Annals of Pure and Applied Logic 54 (1991) 101–120
North-Holland

A slow growing analogue to Buchholz' proof*

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Communicated by D. van Dalen
Received 19 November 1989
Revised 30 October 1990

Abstract

Arai, T., A slow growing analogue to Buchholz' proof, Annals of Pure and Applied Logic 54 (1991) 101–120.

In this journal, W. Buchholz gave an elegant proof of a characterization theorem for provably total recursive functions in the theory ID_ν for the ν-times iterated inductive definitions ($0 \leq \nu \leq \omega$). He characterizes the classes of functions by Hardy functions. In this note we will show that a slow growing analogue to the theorem can be obtained by a slight modification of Buchholz' proof.

In [3], W. Buchholz gave, among other things, an elegant proof of a boundedness theorem for provably total recursive functions in the theory ID_ν for the ν-times iterated inductive definitions ($0 \leq \nu \leq \omega$):

Theorem (Buchholz [3], cf. also Buchholz and Wainer [5]). *Every provably total recursive function in* ID_ν *is dominated by a Hardy function* $\lambda n H_a(1)$ *with* $a = D_0 D_\nu^n 0$.

In this note, we will show that a slow growing version of the theorem can be obtained by a slight modification of Buchholz's proof: we regard the set ω of natural numbers (or formally the corresponding predicate constant N) as inductively generated. Then for a finite ν, ID_ν is interpretable into $ID_{\nu+1}$ minus the scheme of complete induction. Also ID_ω is interpretable into $ID_{<*}$ minus complete induction, where $ID_{<*}$ denotes a theory in which inductive definitions are permissible along the accessible part $\mathbb{N}$ of the arithmetic 'less than' relation $<$. For these theories proof theory is well developed in [1] and [3] by Buchholz.

* This paper was presented at the International Symposium on Mathematical Logic and its Applications, Nagoya, Japan, November 7–11, 1988.

0168-0072/91/$03.50

Hence it is easy to show our theorem. Let GID_ν denote the theory ID_{p-1} if ν is a positive integer p and the theory ID_ω if $\nu = \Omega$. Then our theorem runs as follows:

Theorem A. $\mathrm{GID}_\nu \vdash \forall n\, \exists m\, (a[n]^m = 0)$ *for each* $a \in T_0(\nu)$.
$(a[n]^m := a[n][n] \cdots [n]$ with m $[n]$'s.)

Theorem B. *Assume a Π^0_2-sentence $\forall x\, \exists y\, \phi(x, y)$ $(\phi \in \Sigma^0_1)$ is provable in* GID_ν. *Then*

(a) $\exists n_0\, \forall n \geq n_0\, \exists m < G_1(D_0 D^n_\nu 0)\, \phi(n, m)$,
(b) $\exists n_0\, \forall n > 0\, \exists m < G_n(D_0 D^{n_0}_\nu 0)\, \phi(n, m)$,
(c) $\exists n_0\, \forall n \geq n_0\, \exists m < G_n(D_0 D_{\nu+1} 0)\, \phi(n, m)$.

Thus every provably total recursive function in GID_ν is dominated by a function $\lambda n G_n(D_0 D^m_\nu 0)$ for some $m \in \omega$ and by the function $\lambda n G_n(D_0 D_{\nu+1} 0)$. Also every provably total recursive function in $\mathrm{ID}_{<\omega}$ is dominated by the function $\lambda n G_n(D_0 D_\omega 0)$. Theorems A and B yield a precise characterization of the provably total recursive functions of GID_ν in terms of the slow growing hierarchy.

Corollary 1. *A recursive function f is provably total recursive in* GID_ν *if, and only if, it is primitive recursive in* $\lambda n G_n(D_0 D^m_\nu 0)$ *for some* $m \in \omega$.

Corollary 2. (a) $\psi_0 \Omega_{\nu+1} = \min\{\alpha \in \mathrm{OT}(\Omega)\colon \mathrm{GID}_\nu \nvdash \forall n\, \exists m\, \alpha[n]^m = 0\}$,
(b) $\psi_0 \Omega_\omega = \min\{\alpha \in \mathrm{OT}(\Omega)\colon \mathrm{ID}_{<\omega} \nvdash \forall n\, \exists m\, \alpha[m]^m = 0\}$,

where $\mathrm{OT}(\Omega)$ *denotes the set of ordinal terms defined in* [3], [4] *and* $\psi_0\Omega_{\nu+1}$, $\psi_0\Omega_\omega$ *are ordinals also defined in* [3], [4]. (The definition of the fundamental sequence $\{\alpha[n]\}_{n\in\omega}$ for a countable ordinal α in [4] differs from ours for $\alpha > \psi_0\Omega_\omega$. Cf. Remark in Section 3.)

Part I. Finite cases

Throughout this part, p will denote an arbitrary but fixed positive integer.

1. The term structure $(T(p), \cdot[\cdot])$

In this section we will define a term structure $(T(p), \cdot[\cdot])$. $T(p)$ denotes a set of finite sequences of the symbols 0 and D.

Inductive definition of the sets $PT(p)$ and $T(p)$

(T0) $PT(p) \subseteq T(p)$.
(T1) $0 \in T(p)$.
(T2) If $a \in T(p)$ and $u \in \{0, \ldots, p\}$, then $D_u a \in PT(p)$.
(T3) If $a_0, \ldots, a_k \in PT(p)$ $(k > 0)$, then $(a_0, \ldots, a_k) \in T(p)$.

The letters a, b, c, z now always denote elements of $T(p)$ and u, v, w denote elements of $\{0, \ldots, p\}$. $a = b$ means that a is identical with b.

For $a_0, \ldots, a_k \in PT(p)$ and $k \in \{-1, 0\}$, we set

$$(a_0, \ldots, a_k) := \begin{cases} 0, & \text{if } k = -1, \\ a_0, & \text{if } k = 0. \end{cases}$$

Definition of $a + b$ and $a \cdot n \in T(p)$ for $a, b \in T(p)$ and $n \in \omega$

$$a + 0 := 0 + a := a,$$

$$(a_0, \ldots, a_k) + (b_0, \ldots, b_m) := (a_0, \ldots, a_k, b_0, \ldots, b_m) \quad (k, m \geq 0),$$

$$a \cdot 0 := 0, \qquad a \cdot (n + 1) := a \cdot n + a.$$

Convention. We identify ω with the subset $\{0, 1, 1 + 1, \ldots\}$ of $T(p)$. ($1 := D_0 0$.)

Definition of $T_u(p)$ for $u \leq p$

$$T_u(p) := \{(D_{u_0} a_0, \ldots, D_{u_k} a_k) : k \geq -1, a_0, \ldots, a_k \in T(p), u_0, \ldots, u_k \leq u\}.$$

Now we define, for every $a \in T(p)$, a subset $\mathrm{dom}(a)$ of $T(p)$ and a function $z \mapsto a[z]$ from $\mathrm{dom}(a)$ into $T(p)$.

Definition of $\mathrm{dom}(a)$ *and $a[z]$ for $a \in T(p)$ and $z \in \mathrm{dom}(a)$*

([ ].0) $\mathrm{dom}(0) := \emptyset$.
([ ].1) $\mathrm{dom}(1) := \{0\}$; $1[0] := 0$.
([ ].2) $\mathrm{dom}(D_{u+1} 0) := T_u(p)$; $(D_{u+1} 0)[z] := z$.
([ ].3) Let $a = D_v b$ with $b \neq 0$.
- 3.1. If $b = b_0 + 1$, then $\mathrm{dom}(a) := \omega$ and $a[n] := (D_v b_0) \cdot (n + 1)$.
- 3.2. If $\mathrm{dom}(b) \in \{\omega\} \cup \{T_u(p) : u < v\}$, then $\mathrm{dom}(a) := \mathrm{dom}(b)$, $a[z] := D_v b[z]$.
- 3.3. If $\mathrm{dom}(b) = T_u(p)$ with $v \leq u < p$, then $\mathrm{dom}(a) := \omega$, $a[n] := D_v b[b_n]$, where $b_0 := 1$ and $b_{m+1} := D_u b[b_m]$.

([ ].4) Let $a = (a_0, \ldots, a_k)$ with $k > 0$. $\mathrm{dom}(a) := \mathrm{dom}(a_k)$; $a[z] := (a_0, \ldots, a_{k-1}) + a_k[z]$.

Remark. The definition of $a[z]$ is the same as that given in [3] except 3.3. Also it is a variant of the fundamental sequences in [4, §5] when we restrict $a[z]$ to the ordinal terms a, $z \in \mathrm{OT}(p)$ in [2]. Hence, as in [3], [4], we have the following proposition:

Proposition. (a) $c, a \in \mathrm{OT}(p)\ \&\ c < a \Rightarrow \exists z \in \mathrm{dom}(a) \cap \mathrm{OT}(p)\ (c \leq a[z])$.

(b) $c, a \in \mathrm{OT}(p) \cap T_0(p)\ \&\ c < a\ \Rightarrow$ *the function* $\lambda n G_n(c)$ *is majorized by* $\lambda n G_n(a)$.

