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# Free product decompositions in images of certain free products of groups

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**Abstract**

Let $F$ be the free product of $n$ groups and let $R$ be a normal subgroup generated (as a normal subgroup) by $m$ elements of $F$, where $m < n$. The Main Theorem gives sufficient conditions for families of fewer than $n-m$ subgroups in certain quotients of $F/R$ to generate their free product. This leads to a more direct proof of a result of the first author, that if $G$ is a group having a presentation with $n$ generators and $m$ relators, where $m < n$, then any generating set for $G$ contains $n-m$ elements that freely generate a free subgroup of $G$. Another consequence is that an $n$-generator one-relator group cannot be generated by fewer than $n-1$ subgroups each having a non-trivial abelian normal subgroup.


*Keywords:* Free products; Relations; Magnus Freiheitssatz

## 1. Introduction

In [8], the second author proved the following result:

**Theorem 1.** *Let $G$ be a group which has a presentation with $n$ generators $x_1, \dots, x_n$ and $m$ relators, where $m < n$, and let $S$ be any generating set for $G$. Then some subset of $n-m$ elements of $S$ freely generates a free group.*

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The history of this result dates back to 1930, when Magnus [4] published his Freiheitssatz, which is essentially the case of Theorem 1 in which $S = \{x_1, \ldots, x_n\}$ and $m = 1$. In 1978 the first author [5] generalized the result of Magnus by proving the case of Theorem 1 in which $S = \{x_1, \ldots, x_n\}$ and $m$ is any integer less than $n$. The proof of Theorem 1 was indirect, relying on another result of the first author [6].

Here we shall give a direct proof of a considerably more general result. Roughly speaking, the improvement consists of the replacement of the elements $x_i$ by subgroups, of the members of $S$ by suitably small subgroups, and of the hypothesis that $S$ generates $G$ by a weaker hypothesis. Before stating our Main Theorem we illustrate its power by stating two further consequences.

**Theorem 2.** *Let $G = F/R$, where $F$ is a free product of groups $A_1, \ldots, A_n$ and $R$ is generated as a normal subgroup of $F$ by $m$ elements and where $m < n$. Suppose also that each group $A_i$ has a non-trivial torsion-free abelian image. Let $B_1, \ldots, B_{n-m-1}$ be torsion-free subgroups that generate $G$. Then some subgroup $B_i$ has a non-cyclic free subgroup.*

**Theorem 3.** *Let $G$ be a torsion-free group having a presentation with $n$ generators $x_1, \ldots, x_n$ and one relation, where $n \geqslant 2$.*

(a) *Suppose that $H$ is a subgroup of $G$ that can be generated by fewer than $n - 1$ subgroups each having a non-trivial abelian normal subgroup. Then, for some $x_i$ with non-trivial image $\bar{x}_i$ in $G$, the subgroups $\langle \bar{x}_i \rangle$ and $H$ of $G$ generate their free product.*
(b) *If $G$ can be generated by subgroups $B_1, \ldots, B_{n-1}$, each having a non-trivial abelian normal subgroup, then $G$ is the free product $B_1 * \cdots * B_{n-1}$ and all but one of the subgroups $B_i$ are cyclic.*

All of the above results concern subgroups of groups of the form $F/R$ where $F$ is a free product of $n$ groups and $R$ is generated as a normal subgroup by fewer than $n$ elements. Our Main Theorem makes strong assertions about families of subgroups not usually of $F/R$ but of certain images of $F/R$. We begin by describing the condition that will be imposed on the subgroups.

We say that a finitely generated torsion-free group is *linked* if it lies in the smallest class $\mathcal{X}$ of groups containing all infinite cyclic groups and having the property that $\langle A, B \rangle \in \mathcal{X}$ whenever $A, B \in \mathcal{X}$ and $\langle A, B \rangle$ is not the free product of $A$, $B$. Every such group $G$ can be constructed by finitely many operations of taking joins of previously constructed subgroups $A$, $B$ whose join is not the free product of $A$, $B$, starting from cyclic groups; the minimal number of operations required is called the *height* of $G$.

We call an arbitrary group linked if every finitely generated subgroup is contained in a linked finitely generated subgroup.

The class of linked groups is rather extensive. It contains all torsion-free 2-generator groups that are not free and all torsion-free groups having no free subgroups of rank 2. We shall show that it also contains every torsion-free group having a non-trivial subnormal subgroup with no free subgroups of rank 2. This, and other properties of linked groups, are discussed in Section 2 below.

Now we shall describe the images of free products with relations to which our results apply. The following notation will be used in the Main Theorem.

**Hypothesis.** Let $F$ be the free product of a finite family $\mathcal{A}$ of groups. Write $n = |\mathcal{A}|$, let $m < n$, and let $R$ be a normal subgroup of $F$ generated (as a normal subgroup) by $m$ elements of $F$.

