and c_{ij} is the length of the path along Γ_i from v in the direction o_i to the first point of intersection, with Γ_i, of Γ_j, moving from v on Γ_j in the direction $-o_j$. Of course, c_{ij} is again a residue class $\bmod\, l_i$. The set of codes depends only on the isomorphism class and nonisomorphic maps do not share codes.

For switching from (l_i, c_{ij}, m_i) to (l_j, c_{ji}, m_j), we can use (3): $m_j = \gcd(l_i, c_{ij})$, $l_j = m_i l_i / \gcd(l_i, c_{ij}), c_{ji} \equiv 0 \pmod{m_i}, c_{ij} c_{ji} \equiv m_i m_j \pmod{m_i l_i}$. Nontrivial automorphisms correspond to the solutions of: 1. $c_{ij} + c_{ij} \equiv 0 \pmod{l_i}$; 2. $m_i = \gcd(l_i, c_{ij})$, $c_{ij}^2 \equiv 1 \pmod{n/m_i^2}$; 3. $m_i = \gcd(l_i, c_{ij}), c_{ij}^2 \equiv -1 \pmod{n/m_i^2}$. Each of 1., 2., 3. is equivalent to its counterpart obtained by the transposition $i \leftrightarrow j$. The sign of the automorphism associated with 1. or 3. (2.) is -1 (1). The sign of an automorphism is not revealed by the mere coincidence pattern of codes. 1. involves automorphisms preserving each $\mathfrak{C}_i$. The analogs of $T(G)$ and $P(G,v)$ behave similarly to the case of the type $(3,6)$. However, $P(G,v)$ is now isomorphic to either the two-element group or its direct product with itself.

Kurth [14, Lemmas 3.1 and 3.2] gives the value

$$N_q(n) = \frac{1}{4}\left[\sigma(n) + 2d(n) + 2d\left(\frac{n}{4}\right) + (\gamma * \rho_{0,1})(n)\right] \tag{17}$$

for the number $N_q(n)$ of isomorphism classes of toroidal maps of type $(4,4)$ with n vertices. Section 2, combined with a technique similar to that applied in the proof of Proposition 1, yields

$$(\gamma * \rho_{0,1})(n) = \gamma(r_{4,-1}(n)) d(r_{4,1}(n)). \tag{18}$$

It is an easy exercise to verify that

$$2d(n) + 2d(n/4) = \max\{2, 4(1 - (\log_2 r_{2,0}(n) + 1)^{-1})\} d(n).$$

Consequently,

$$N_q(n) = \tfrac{1}{4}[\sigma(n) + \max\{2, 4(1 - (\log_2 r_{2,0}(n)+1)^{-1})\}d(n) + \gamma(r_{4,-1}(n))d(r_{4,1}(n))]. \tag{19}$$

Let $N_{qh}(n)$ denote the number of toroidal maps of type $(4,4)$ with a Hamiltonian normal cycle. For $n > 2$ the exact formula

$$N_{qh}(n) = \tfrac{1}{4}[4\,\mathrm{int}(n/2) + 2 - \phi(n) + \rho_{0,-1}(n) + \rho_{0,1}(n)] \tag{20}$$

can be obtained similarly to Theorem 5. We omit the (easy) details.

From formula (19) and from the asymptotics of the functions appearing in it we obtain an asymptotic formula for the number $\bar{N}_q(n)$ of toroidal maps of type $(4,4)$ with at most n vertices. Using this asymptotic formula, a technique like in Section 6 provides an asymptotic formula for the number $\bar{N}_{q1}$ of toroidal maps of type $(4,4)$ with multiplicity 1 and with at most n vertices. In contrast to the case of type $(3,6)$, we also obtain an asymptotic formula for the number $\bar{N}_{qh}(n)$ of toroidal maps of type $(4,4)$ with at most n vertices and with a Hamiltonian normal cycle. For this we only have to take account of (20) together with the asymptotics of $\phi(n), d(n)$ and the inequalities $\rho_{0,-1}(n), \rho_{0,1}(n) \leqslant d(n)$. The asymptotic formulas are:

$$\bar{N}_q(n) \sim \pi^2 n^2/48, \quad \bar{N}_{q1}(n) \sim 15n^2/(8\pi^2), \quad \bar{N}_{qh}(n) \sim (1/4 - 3/(4\pi^2))n^2 \tag{21}$$

with remainders of $O(n \log n)$. The number of codes for at most n vertices is $\sigma(n)$ again.

In the case of $\bar{N}(n)$ and $\bar{N}_1(n)$ the coefficient of n^2 is 3/2 times the corresponding coefficient for triangular maps, reflecting the fact that 3 (possibly isomorphic) quadrangular maps can be constructed from every triangular map and 2 (possibly isomorphic) triangular maps from every quadrangular map. The case of $\bar{N}_h(n)$ is different because it can happen that only one family of normal cycles consists of Hamiltonian cycles in a triangular map. In that case, the quadrangular map constructed from the other two families does not contain a Hamiltonian normal cycle. Hence less than 92% of quadrangular maps of multiplicity 1 contain a Hamiltonian normal cycle as opposed to triangular maps, for which this percentage is 'almost 99%'. Of course, the same circumstance forced us to consider the function $\hat{\phi}(n)$ in studying the asymptotics of $\bar{N}_h(n)$ for triangular maps.

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