Proposition 1. (a) *$a \in T_v(p) \Rightarrow \mathrm{dom}(a) \in \{\phi, \{0\}, \omega\} \cup \{T_u(p)\colon u<v\}$, and $a[z] \in T_v(p)$ for all $z \in \mathrm{dom}(a)$.*

(b) *$\mathrm{dom}(a+b) = \mathrm{dom}(b)$ & $(a+b)[z] = a + b[z]$ if $b \neq 0$.*

As in the Introduction, GID_p will denote the theory ID_{p-1}. (ID_0 is another name of PA, the first-order arithmetic.) The theory ID_{p-1} is defined in Section 2.

Convention. $0[n] := 0$ and $(a+1)[n] := a$ for each $n \in \omega$ and any $a \in T(p)$.

Definition. For $a \in T_0(p)$ and $n, m \in \omega$, we set

$$a[n]^0 := a; \qquad a[n]^{m+1} := (a[n]^m)[n] \qquad \text{(cf. Proposition 1(a)).}$$

We will prove Theorem A. In what follows, we will work in GID_p. Let n be a fixed natural number.

Iterated inductive definition of sets $W_{un} \subseteq T_u(p)$ $(u<p)$

(W1) $0 \in W_{un}$.
(W2) $a \in T_u(p)$, $\mathrm{dom}(a) \in \{\{0\}, \omega\}$, $a[n] \in W_{un} \Rightarrow a \in W_{un}$.
(W3) $a \in T_u(p)$, $\mathrm{dom}(a) = T_v(p)$ with $v<u$, $\forall z \in W_{vn}(a[z] \in W_{un}) \Rightarrow a \in W_{un}$.

Proposition 2. (a) $a \in W_{on} \Leftrightarrow \exists m\, (a[n]^m = 0)$ for $a \in T_0(p)$.

(b) $v<u<p \Rightarrow W_{vn} \subseteq W_{un}$.

Abbreviations. Let X range over subsets of $T(p)$ which are definable in the language of GID_p.

1. By $A_{un}(X, a)$ $(u \leq p)$ we denote the following statement:

$$a \in T_u(p) \,\&\, [a = 0 \vee (\mathrm{dom}(a) \in \{\{0\}, \omega\} \,\&\, a[n] \in X) \vee \exists v<u\, (\mathrm{dom}(a) = T_v(p) \,\&\, \forall z \in W_{vn}(a[z] \in X))].$$

2. $A_{un}(X) := \{x \in T(p)\colon A_{un}(X, x)\}$.
3. $X^{(a)} := \{y \in T(p)\colon a + y \in X\}$.
4. $\bar{X} := \{y \in T(p)\colon \forall x \in X(x + D_p y \in X)\}$.
5. $W_n^* := \{x \in T(p)\colon \forall u<p\, (D_u x \in W_{un})\}$.

By the definition of W_{un}, for all $u<p$ we have:

(A1) $A_{un}(W_{un}) = W_{un}$,
(A2) $A_{un}(X) \subseteq X \Rightarrow W_{un} \subseteq X$.

The following lemma can be proved exactly as in [3].

1.1. Lemma. (a) $A_{un}(X) \subseteq X$ & $a \in X \cap T_u(p) \Rightarrow A_{un}(X^{(a)}) \subseteq X^{(a)}$ $(u \leq p)$.
(b) $a, b \in W_{un} \Rightarrow a + b \in W_{un}$ $(u < p)$.

1.2. Lemma. (a) $A_{pn}(X) \subseteq X \Rightarrow \bigcup \{W_{un}: u < p\} \subseteq X$.
(b) $0 \in W_n^*$.
(c) $A_{pn}(W_n^*) \subseteq W_n^*$.

Proof. (a) This follows from (A2) and the fact:

$$A_{pn}(X) \subseteq X \Rightarrow \forall u < p\ (A_{un}(X) \subseteq X).$$

(b) We have to show $\forall u < p\ (D_u 0 \in W_{un})$. Clearly (1) $D_0 0 \in W_{0n}$. If $u = v + 1 < p$, then $W_{vn} \subseteq W_{un}$, $\mathrm{dom}(D_u 0) = T_v(p)$ and $(D_u 0)[z] = z$ for $z \in T_v(p)$. Therefore (2) $\exists v < p\ (u = v + 1) \Rightarrow D_u 0 \in W_{un}$. We are done.

(c) Assume $b \in A_{pn}(W_n^*)$ and $u < p$. We show $a := D_u b \in W_{un}$.

1. $b = 0$: This follows from (b).

2. $b = b_0 + 1$ and $b_0 \in W_n^*$: Then $\mathrm{dom}(a) = \omega$ and $a[n] = (D_u b_0) \cdot (n + 1)$. By $u < p$ and $b_0 \in W_n^*$ we have $D_u b_0 \in W_{un}$. Using 1.1(b) we obtain $\forall m\ ((D_u b_0) \cdot m \in W_{un})$ by induction on m and hence $a \in W_{un}$.

3. $\mathrm{dom}(b) = T_v(p)$, $v < p$, and $b[z] \in W_n^*$ for all $z \in W_{vn}$:

3.1. $v < u$: Then we have $\mathrm{dom}(a) = T_v(p)$ and $a[z] = D_u b[z] \in W_{un}$ for all $z \in W_{vn}$, i.e., $a \in W_{un}$.

3.2. $u \leq v$: Then we have $\mathrm{dom}(a) = \omega$ and $a[n] = D_u b[b_n]$, where $b_0 = 1$ and $b_{m+1} = D_v b[b_m]$. By induction on m we have $\forall m\ (b_m \in W_{vn})$. Therefore $b_n \in W_{vn}$ and $D_u b[b_n] \in W_{un}$. Hence $a \in W_{un}$.

4. $\mathrm{dom}(b) = \omega$: Then $\mathrm{dom}(a) = \omega$ and $a[n] = D_u b[n]$. By $b[n] \in W_n^*$ we have $a[n] \in W_{un}$, i.e., $a \in W_{un}$. □

1.3. Lemma. $A_{pn}(X) \subseteq X \Rightarrow A_{pn}(\bar{X}) \subseteq \bar{X}$.

Proof. Assume $A_{pn}(X) \subseteq X$, $A_{pn}(\bar{X}, b)$ and $a \in X$. We have to show $a + D_p b \in X$, i.e., $D_p b \in X^{(a)}$.

1. $b = 0$: by 1.1(a) and 1.2(a) we have $\cup \{W_u: u < p\} \subseteq X^{(a)}$. $\mathrm{dom}(D_p 0) = T_{p-1}(p)$, $(D_p 0)[z] = z$ for $z \in T_{p-1}(p)$ and $W_{p-1,n} \subseteq X^{(a)}$. Hence $A_{pn}(X^{(a)}, D_p 0)$ and thus $D_p 0 \in X^{(a)}$, since $A_{pn}(X^{(a)}) \subseteq X^{(a)}$, by 1.1(a).

2. $b = b_0 + 1$ and $b_0 \in \bar{X}$: Then $\mathrm{dom}(a + D_p b) = \omega$ and $(a + D_p b)[n] = a + (D_p b_0) \cdot (n + 1)$. By induction we get $(a + D_p b)[n] \in X$ from $b_0 \in X$ and hence $a + D_p b \in X$.

3. $\mathrm{dom}(b) = T_v(p)$, $v < p$, and $b[z] \in \bar{X}$ for all $z \in W_{vn}$: Then $\mathrm{dom}(a + D_p b) = T_v(p)$ and $(a + D_p b)[z] = a + D_p b[z] \in X$ for all $z \in W_{vn}$. Hence $a + D_p b \in X$.

4. $\mathrm{dom}(b) = \omega$ and $b[n] \in \bar{X}$: Similar to 3. □

1.4. Lemma. *For each* $a \in T(p)$,

(a) $a \in T_0(p) \Rightarrow a \in W_{0n}$,

(b) $A_{pn}(X) \subseteq X \Rightarrow a \in X$.

Proof. By simultaneous metainduction on the length of a.

1. $a = 0$: Clear.

2. $a = (a_0, \ldots, a_k)$ $(k > 0)$: By using 1.1 and IH (Induction Hypothesis) we get (a) and (b).

3. $a = D_u b$: By 1.2(c) and IH we have $b \in W_n^*$.

(a) Assume $a \in T_0(p)$, i.e., $u = 0$. By $b \in W_n^*$ and $0 < p$ we get $a \in W_{0n}$.

(b) Assume $A_{pn}(X) \subseteq X$.

Case 1. $u \neq p$: By 1.2(a), $a \in W_{un} \subseteq X$.

Case 2. $u = p$: Then by 1.3 and IH we have $b \in \bar{X}$. By $0 \in X$ we get $a = 0 + D_p b \in X$. □

Lemma 1.4(a) yields Theorem A for $v = p$.