By a *filter* of normal subgroups of a group $G$ we understand a family $\mathcal{F}$ of normal subgroups with the property that for any $K_1, K_2 \in \mathcal{F}$, there is a subgroup $K_3 \in \mathcal{F}$ with $K_3 \leqslant K_1 \cap K_2$. We call a family $\mathcal{F}$ of normal subgroups of $F$ a *strong special filter* if it has the following three properties:

(i) $\mathcal{F}$ is a filter consisting of normal subgroups of $F$ containing $R$;
(ii) for each $K \in \mathcal{F}$ the group ring $\mathbb{Z}(F/K)$ can be embedded in a skew-field $Q_K$;
(iii) if $K \in \mathcal{F}$ and if $L$ is a normal subgroup of $F$ such that $R \leqslant L \leqslant K$ and $K/L$ is a torsion-free abelian group then $L \in \mathcal{F}$.

By a theorem of Kropholler, Linnell and Moody [2], the group ring of a soluble (or even elementary amenable) torsion-free group over a right Ore domain is itself a right Ore domain, and so can be embedded in a skew-field. Therefore the family consisting of all normal subgroups $K \geqslant R$ with $F/K$ soluble and torsion-free is a strong special filter (and similarly with 'soluble' replaced by 'elementary amenable').

In our Main Theorem we shall be concerned with filters of normal subgroups of $F$, but we do not require the full force of condition (iii): it will suffice that conditions (i), (ii) hold and that, for each $K \in \mathcal{F}$, we have $L \in \mathcal{F}$ for just one subgroup $L$ associated with $K$. This subgroup, denoted by $s(K)$, will be defined in Section 3; it depends on the choice of the skew-field $Q_K$. We call a family $\mathcal{F}$ of normal subgroups of $F$ a *special filter* (or an *s-filter* for short) if it satisfies (i), (ii) and the following condition:

(iii)′ if $K \in \mathcal{F}$ then $s(K) \in \mathcal{F}$.

**Main Theorem.** *Let $\mathcal{F}$ be an s-filter and $G = F/R_{\mathcal{F}}$, where $R_{\mathcal{F}} = \bigcap_{K \in \mathcal{F}} K$. For $A \in \mathcal{A}$ write $\bar{A}$ for the image of $A$ in $G$. Suppose that for each $A \in \mathcal{A}$ with $\bar{A} = 1$ the abelianization of $A$ is not a torsion group. Let $\mathcal{B}$ be a family of linked subgroups of $G$, set $J = \langle B \mid B \in \mathcal{B}\rangle$, and suppose that for each $A$ in $\mathcal{A}$ with $\bar{A} \neq 1$, the subgroups $\bar{A}$ and $J$ do not generate their free product in $G$. Then $|\mathcal{B}| \geqslant n - m$, and there are $n - m$ members of $\mathcal{B}$ that generate in $G$ their free product.*

Among the noteworthy special cases are the case in which $F/R$ is residually (soluble and torsion-free) and $\mathcal{F}$ consists of all subgroups $K$ with $R \leqslant K \lhd F$ and $F/K$ soluble and torsion-free (so that $G = F/R$), and the case when $\mathcal{B}$ generates $G$ (so that the condition that $J$ and certain subgroups of $G$ do not generate their free product becomes automatic). We conclude this Introduction by showing that Theorems 1–3 are immediate consequences of the Main Theorem.

Assume the hypotheses of Theorem 1; thus $F$ is free on a set $\{x_1, \ldots, x_n\}$ and $R$ is generated as a normal subgroup of $F$ by $m$ elements. Let $\mathcal{A} = \{\langle x_i\rangle \mid i = 1, \ldots, n\}$, and let $\mathcal{F}$ be the $s$-filter of all normal subgroups $K$ of $F$ containing $R$ such that $F/K$ is torsion-free and soluble. Let $h \mapsto \bar{h}$ be the map $F/R \to F/R_{\mathcal{F}}$. Let $S$ be any generating set for $F/R$. By the Main Theorem, there is a subset $\mathcal{X}$ of $\{\langle\bar{s}\rangle \mid s \in S,\ \bar{s} \neq 1\}$ with $|\mathcal{X}| \geqslant n - m$ that generates its free product; it follows that the set $\{s \mid \langle\bar{s}\rangle \in \mathcal{X}\}$ freely generates a free group.

Next assume the hypotheses of Theorem 2. Define $\mathcal{F}$ exactly as above and let $\mathcal{B}$ be the family consisting of the images in $F/R_{\mathcal{F}}$ of the subgroups $B_i$. Since $|\mathcal{B}| < n - m$, the Main Theorem implies that not all groups in $\mathcal{B}$ are linked, so that (e.g., by Proposition 1 below) one of them has a free subgroup of rank 2, and Theorem 2 follows.

Finally we show that Theorem 3 holds, with the requirement that subgroups have non-trivial abelian normal subgroups replaced by the much weaker requirement that they are linked. Let $G$ be isomorphic to $F/R$, where $F$ is free of rank $n$ and $R$ is generated (as a normal subgroup) by one element of $F$. Since $\mathbb{Z}G$ can be embedded in a skew-field by a theorem of J. Lewin and T. Lewin [1], the family $\{R\}$ is an $s$-filter. Assertion (a) and the fact that $G = B_1 * \cdots * B_{n-1}$ in (b) now follow directly from the Main Theorem, and the rest of (b) follows since $G$ can be generated by $n$ elements.