Using 1.2(c) and 1.4(b), we get $a \in W_n^*$ for each $a \in T(p)$. Now we work outside GID_p. Then since GID_p is sound, we have:

1.5. Lemma. $\forall u < p\ (T_u(p) = W_{un})$.

For each $n \in \omega$, let $<_n$ denote the following relation over $T(p)$:

$$b <_n a \quad :\Leftrightarrow \quad a \neq 0 \ \& \ (\mathrm{dom}(a) \in \{\{0\}, \omega\} \Rightarrow b = a[n])$$
$$\& \ (\mathrm{dom}(a) \in \{T_u(p) : u < p\} \Rightarrow \exists z \in \mathrm{dom}(a)\ (b = a[z])).$$

Let X_n denote the accessible part of $<_n$:

$$X_n = \bigcap \{X \subseteq T(p) : \forall a\ (\forall b <_n a\ (b \in X) \Rightarrow a \in X)\}.$$

Then by 1.5 we have $A_{pn}(X_n) \subseteq X_n$ and hence $X_n = T(p)$. Therefore

1.6. Lemma. *The relation* $<_n$ *is well founded for all* $n \in \omega$.

In what follows transfinite induction over $T(p)$ or on $a \in T(p)$ means a transfinite induction with respect to $<_n$ for an n.

Definition of $c \ll_k a$ *by transfinite induction on* $a \in T(p)$

$$c \ll_k a \quad :\Leftrightarrow \quad a \neq 0 \ \& \ \forall z \in d_k(a)\ (c \underline{\ll}_k a[z])$$

where

$$d_k(a) := \begin{cases} \{k\}, & \text{if } \mathrm{dom}(a) \in \{\{0\}, \omega\}, \\ \{D_u e : 0 \neq e \in T(p)\}, & \text{if } \mathrm{dom}(a) = T_u(p), \end{cases}$$

and

$$c \underline{\ll}_k a \quad :\Leftrightarrow \quad c \ll_k a \quad \text{or} \quad c = a.$$

Definition of the function $G_k : T_0(p) \to \omega$ by transfinite induction over $T_0(p)$

(G1) $G_k(0) = 0$.
(G2) $G_k(a+1) = G_k(a) + 1$.
(G3) $G_k(a) = G_k(a[k])$ if $\mathrm{dom}(a) = \omega$.

Definition of $D^n_u a$ for $u \leq p$, $n \in \omega$ and $a \in T(p)$

$$D^0_u a := a, \qquad D^{n+1}_u a := D_u D^n_u a.$$

The following lemma is proved by transfinite induction over $T(p)$ (cf. 3.1, 3.2, 3.5 and 3.7 in [3]):

1.7. Lemma. (a) $a \neq 0 \Rightarrow 1 \underline{\ll}_k a$.
(b) $c \ll_k a \;\&\; a \ll_k b \Rightarrow c \ll_k b$.
(c) $c \ll_k b \Rightarrow a + c \ll_k a + b$.
(d) $b \neq 0 \Rightarrow a \ll_k a + b$.
(e) $a \ll_k b \Rightarrow D_u a \ll_k D_u b$.
(f) $c \ll_k a \Rightarrow G_k(c) \leq G_k(a)$ *for* $c, a \in T_0(p)$.
(g) $\mathrm{dom}(a) = T_v(p) \;\&\; u \leq v \;\&\; 1 \leq k \Rightarrow (D_u a)[1] \ll_k D_u a$.
(h) $D_u a + 1 \ll_k D_u(a+1)$ *for* $k \geq 1$.
(i) $(D^m_u a) \cdot (k+1) \ll_k D^m_u(a+1)$ *for* $m > 0$.
(j) $(D^m_u 0) \cdot (k+1) \ll_k D^{m+1}_u 0$.

Definition of $a \to^k_n b$ for $a, b \in T(p)$ and $k, n \in \omega$

$$a \to^k_n b \quad :\Leftrightarrow \quad \exists a_0, \ldots, a_n\, [a = a_0 \;\&\; b = a_n \;\&\; \forall i < n\, (a_i + 1 \underline{\ll}_k a_{i+1})].$$

Clearly $G_k(a) = \max\{n \in \omega : 0 \to^k_n a\}$.

1.8. Lemma. (a) $a \to^k_n b \Rightarrow D_u a \to^k_n D_u b$.
(b) $0 \to^k_{g_p(k)} D_p 0$ *with* $g_p(k) = (k+1)^p$.
(c) $0 \to^k_n a \Rightarrow 0 \to^k_{(k+1)^{p+n}} D_p a$.
(d) $0 \to^k_{f_p(k,l)} D^l_p 0$ *with* $f_p(k, 0) = 0$, $f_p(k, l+1) = (k+1)^{p + f_p(k,l)}$.

Proof. (b) $0 \to^k_1 D_0 0$ and $(D_u 0) \cdot (k+1) \ll_k D_u 1 \ll_k D_{u+1} 0$ for $u < p$. Hence by induction on $u \leq p$ we get the assertion.

(c) By induction on n. The case $n = 0$ follows from (b). The induction step is seen easily.

(d) By induction on l using (b) and (c). □

2. The infinitary system GID_p^∞

Let L denote the first order language consisting of the following symbols:

(i) logical constants $\neg$, $\wedge$, $\vee$, $\forall$ $\exists$.

(ii) number variables (indicated by x, y).

(iii) a constant 0 (zero) and a unary function symbol $'$ (successor).

(iv) constants for primitive recursive predicates (among them the symbol $<$ for the arithmetic 'less than' relation).

By $s, t, \ldots$ we denote arbitrary L-terms. The constant terms $0, 0', 0'', \ldots$ are called numerals; we identify numerals and natural numbers and denote them by k, m, n. A formula of the shape $Rt_1 \cdots t_n$ or $\neg Rt_1 \cdots t_n$ with an n-ary predicate symbol R of L, is called an arithmetic prime formula (abbreviated by a.p.f.).

Let X be a unary and Y a binary predicate variable. A positive operator form is a formula $\mathfrak{A}(X, Y, y, x)$ of $L(X, Y)$ in which only X, Y, y, x occur free and all occurrences of X are positive. The language L_{ID} is obtained from L by adding a binary predicate constant $P^{\mathfrak{A}}$ and a 3-ary predicate constant $P^{\mathfrak{A}}_<$ for each positive operator form $\mathfrak{A}$.

Abbreviations

$$t \in P^{\mathfrak{A}}_s :\equiv P^{\mathfrak{A}}_s t :\equiv P^{\mathfrak{A}} st, \qquad t \notin P^{\mathfrak{A}}_s :\equiv \neg(t \in P^{\mathfrak{A}}_s), \qquad P^{\mathfrak{A}}_{<s} t_0 t_1 :\equiv P^{\mathfrak{A}}_< s t_0 t_1,$$

$$\mathfrak{A}_s(X, x) :\equiv \mathfrak{A}(X, P^{\mathfrak{A}}_{<s}, s, x), \qquad P^{\mathfrak{A}}_y \subseteq F :\equiv \forall x\,(x \in P_y \to F(x)),$$

$$\mathfrak{A}_y(F) \subseteq F :\equiv \forall x\,(\mathfrak{A}_y(F, x) \to F(x)) \quad \text{for each formula } F(x).$$

The formal theory GID_p is an extension of Peano Arithmetic, formulated in the language L_{ID}, by the following axioms;

($P^{\mathfrak{A}}.1$) $\forall y < p-1\,(\mathfrak{A}_y(P^{\mathfrak{A}}_y) \subseteq P^{\mathfrak{A}}_y)$.

($P^{\mathfrak{A}}.2$) $\forall y < p-1\,(\mathfrak{A}_y(F) \subseteq F \to P^{\mathfrak{A}}_y \subseteq F)$ for every L_{ID}-formula $F(x)$.

($P^{\mathfrak{A}}_<$) $\forall y < p-1\,\forall x_0 \forall x_1\,(P^{\mathfrak{A}}_{<y} x_0 x_1 \leftrightarrow x_0 < y \wedge x_1 \in P^{\mathfrak{A}}_{x_0})$.

PA formulated in L_{ID} means that GID_p has the following scheme of complete induction;

(CI) $\quad F(0) \wedge \forall x\,(F(x) \to F(x')) \to \forall x\, F(x)$, for every L_{ID}-formula $F(x)$.

The infinitary system GID_p^∞ will be formulated in the language $L_{\mathrm{ID}}(N)$ which arises from L_{ID} by adding a new unary predicate constant N. We assume all formulas to be in negation normal form, i.e., the formulas are built up from atomic and negated atomic formulas by means of $\wedge$, $\vee$, $\forall$, $\exists$. If A is a complex formula we consider $\neg A$ as a notation for the corresponding negation normal form.

Definition of the length $|A|$ of an $L_{\mathrm{ID}}(N)$-formula A

1. $|A| := |\neg A| := 0$, if A is an a.p.f. or a formula of the form Nt, $P^{\mathfrak{A}}st$.
2. $|P^{\mathfrak{A}}_{<}st_0t_1| := |\neg P^{\mathfrak{A}}_{<}st_0t_1| := 1$.
3. $|A \$ B| := \max\{|A|, |B|\} + 1$ for $\$ \in \{\wedge, \vee\}$.
4. $|\mathrm{Q}x\,A| := |A| + 1$ for $\mathrm{Q} \in \{\forall, \exists\}$.

Clearly $|\neg A| = |A|$, for each $L_{\mathrm{ID}}(N)$-formula A.

Inductive definition of formula sets Pos_u $(u < p)$

1. All a.p.f.'s belong to Pos_u.
2. $Nt \in \mathrm{Pos}_u$.
3. $\neg Nt \in \mathrm{Pos}_u \Leftrightarrow 0 < u$, i.e., $u \neq 0$.
4. $P^{\mathfrak{A}}_n t$, $(\neg)P^{\mathfrak{A}}_{<n}t_0t_1 \in \mathrm{Pos}_u \Leftrightarrow n + 1 \leq u$.
5. $\neg P^{\mathfrak{A}}_n t \in \mathrm{Pos}_u \Leftrightarrow n + 1 < u$.
6. $A \$ B \in \mathrm{Pos}_u \Leftrightarrow A, B \in \mathrm{Pos}_u$, $\$ \in \{\wedge, \vee\}$.
7. $\mathrm{Q}x\,A \in \mathrm{Pos}_u \Leftrightarrow A \in \mathrm{Pos}_u$, $\mathrm{Q} \in \{\forall, \exists\}$.

Remark. If a term s contains a variable, then formulas $(\neg)P^{\mathfrak{A}}_s t$, $(\neg)P^{\mathfrak{A}}_{<s}t_0t_1$ do not belong to Pos_u.

Notations. (1) In the following A, B, C always denote closed $L_{\mathrm{ID}}(N)$-formulas.

(2) Γ and Δ denote finite sets of closed $L_{\mathrm{ID}}(N)$-formulas, we write Γ, Δ, A for $\Gamma \cup \Delta \cup \{A\}$.

(3) A^N denotes the results of restricting all quantifiers in A to N.

(4) $t \in N :\equiv Nt$, $t \notin N :\equiv \neg Nt$.

Basic inference rules

$(\wedge)$	$A_0, A_1 \vdash A_0 \wedge A_1$.
$(\vee)$	$A_i \vdash A_0 \vee A_1$ $(i = 0, 1)$.
$(\forall^\infty)$	$(A(n))_{n\in\omega} \vdash \forall x\,A(x)$.
$(\exists)$	$A(n) \vdash \exists x\,A(x)$.
$(P^{\mathfrak{A}}_{<})$	$P^{\mathfrak{A}}_j t \vdash P^{\mathfrak{A}}_{<n}jt$, if $j < n$.
$(\neg P^{\mathfrak{A}}_{<})$	$\neg P^{\mathfrak{A}}_j t \vdash \neg P^{\mathfrak{A}}_{<n}jt$, if $j < n$.
(N)	$n = 0 \vee (n = m' \wedge Nm) \vdash Nn$.
$(P^{\mathfrak{A}})$	$t \in N \wedge \mathfrak{A}^N_n(P^{\mathfrak{A}}_n, t) \vdash P^{\mathfrak{A}}_n t$.

Every instance $(A_i)_{i\in I} \vdash A$ of these rules is called a basic inference. If $(A_i)_{i\in I} \vdash A$ is a basic inference with $A \in \mathrm{Pos}_u$ $(u < p)$, then $A_i \in \mathrm{Pos}_u$ for all $i \in I$. We divide the basic inferences into three kinds: Every instance of rules $(\wedge)$, $(\forall^\infty)$, $(P^{\mathfrak{A}}_{<})$ is said to be of kind 1. Every instance of rules $(\vee)$, $(\exists)$, $(\neg P^{\mathfrak{A}}_{<})$ is said to be of kind 2. Every instance of rules (N), $(P^{\mathfrak{A}})$ is said to be of kind 3. Next we define a derivability relation $k \vdash^a_m \Gamma$ for GID^∞_p by an iterated inductive definition.

Inductive definition of $k \vdash^a_m \Gamma$ ($a \in T(p)$ and $k, m \in \omega$)

(Ax1) $k \vdash^a_m \Gamma, A$ if A is a true a.p.f. or $A \equiv \neg P^{\mathfrak{A}}_{<n} jt$ with $n \leq j$.

(Ax2) $k \vdash^a_m \Gamma, \neg A, A$ if $A \in Nn$ *or* $A \equiv P^{\mathfrak{A}}_n t$.

(Bas1) If $(A_i)_{i \in I} \vdash A$ is a basic inference of kind 1 with $A \in \Gamma$ and $\forall i \in I$ $(k \vdash^a_m \Gamma, A_i)$, then $k \vdash^a_m \Gamma$.

(Bas2,3) If $(A_i)_{i \in I} \vdash A$ is a basic inference of kind 2 or of kind 3 with $A \in \Gamma$ and $\forall i \in I\, (k \vdash^a_m \Gamma, A_i)$, then $k \vdash^{a+1}_m \Gamma$.

(Cut) $k \vdash^a_m \Gamma, \neg C$ and $k \vdash^a_m \Gamma, C$ and $|C| < m \Rightarrow k \vdash^{a+1}_m \Gamma$.

(Ω_{u+1}) If Rt is an atomic formula of the shape Nt or $P^{\mathfrak{A}}_n t$, and the following four conditions hold, then $k \vdash^a_m \Gamma$.

(1) $\mathrm{dom}(a) = T_u(p)$.

(2) $k \vdash^{a[1]}_m \Gamma, Rt$.

(3) $\forall z \in T_u(p)\, \forall \Delta \subseteq \mathrm{Pos}_u\, (k \vdash^z_0 \Delta, Rt \Rightarrow k \vdash^{a[z]}_m \Delta, \Gamma)$.

(4) $Rt \in \mathrm{Pos}_u$.

($\ll_k$) $k \vdash^b_m \Gamma$ and $b \ll_k a \Rightarrow k \vdash^a_m \Gamma$.

2.1. Lemma (Inversion). *Let* $(A_i)_{i \in I} \vdash A$ *be a basic inference of kind* 1. *Then* $k \vdash^a_m \Gamma, A$ *implies* $\forall i \in I\, (k \vdash^a_m \Gamma, A_i)$.

Proof. By transfinite induction on $k \vdash^a_m \Gamma, A$. Precisely speaking, if $k \vdash^a_m \Gamma, A$ holds in a stage of inductive definition of the derivability relation, then $k \vdash^a_m \Gamma, A_i$ holds in the same or a previous stage for all i. □

2.2. Lemma (Reduction). *Suppose* $k \vdash^a_m \Gamma_0, \neg C$ *and* $|C| \leq m$, *where* C *is a formula of the shape* $A \vee B$ *or* $\exists x\, A$ *or* $\neg P^{\mathfrak{A}}_{<n} jt$ *or* $\neg P^{\mathfrak{A}}_n t$ *or* Nn *or a false a.p.f. Then* $k \vdash^b_m \Gamma, C$ *implies* $k \vdash^{a+b}_m \Gamma_0, \Gamma$.

Proof. By transfinite induction on $k \vdash^b_m \Gamma, C$ using 2.1, Proposition 1 and 1.7 in Section 1. If $k \vdash^b_m \Gamma, C$ holds by (Ax1) and $C \equiv \neg P^{\mathfrak{A}}_{<n} jt$ with $n \leq j$, then use the following proposition. Cf. Lemma 4.4 in [3] for other cases. □

Proposition. $k \vdash^a_m \Gamma_0, P^{\mathfrak{A}}_{<n} jt$ & $n \leq j \Rightarrow k \vdash^a_m \Gamma_0$.

2.3. Theorem (Cut elimination). $k \vdash^a_{m+1} \Gamma \Rightarrow k \vdash^{D_p a}_m \Gamma$ *for* $k \geq 1$.

Proof. By transfinite induction on $k \vdash^a_{m+1} \Gamma$ using 1.7 and 2.2. Cf. Theorem 4.5 in [3]. □

2.4. Lemma. $k \vdash^a_0 n \notin N, \Gamma$ *and* $0 \rightarrow^k_m b$ *and* $2n + 2 \leq m \Rightarrow k \vdash^{b+a}_0 \Gamma$.

Proof. This follows from the fact;

$$k \vdash^b_0 n \in N \quad \text{if } 0 \rightarrow^k_m b \text{ and } 2n + 2 \leq m. \qquad \square$$

2.5. Theorem (Collapsing). (a) $k \vdash_0^a \Gamma \;\&\; \Gamma \subseteq \mathrm{Pos}_u \Rightarrow k \vdash_0^{D_u a} \Gamma$ *for* $k \geqslant 1$.
(b) $k \vdash_0^a \Gamma \;\&\; a \in T_0(p) \Rightarrow k \vdash_0^{G_k(a)} \Gamma$ *for* $k \geqslant 1$.

Proof. By transfinite induction on $k \vdash_0^a \Gamma$. Cf. Theorem 4.6 in [3] and 1.7(g). □

Definition. For $\Gamma = \{A_1, \ldots, A_n\} \subseteq \mathrm{Pos}_0$ we define: $\models \Gamma(j) :\Leftrightarrow A_1 \vee \cdots \vee A_n$ is true in the standard model when N is interpreted as $\{i \in \omega : i < j\}$.

2.6. Lemma (Truth). $k \vdash_0^j \Gamma \;\&\; \Gamma \subseteq \mathrm{Pos}_0 \;\&\; j \in \omega \Rightarrow \models \Gamma(j)$.

Proof. This follows from the fact:

$$\models n = 0 \vee (n = m' \wedge Nm)(j) \Rightarrow \models Nn(j+1). \qquad \square$$

In the remainder of this section we show that GID_p can be embedded into GID_p^∞.

2.7. Lemma. (a) $0 \rightarrow_n^k a \;\&\; k \vdash_0^n \Gamma \Rightarrow k \vdash_0^a \Gamma$.
(b) $k \vdash_0^a \Gamma, A \vee B \Rightarrow k \vdash_0^a \Gamma, A, B$.
(c) $k \vdash_0^a \Gamma, A$ & A *is a false a.p.f.* $\Rightarrow k \vdash_0^a \Gamma$.
(d) $k \vdash_0^0 \Gamma, A \vee B \Rightarrow k \vdash_0^0 \Gamma$.
Furthermore, in each case except (a), if $k \vdash_0^a \Delta$ *holds under the condition, then* $k \vdash_0^b \Delta'$ *holds in the same or a previous stage of the inductive definition of the derivability relation.*

Definition. $\bar{n} := D_p^n 0$.

2.8. Lemma. *For any* $k \geqslant 1$, $k \vdash_0^{\bar{l}} \neg A, A$ *where* $l := |A|$.

Proof. By induction on the length of A we see that $k \vdash_0^l \neg A, A$ if $l = |A|$. By 1.8(d), $0 \rightarrow_l^k \bar{l}$. Hence the assertion follows from 2.7(a). □

Definition. In the following proposition and lemma we will use the following notations: R denotes a 'predicate constant' of the form N or $P_n^{\mathfrak{A}}$. $F(x)$ denotes a formula with a variable x. For $A \in \mathrm{Pos}_u$ $(u < p)$ let A^* denote the result of replacing all occurrences of R in A by $F(\cdot)$. $\{A_1, \ldots, A_m\}^* := \{A_1^*, \ldots, A_m^*\}$. Further let Θ denote the following set:

$$\Theta := \begin{cases} \{\neg F(0), \neg \forall x \in N\, (F(x) \rightarrow F(x'))\}, & \text{if } R \equiv N, \\ \{\neg \forall x \in N\, (\mathfrak{A}_n^N(F, x) \rightarrow F(x))\}, & \text{if } R \equiv P_n^{\mathfrak{A}}. \end{cases}$$

Proposition. $\Delta \cup \Delta_0 \subseteq \mathrm{Pos}_u$, $\neg Rt \notin \mathrm{Pos}_u$, $z \in T_u(p)$, $k \vdash_0^z \Delta$, Δ_0 *and* $l = |F| \Rightarrow$ $k \vdash_0^{\bar{l}+z} \Delta, \Theta, \Delta_0^*$ *for* $k \geqslant 1$.

Proof. By transfinite induction on $k \vdash_0^z \Delta, \Delta_0$.

1. If $k \vdash_0^z \Delta, \Delta_0$ holds by (Ax2), then also $k \vdash_0^{\bar{l}+z} \Delta, \Theta, \Delta_0^*$ by (Ax2), since $\neg Rt \notin \Delta \cup \Delta_0$.

2. Suppose $z = w+1$, $Nm \in \Delta_0$, $R \equiv N$ and $k \vdash_0^w \Delta, \Delta_0, m = 0 \vee (m = j' \wedge Nj)$. Then $F(m) \in \Delta_0^*$. By 2.8

$$(1) \qquad k \vdash_0^{\bar{l}} \neg F(m), F(m).$$

2.1. $m = 0$: Then by (1) and $F(0) \in \Theta$ we get the assertion.

2.2. $m \neq 0$ & $m \neq j+1$: Then by 2.7(b),(c) and 2.1, we have $k \vdash_0^w \Delta, \Delta_0$. By IH we get the assertion.

2.3. $m = j+1$; Then again by 2.7(b),(c) and 2.1, we have $k \vdash_0^w \Delta, \Delta_0, Nj$. By IH we have

$$(2) \qquad k \vdash_0^{\bar{l}+w} \Delta, \Theta, \Delta_0^*, Nj \quad \text{and} \quad (3) \quad k \vdash_0^{\bar{l}+w} \Delta, \Theta, \Delta_0^*, F(j).$$

By (1), (2), (3) using ($\wedge$) and ($\exists$) we get the assertion.

3. Other cases can be treated as in [3]. □

From this proposition and (Ω_{u+1}) we get:

2.9. Lemma. *If $l = |F|$ and $u = 0$ if $R \equiv N$, $u = n+1$ if $R \equiv P_n$, then*

$$k \vdash_0^{\bar{l}+D_{u+1}0} \Theta, \neg Rt, F(t) \quad \textit{for any } k \geq 1, \textit{ and } n < p-1.$$

2.10. Lemma. *For each universal closure A of an axiom of the theory GID_p there exists an $n \in \omega$ such that $k \vdash_0^{\bar{n}} A^N$ for any $k \geq 1$.*

Proof. This follows from 1.7 and 2.9. □

2.11. Theorem. *If a sentence A is provable in GID_p, then there exists an $n \in \omega$ such that $k \vdash_n^{\bar{n}} A^N$ for any $k \geq 1$.*

Proof. This follows from 1.7 and 2.10. (Cf. Theorem 4.14 in [3].) □

Proof of Theorem B. Assume $\mathrm{GID}_p \vdash \forall x\, \exists y\, \phi(x, y)$ ($\phi \in \Sigma_1^0$). By 2.11 and 2.3 (also by 2.1 and 2.7(b)), there exists an n_1 such that

$$k \vdash_0^{\bar{n}_1} n \notin N, \exists y \in N\, \phi^N(n, y) \quad \text{for all } n \text{ and all } k \geq 1.$$

W.l.o.g. we can assume $n_1 \geq 2$.

(a) Put $n_0 := n_1 + 1$ and $k := 1$. If $3 \leq n_0 \leq n$, then $2n + 2 \leq f_p(1, n-1)$. Hence by 1.8(d) and 2.4 we have

$$1 \vdash_0^{a_n} \exists y \in N\, \phi^N(n, y) \quad \text{with } a_n = D_p^{n-1}0 + D_p^n 0.$$

By 1.7, we have $D_0 a_n \ll_1 D_0 D_p^n 0$ and hence by 2.5 and 2.6 we conclude

$$\models \exists y \in N\, \phi^N(n, y)(G_1(D_0 D_p^n 0)),$$

and by persistency of Σ_1^0 formulas we get the assertion.

(b) Put $n_0 := n_1 + 1$ and $k := n > 0$. If $n \neq 0$, then $2n + 2 \leq f_p(n, 2) \leq f_p(n, n_1)$. Therefore by 1.8(d) and 2.4 we have

$$n \vdash_0^a \exists y \in N\, \phi^N(n, y) \quad \text{with } a = D_p^{n_0} 0 \cdot 2.$$

By 1.7, $D_0 a \ll_n D_0 D_p^{n_0} 0$ and hence by 2.5 and 2.6 we conclude that

$$\models \exists y \in N\, \phi^N(n, y)(G_n(D_0 D_p^{n_0} 0)).$$

(c) By definition $(D_0 D_{p+1} 0)[n] := D_0 D_p^n 1$. Thus this follows from (b) and 1.7. □

Remark. As in [3; 3.3, 3.4 and 3.6], we can show that, for any $a \in T_0(p)$, the function $\lambda n G_n(a)$ is weakly monotonic, i.e., $G_n(a) \leq G_{n+1}(a)$. From this and Theorem B(a), Theorem B(c) follows.

Part II. The case $\nu = \Omega$

In this part we prove Theorems A and B for the case $\nu = \Omega$. The proof is obtained by a slight modification and extension of that for the case $\nu = p < \omega$, $p \neq 0$. We will give only necessary changes.

3. The term structures $(T(\Omega), \cdot[\cdot]_n)_{n \in \omega}$

Inductive definition of the sets $T_0(\Omega)$, $PT(\Omega)$ and $T(\Omega)$

(T0) $T_0(\Omega) \cup PT(\Omega) \subseteq T(\Omega)$.

(T1) 1. If $a \in T(\Omega)$ and $u \in T_0(\Omega) \cup \{\Omega\}$, then $D_u a \in PT(\Omega)$ ($\Omega := D_1 0$).
  2. If $a \in T(\Omega)$, then $D_0 a \in T_0(\Omega) \cap PT(\Omega)$.

(T2) 1. If $a_0, \ldots, a_k \in PT(\Omega)$ $(k > 0)$, then $(a_0, \ldots, a_k) \in T(\Omega)$.
  2. If $a_0, \ldots, a_k \in T_0(\Omega) \cap PT(\Omega)$ $(k > 0)$, then $(a_0, \ldots, a_k) \in T_0(\Omega)$.

The letters a, b, c, z now always denote elements of $T(\Omega)$, and u, v, w denote elements of $T_0(\Omega) \cup \{\Omega\}$.

Now we define, for each $n \in \omega$, four partial recursive functions;

1. $G_n : T_0(\Omega) \cup \{\Omega\} \to \omega + 1$, 2. $\mathcal{T}_n : (T_0(\Omega) \cup \{\Omega\}) \times T(\Omega) \to \{0, 1\}$,

3. $\mathrm{dom}_n : T(\Omega) \to \{\emptyset, \{n\}\} \cup T_0(\Omega)$, 4. $\cdot[\cdot]_n : T(\Omega) \times T(\Omega) \to T(\Omega)$.

Convention. These functions except $\cdot[\cdot]_n$ turn out to be total. So

1. $a \in T_u(\Omega) :\Leftrightarrow \mathcal{T}_n(u, a) \simeq 0$. (Even if $u = 0$, this is consistent. See Proposition 3(a) below.)
2. If $\mathrm{dom}_n(a) \simeq u$ for some $u \in T_0(\Omega)$, then we write

$$b \in \mathrm{dom}_n(a) :\Leftrightarrow b \in T_u(\Omega) \quad \text{and} \quad \mathrm{dom}_n(a) \simeq T_u(\Omega) :\Leftrightarrow \mathrm{dom}_n(a) \simeq u.$$

Strictly speaking, the 'set' $T_u(\Omega)$ depends on the subscript n.

Definition of G_n, $T_u(\Omega)$, dom_n *and* $\cdot[\cdot]_n$

(G1) $G_n(0) \simeq 0$.

(G2) $G_n(u+1) \simeq G_n(u) + 1$.

(G3) $G_n(u) \simeq G_n(u[n]_n)$ if $u \notin \{0, \Omega\} \cup \{v + 1 : v \in T_0(\Omega)\}$.

(G4) $G_n(\Omega) \simeq \omega$.

(T_u) Let $a = (D_{u_0}a_0, \ldots, D_{u_k}a_k)$ $(k \geq -1)$.

1. $a \in T_u(\Omega) :\Leftrightarrow \mathcal{T}_n(u, a) \simeq 0 :\Leftrightarrow G_n(u_0) \leq G_n(u) \,\&\, \cdots \,\&\, G_n(u_k) \leq G_n(u)$.
2. $a \notin T_u(\Omega) :\Leftrightarrow \mathcal{T}_n(u, a) \simeq 1 :\Leftrightarrow G_n(u_0) > G_n(u) \vee \cdots \vee G_n(u_k) > G_n(u)$.
3. $\mathcal{T}_n(u, a)$ is undefined $:\Leftrightarrow$ one of $G_n(u), G_n(u_0), \ldots, G_n(u_k)$ is undefined.

([ ]1) $\mathrm{dom}_n(0) \simeq \emptyset$; $0[z]_n$ is always undefined.

([ ]2) $\mathrm{dom}_n(1) \simeq \{n\}$; $1[z]_n \simeq c \Leftrightarrow z = n \,\&\, c = 0$.

([ ]3) $\mathrm{dom}_n(D_{u+1}0) \simeq T_u(\Omega)$; $(D_{u+1}0)[z]_n \simeq c \Leftrightarrow z \in T_u(\Omega) \,\&\, z = c$.

([ ]4) $\mathrm{dom}_n(D_u0) \simeq \mathrm{dom}_n(u)$ if $u \notin \{0\} \cup \{v+1 : v \in T_0(\Omega)\}$; $(D_u0)[z]_n \simeq D_{u[z]_n}0$.

([ ]5) Let $a = D_v b$ with $b \neq 0$.

5.1. If $b = b_0 + 1$, then $\mathrm{dom}_n(a) \simeq \{n\}$ and $a[z]_n \simeq c \Leftrightarrow z = n \,\&\, (D_v b_0) \cdot (n+1) = c$.

5.2. If $\mathrm{dom}_n(b) \simeq \{n\}$ and $b \notin \{b_0 + 1 : b_0 \in T(\Omega)\}$, then $\mathrm{dom}_n(a) \simeq \{n\}$ and $a[z]_n \simeq D_v b[z]_n$.

5.3. If $\mathrm{dom}_n(b) \simeq T_u(\Omega)$ with $G_n(u) < G_n(v)$, then $\mathrm{dom}_n(a) \simeq T_u(\Omega)$ and $a[z]_n \simeq D_v b[z]_n$.

5.4. If $\mathrm{dom}_n(b) \simeq T_u(\Omega)$ with $G_n(v) \leq G_n(u)$, then $\mathrm{dom}_n(a) \simeq \{n\}$ and $a[z]_n \simeq c \Leftrightarrow z = n \,\&\, \exists b_0, \ldots, b_n \forall m < n\, (b_0 = 1 \,\&\, b_{m+1} \simeq D_v b[b_m]_n \,\&\, D_v b[b_n]_n \simeq c)$.

5.5. If $\mathrm{dom}_n(b)$ is undefined or $\mathrm{dom}_n(b) \simeq T_u(\Omega)$ but either $G_n(u)$ or $G_n(v)$ is undefined, then $\mathrm{dom}_n(a)$ and $a[z]_n$ are undefined.

([ ]6) Let $a = (a_0, \ldots, a_k)$ with $k > 0$. $\mathrm{dom}_n(a) \simeq \mathrm{dom}_n(a_k)$; $a[z]_n \simeq (a_0, \ldots, a_{k-1}) + a_k[z]_n$.

Convention. $0[n]_n := 0$.

Remark. The definition of $a[z]_n$ is similar to that in Part I and the fundamental sequences given in [4]. The only essential difference lies in ([ ]5), 5.3 and 5.4, i.e., in which case we apply 5.3 or 5.4. In [4], the decision is made by comparing u and v with respect to the relation $<$ given in [2]. Here we consider u to be smaller than v with respect to n when $G_n(u) < G_n(v)$. It seems that this view is consistent with the idea of the slow growing functions. The price to pay is that we lose the proposition in the corresponding Remark in Part I and also the weak monotonicity of the function $\lambda n G_n(a)$. The author does not know whether these hold for this case.

Definition. Let t be an expression of the form $G_n(u)$, $\mathrm{dom}_n(a)$ or $a[z]_n$. Then we set: $t\downarrow :\Leftrightarrow t$ is defined.

Proposition 3. (a) $G_n(u) \simeq 0 \Rightarrow u = 0$. *Hence* $a \in T_0(\Omega) \Leftrightarrow \mathcal{T}_n(0, a) \simeq 0$.
(b) $G_n(u)\downarrow \;\&\; u \neq \Omega \Rightarrow G_n(u) < \omega$.
(c) $a \in T_0(\Omega) \;\&\; \mathrm{dom}_n(a)\downarrow \Rightarrow \mathrm{dom}_n(a) \in \{\emptyset, \{n\}\}$.
(d) $\mathrm{dom}_n(a)\downarrow \Rightarrow \forall z \in \mathrm{dom}_n(a)(a[z]_n\downarrow)$.
(e) $a[z]_n\downarrow \Rightarrow \mathrm{dom}_n(a)\downarrow \;\&\; z \in \mathrm{dom}_n(a)$.
(f) $a \in T_v(\Omega) \;\&\; v \neq \Omega \;\&\; \mathrm{dom}_n(a)\downarrow \Rightarrow \forall z \in \mathrm{dom}_n(a)(a[z]_n \in T_v(\Omega))$.
(g) $a \in T_v(\Omega) \;\&\; \mathrm{dom}_n(a) \simeq T_u(\Omega) \Rightarrow G_n(u) < G_n(v)$.
(h) $\mathrm{dom}_n(a+b) \simeq \mathrm{dom}_n(b)$ *and* $(a+b)[z]_n \simeq a + b[z]_n$ *if* $b \neq 0$.

Definition. For $a \in T_0(\Omega)$ and $n, m \in \omega$, we set:

$$a[n]^0 := a; \qquad a[n]^{m+1} := (a[n]^m)[n]_n \qquad \text{(cf. Proposition 3(c)).}$$

As in the Introduction GID_Ω will denote the theory ID_ω. In what follows we will work in GID_Ω. Let n be a fixed natural number.

Let U_n denote the following set:

$$U_n := \{u \in T_0(\Omega) : \exists m\, (u[n]^m \simeq 0)\}.$$

Clearly we have: $u \in U_n \Rightarrow G_n(u)\downarrow$.

Iterated inductive definition of sets $W_{un} \subseteq T_u(\Omega)$ $(u \in U_n)$

(W1) $0 \in W_{un}$.
(W2) $a \in T_u(\Omega)$, $\mathrm{dom}_n(a) \simeq \{n\}$, $a[n]_n \in W_{un} \Rightarrow a \in W_{un}$.
(W3) $a \in T_u(\Omega)$, $\exists v \in U_n(G_n(v) < G_n(u) \;\&\; \mathrm{dom}_n(a) \simeq T_v(\Omega) \;\&\; \forall z \in W_{vn}\, (a[z]_n \in W_{un})) \Rightarrow a \in W_{un}$.

Remark. It seems that this does not fit with an ω-times iterated inductive definition at first sight. Formally W_{un} is defined by

$W_{un} := \{a \in T(\Omega) : \langle a, u, n\rangle \in \mathfrak{W}_k\}$ where $k := G_n(u)$,

$\mathfrak{W}_k := \bigcap \{Y \subseteq T(\Omega) \times T_0(\Omega) \times \omega : \forall a, u, n\, (\mathfrak{A}_k(Y, a, u, n) \rightarrow \langle a, u, n\rangle \in Y)\}$,

$\mathfrak{A}_k(Y, a, u, n) :\equiv u \in U_n \;\&\; G_n(u) \simeq k \;\&\; \mathcal{T}_n(u, a) \simeq 0 \;\&\; (a = 0$
or $\{\mathrm{dom}_n(a) \simeq \{n\} \;\&\; \langle a[n]_n, u, n\rangle \in Y\}$
or $\exists v \in U_n\, \exists m < k\, \{G_n(v) \simeq m \;\&\; \mathrm{dom}_n(a) \simeq T_v(\Omega)$
$\&\; \forall z\, (\langle z, v, n\rangle \in \mathfrak{W}_m \Rightarrow \langle a[z]_n, u, n\rangle \in \mathfrak{W}_k)\})$.

(The remark follows the reviewer's suggestion.)

Proposition 4. (a) $a \in W_{0n} \Leftrightarrow \exists m\, (a[n]^m \simeq 0)$ *for* $a \in T_0(\Omega)$, *i.e.*, $W_{0n} = U_n$.

(b) $v, u \in U_n\ \&\ G_n(v) < G_n(u) \Rightarrow W_{vn} \subseteq W_{un}$.

Abbreviations. Let X range over subsets of $T(\Omega)$ which are definable in the language of GID_Ω.

1. By $A_{un}(X, a)$ ($u \in U_n \cup \{\Omega\}$) we denote the following statement:

$$a \in T_u(\Omega)\ \&\ \{a = 0 \vee (\mathrm{dom}_n(a) \simeq \{n\}\ \&\ a[n]_n \in X) \vee \exists v \in U_n\, (G_n(v) < G_n(u)$$
$$\&\ \mathrm{dom}_n(a) \simeq T_v(\Omega)\ \&\ \forall z \in W_{vn}\, (a[z]_n \in X))\}.$$

2. $A_{un}(X) := \{x \in T(\Omega)\colon A_{un}(X, x)\}$.
3. $X^{(a)} := \{y \in T(\Omega)\colon a + y \in X\}$.
4. $\bar{X} := \{y \in T(\Omega)\colon \forall x \in X \cap T_\Omega(\Omega)\, (x + D_\Omega y \in X)\}$.
5. $W_n^* := \{x \in T(\Omega)\colon \forall u \in U_n\, (D_u x \in W_{un})\}$.

Note that, in 4, we can not assume $T_\Omega(\Omega) = T(\Omega)$ until Theorem A is proved. By the definition of W_{un}, for all $u \in U_n$ we have:

(A1) $A_{un}(W_{un}) = W_{un}$,
(A2) $A_{un}(X) \subseteq X \Rightarrow W_{un} \subseteq X$.

As in Part I we have the folowing lemma.

3.1. Lemma. (a) $A_{un}(X) \subseteq X\ \&\ a \in X \cap T_u(\Omega) \Rightarrow A_{un}(X^{(a)}) \subseteq X^{(a)}$ ($u \in U_n \cup \{\Omega\}$).

(b) $a, b \in W_{un} \Rightarrow a + b \in W_{un} (u \in U_n)$.

3.2. Lemma. (a) $A_{\Omega n}(X) \subseteq X \Rightarrow \bigcup \{W_{un} : u \in U_n\} \subseteq X$.

(b) $0 \in W_n^*$.

(c) $A_{\Omega n}(W_n^*) \subseteq W_n^*$.

Proof. (a) and (c) are proved exactly as in 1.2.

(b) We have to show $\forall u \in U_n\, (D_u 0 \in W_{un})$. As in 1.2, we have

(1) $1 \in W_{0n}$ and (2) $\exists v \in U_n\, (u = v + 1) \Rightarrow D_u 0 \in W_{un}$.

If $u \in U_n$ and $u \notin \{0\} \cup \{v + 1\colon v \in U_n\}$, then $\mathrm{dom}_n(D_u 0) \simeq \{n\}$; $(D_u 0)[n]_n \simeq D_{u[n]_n} 0$ and $W_{u[n]_n, n} \subseteq W_{un}$. Hence

(3) $u \in U_n$, $u \notin \{0\} \cup \{v + 1\colon v \in U_n\}$, $D_{u[n]_n} 0 \in W_{u[n]_n, n} \Rightarrow D_u 0 \in W_{un}$.

By induction on $u \in U_n$ we get the assertion. □

3.3. Lemma. $A_{\Omega n}(X) \subseteq X \Rightarrow A_{\Omega n}(\bar{X}) \subseteq \bar{X}$.

Proof. Assume $A_{\Omega n}(X) \subseteq X$, $A_{\Omega n}(\bar{X}, b)$ and $a \in X \cap T_\Omega(\Omega)$. We have to show $a + D_\Omega b \in X$. Except the case $b = 0$, the same proof as in 1.3 works.

So it suffices to show $a + D_\Omega 0 \in X$. By 3.1(a) and 3.2(a) we have $\bigcup \{W_{un}: u \in U_n\} \subseteq X^{(a)} \cdot \mathrm{dom}_n(D_\Omega 0) = T_0(\Omega)$ and $(D_\Omega 0)[z]_n = D_z 0$ for $z \in T_0(\Omega)$. By 3.2(b) we have $\forall z \in W_{0n}\, (D_z 0 \in W_{zn} \subseteq X^{(a)})$. Hence $A_{\Omega n}(X^{(a)}, D_\Omega 0)$ and thus $D_\Omega 0 \in X^{(a)}$, since $A_{\Omega n}(X^{(a)}) \subseteq X^{(a)}$ by 3.1(a).

3.4. Lemma. *For each* $a \in T(\Omega)$,

(a) $a \in T_0(\Omega) \Rightarrow a \in W_{0n}$.

(b) $A_{\Omega n}(X) \subseteq X \Rightarrow a \in X$.

Proof. Again by simultaneus metainduction on the length of a. Assume $a = D_u b$. By 3.2(c) and IH we have $b \in W_n^*$. Assume $A_{\Omega n}(X) \subseteq X$ and $u \neq \Omega$. Then $u \in T_0(\Omega)$ and by IH, $u \in W_{0n} = U_n$. Hence by 3.2(a), $a \in W_{un} \subseteq X$. Other cases are seen as in 1.4. □

Lemma 3.4(a) yields Theorem A.

Now we work outside GID_Ω. By the soundness of the theory GID_Ω we have the following lemma.

3.5. Lemma. (a) $T_0(\Omega) = W_{0n}$.

(b) $\forall u \in T_0(\Omega)\, (T_u(\Omega) = W_{un})$.

(c) $\forall n \in \omega\, \{\forall u \in T_0(\Omega)\, (G_n(u)\downarrow)\ \&\ \forall a \in T(\Omega)\, (\mathrm{dom}_n(a)\downarrow)\}$.

Let $<_n$ denote the following relation over $T(\Omega)$:

$$b <_n a \quad :\Leftrightarrow \quad \exists z \in \mathrm{dom}_n(a)\, (b = a[z]_n).$$

Then we see that the relation $<_n$ is well founded for all $n \in \omega$.

Definition of $c \ll_k a$ *by transfinite induction on* $a \in T(\Omega)$

$$c \ll_k a \quad :\Leftrightarrow \quad a \neq 0\ \&\ \forall z \in d_k(a)\, (c \underline{\ll}_k a[z]_k)$$

where

$$d_k(a) := \begin{cases} \{k\}, & \text{if } \mathrm{dom}_k(a) = \{k\}, \\ \{D_u e : 0 \neq e \in T(\Omega)\}, & \text{if } \mathrm{dom}_k(a) = T_u(\Omega). \end{cases}$$

and

$$c \underline{\ll}_k a \quad :\Leftrightarrow \quad c \ll_k a \quad \text{or} \quad c = a.$$

For $u \in T_0(\Omega) \cup \{\Omega\}$, $a \in T(\Omega)$, $b \in T(\Omega)$ and $n, k \in \omega$, $D_u^n a \in T(\Omega)$ and $a \to_n^k b$ are defined as in Part I. Then 1.7 holds also in this case.

3.6. Lemma. (a) $\text{dom}_k(a) = T_v(\Omega)$ & $G_k(u) \leq G_k(v)$ & $1 \leq k \Rightarrow (D_u a)[1] \ll_k D_u a$.
(b) $n+1 < G_k(z)$ $(z \in T_0(\Omega)) \Rightarrow \exists u\ (n+1 = G_k(u)$ & $u+1 \underline{\ll}_k z)$.
(c) $a, z \in T_0(\Omega)$ & $a \ll_k z \Rightarrow D_a 0 \ll_k D_z 0$.
(d) $0 \to^k_{g(k)} D_\Omega 0$ with $g(k) = (k+1)^{k+1}$.
(e) $0 \to^k_n a \Rightarrow 0 \to^k_{(k+1)^{k+1+n}} D_\Omega a$.
(f) $0 \to^k_{f_\Omega(k,l)} D^l_\Omega 0$ with $f_\Omega(k, 0) = 0$, $f_\Omega(k, l+1) = (k+1)^{k+1+f_\Omega(k,l)}$.
(g) $0 \to^k_{h(k,u)} D_u 0$ with $h(k, u) = (k+1)^{G_k(u)}$.

Proof. Cf. 1.8. □

4. The infinitary system $\mathbf{GID}^\infty_\Omega$

The theory GID_Ω is an extension of PA by the following axioms;

$(P^{\mathfrak{A}}.1)$ $\forall y\ (\mathfrak{A}_y(P^{\mathfrak{A}}_y) \subseteq P^{\mathfrak{A}}_y)$.
$(P^{\mathfrak{A}}.2)$ $\forall y\ (\mathfrak{A}_y(F) \subseteq F \to P^{\mathfrak{A}}_y \subseteq F)$ for every L_{ID}-formula $F(x)$.
$(P^{\mathfrak{A}}_<)$ $\forall y\ \forall x_0\ \forall x_1\ (P^{\mathfrak{A}}_{<y} x_0 x_1 \leftrightarrow x_0 < y \wedge x_1 \in P^{\mathfrak{A}}_{x_0})$.

Again GID^∞_Ω is formulated in the language $L_{\text{ID}}(N)$. The length of a formula, basic inference rules, the set Pos_{uk} of formulas and the derivability relation $k \vdash^a_m \Gamma$ for GID^∞_Ω are defined mutatis mutandis.

Inductive definition of formula set Pos_{uk} $(u \in T_0(\Omega))$

1. All a.p.f.'s belong to Pos_{uk}.
2. $Nt \in \text{Pos}_{uk}$.
3. $Nt \in \text{Pos}_{uk} \Leftrightarrow 0 < G_k(u)$, i.e., $u \neq 0$.
4. $P^{\mathfrak{A}}_n t$, $(\neg)P^{\mathfrak{A}}_{<n} t_0 t_1 \in \text{Pos}_{uk} \Leftrightarrow n+1 \leq G_k(u)$.
5. $\neg P^{\mathfrak{A}}_n t \in \text{Pos}_{uk} \Leftrightarrow n+1 < G_k(u)$.
6. $A \$ B \in \text{Pos}_{uk} \Leftrightarrow A, B \in \text{Pos}_{uk}$, $\$ \in \{\wedge, \vee\}$.
7. $\text{Q}x\, A \in \text{Pos}_{uk} \Leftrightarrow A \in \text{Pos}_{uk}$, $\text{Q} \in \{\forall, \exists\}$.

The rule (Ω_{u+1}) is defined as follows:

(Ω_{u+1}) If Rt is an atomic formula of the form Nt or $P^{\mathfrak{A}}_n t$, and the following four conditions hold, then $k \vdash^a_m \Gamma$.
(1) $\text{dom}_k(a) = T_u(\Omega)$.
(2) $k \vdash^{a[1]_k}_m \Gamma, Rt$.
(3) $\forall z \in T_u(\Omega)\ \forall \Delta \subseteq \text{Pos}_{uk}\ (k \vdash^z_0 \Delta, Rt \Rightarrow k \vdash^{a[z]_k}_m \Delta, \Gamma)$.
(4) $Rt \in \text{Pos}_{uk}$.

Then 2.1–2.8 hold with $\tilde{n} := D^n_\Omega 0$. 2.9 now runs as follows.

4.1. Lemma. *If* $l=|F|$ *and* $G_k(u)=0$, *if* $R\equiv N$ *and* $G_k(u)=n+1$, *if* $R\equiv P_n^{\mathfrak{A}}$, *then* $k\vdash_0^{\bar{l}+D_{u+1}0}\Theta,\ \neg Rt,\ F(t)$ *for any* $k\geq 1$. (Θ denotes the set defined in the definition after 2.8.)

4.2. Lemma. *If* $z\in T_0(\Omega)$, $\Delta\subseteq \mathrm{Pos}_{0k}$, $k\vdash_0^{z}\Delta,\ Nn$ *and* $G_k(z)<n+1$, *then* $k\vdash_0^{z}\Delta$.

Proof. By transfinite induction on $k\vdash_0^{z}\Delta,\ Nn$.

1. Suppose $z=w+1$ and $k\vdash_0^{w}\Delta,\ Nn,\ n=0\vee(n=m'\wedge Nm)$. By IH we have $k\vdash_0^{w}\Delta,\ n=0\vee(n=m'\wedge Nm)$.

1.1. $n=0$: Then $G_k(w)+1=G_k(z)\leq 1$ and hence $w=0$. By 2.7(d) we have $k\vdash_0^{0}\Delta$.

1.2. $n\neq 0$ and $n\neq m+1$: By 2.7(b),(c) and 2.1, we have $k\vdash_0^{w}\Delta$.

1.3. $n=m+1$: Again by 2.7(b),(c) and 2.1, we have $k\vdash_0^{w}\Delta,\ Nm$ and $G_k(w)\leq m+1$. Hence IH (since $k\vdash_0^{w}\Delta,\ Nm$ holds in a previous stage of the derivability relation) we get $k\vdash_0^{w}\Delta$.

2. Other cases are easy. □

4.3. Lemma. *Put* $\Gamma:=\{\neg(\mathfrak{A}_n(F)\subseteq F)^N,\ n\notin N,\ t\notin P_n^{\mathfrak{A}},\ F(t)\}$ *and* $l:=|F|$. *Then, for any* $k\geq 1$,

$$k\vdash_0^{\bar{l}+D_\Omega 0}\Gamma.$$

Proof. Put $a:=\bar{l}+D_\Omega 0$. Then

(1) $\mathrm{dom}_k(a)=T_0(\Omega)$, (2) $k\vdash_0^{a[1]_k}\Gamma,\ Nn$ and (4) $Nn\in\mathrm{Pos}_{0k}$.

Also $a[z]_k=\bar{l}+D_z 0$ for $z\in T_0(\Omega)$. By (Ω_1) it remains to show

(3) $\forall z\in T_0(\Omega)\,\forall\Delta\subseteq\mathrm{Pos}_{0k}\,(k\vdash_0^{z}\Delta,\ Nn\Rightarrow k\vdash_0^{a[z]_k}\Delta,\ \Gamma)$.

Assume that $z\in T_0(\Omega)$, $\Delta\subseteq\mathrm{Pos}_{0k}$ and $k\vdash_0^{z}\Delta,\ Nn$.

1. $G_k(z)\leq n+1$: Then by 4.2, $k\vdash_0^{z}\Delta$. By 2.5(b), 2.7(a) and 3.6(g) we have $k\vdash_0^{D_z 0}\Delta$ and hence $k\vdash_0^{a[z]_k}\Delta,\ \Gamma$.

2. $n+1<G_k(z)$: Then by 3.6(b), $n+1=G_k(u)$ and $u+1\ll_k z$ for some u. By 4.1 we have $k\vdash_0^{a[u+1]_k}\Gamma$. By 3.6(c), $D_{u+1}0\ll_k D_z 0$ and hence $k\vdash_0^{a[z]_k}\Gamma$. □

We have $D_1 0\ll_k D_{k+1}0\ll_k D_\omega 0\ll_k D_\Omega 0$. Thus by 4.1 and 4.3 we get:

4.4. Lemma. *For each universal closure A of an axiom of the theory* GID_Ω *there exists an* $n\in\omega$ *such that* $k\vdash_0^{\bar{n}}A^N$ *for any* $k\geq 1$.

4.5. Lemma. *If a sentence A is provable in* GID_Ω, *then there exists an* $n\in\omega$ *such that* $k\vdash_0^{\bar{n}}A^N$ *for any* $k\geq 1$.

Proof of Theorem B. Assume $\mathrm{GID}_\Omega\vdash\forall x\,\exists y\,\phi(x,y)$ ($\phi\in\Sigma_1^0$). By 4.5 there exists an n_1 such that $k\vdash_0^{\bar{n}_1}n\notin N,\ \exists y\in N\,\phi^N(x,y)$ for all n and all $k\geq 1$. Assume $n_1\geq 2$.

(a) Put $n_0 := n_1 + 1$ and $k := 1$. If $3 \leqslant n_0 \leqslant n$, then $2n + 2 \leqslant f_\Omega(1, n-1)$. From this and 3.6(f), we see that the theorem is true.

(b) Put $n_0 := n_1 + 1$ and $k := n > 0$. If $n \neq 0$, then $2n + 2 \leqslant f_\Omega(n, 2) \leqslant f_\Omega(n, n_1)$. From this and 3.6(f) we see that the theorem is true.

(c) By definition $(D_0 D_{\Omega+1} 0)[n]_n := D_0 D_\Omega^n 1$. (c) follows from (b). □

Acknowledgements

I wish to express my heart-felt thanks to Professor W. Buchholz and the referee for invaluable comments and suggestions.

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