## 2. Linked groups

Let $G$ be a non-trivial torsion-free group. For $a, b \in G \setminus \{1\}$ write $a \sim b$ if and only if $a, b$ are contained in some (finitely generated) linked subgroup. Clearly $\sim$ is an equivalence relation on $G \setminus \{1\}$. The union of $\{1\}$ and the equivalence class containing $a$ is called the linked component of $G$ containing $a$.

**Lemma 1.** *Let $G$ be a non-trivial torsion-free group.*

(a) *Each linked component is a subgroup of $G$.*
(b) *If $G_1$, $G_2$ are distinct linked components of $G$ then $\langle G_1, G_2\rangle = G_1 * G_2$.*
(c) *Let $\{G_i \mid i \in I\}$ be a family of linked subgroups of $G$ such that $G = \bigcup_{i\in I} G_i$ and $\langle G_i, G_j\rangle = G_i * G_j$ if $i \neq j$. Then the subgroups $G_i$ are exactly the linked components of $G$.*

**Proof.** (a) follows immediately from the definition of the equivalence relation $\sim$.

Suppose that $G_1$, $G_2$ are linked components and that $G_1$, $G_2$ do not generate their free product. Then there are finitely generated linked subgroups $H_1 \leqslant G_1$ and $H_2 \leqslant G_2$ such that $\langle H_1, H_2\rangle \neq H_1 * H_2$. Thus $\langle H_1, H_2\rangle$ is a linked subgroup and it is contained in one linked component. It follows that $G_1 = G_2$, and assertion (b) follows.

(c) It is sufficient to prove that each finitely generated linked subgroup $H$ is contained in some $G_i$. If $H$ is cyclic this is clear. Otherwise, we may write $H = \langle A, B\rangle \neq A * B$ where $A$, $B$ are finitely generated linked subgroups with heights less then the height of $H$. By induction we can suppose that there are indices $i, j$ such that $A \leqslant G_i$, $B \leqslant G_j$. If $G_i \neq G_j$ then $\langle G_i, G_j\rangle = G_i * G_j$ so that $\langle A, B\rangle = A * B$, and the result follows from this contradiction. □

**Lemma 2.** *Let $G$ be a torsion-free group with a non-trivial linked subgroup $H$ such that $\langle g, H\rangle \neq \langle g\rangle * H$ for all $g \in G \setminus \{1\}$. Then $G$ is linked.*

**Proof.** Fix $h_0 \in H \setminus \{1\}$, and let $g \in G \setminus \{1\}$. By hypothesis, $H$ has a finitely generated linked subgroup $A$ containing $h_0$ such that $\langle g, A\rangle \neq \langle g\rangle * A$. Clearly $\langle g, A\rangle$ is linked, so that $g \sim h_0$. Thus $G$ has a single equivalence class and the result follows. □

Since $H$ is not a subnormal subgroup of the free product $H * K$ unless $H = 1$ or $K = 1$, Lemma 2 has the following immediate consequence.

**Proposition 1.** *Let $G$ be a torsion-free group.*

(a) *If $G$ has a non-trivial subnormal linked subgroup then $G$ is linked.*
(b) *If $G$ has a non-trivial subnormal subgroup which contains no free subgroups of rank 2, then $G$ is linked.*

**Lemma 3.** *Let $G$ be a group and $\mathcal{K}$ be a filter of normal subgroups of $G$ such that $\bigcap_{K\in\mathcal{K}} K = 1$. Let $\mathcal{L} = \{L \lhd G \mid G/L$ is torsion-free$\}$. Let $B$ be a finitely generated linked subgroup of $G$. Then $\mathcal{K}$ contains a subgroup $K_0$ such that the image of $B$ in $G/L$ is linked whenever $L \in \mathcal{L}$ and $L \leqslant K_0$.*

**Proof.** We argue by induction on the height of $B$. The result is clear if $B$ is cyclic. Assume therefore that $B = \langle D_1, D_2\rangle$, where each of $D_1$, $D_2$ is linked and $B$ is not the free product of $D_1$, $D_2$. By induction there are subgroups $K_1, K_2 \in \mathcal{K}$ such that $D_1L/L$ is linked for all $L \in \mathcal{L}$ with $L \leqslant K_1$ and $D_2L/L$ is linked for all $L \leqslant K_2$. Since $B$ is not the free product of $D_1$, $D_2$, we can find a non-trivial element $w$ of $D_1 * D_2$ that maps to 1 in $B$. Now $w$ is a product of a sequence $Y$ of elements coming alternately from $D_1 \setminus \{1\}$ and $D_2 \setminus \{1\}$. Let $K_3$ be a subgroup in $\mathcal{K}$ such that all distinct elements of $Y$ are non-congruent modulo $K_3$; then for each $L \leqslant K_3$ the images of $w$ in $D_1L/L * D_2L/L$ and $BL/L$ are respectively non-trivial and trivial. We conclude that $BL/L$ is linked for each $L \in \mathcal{L}$ with $L \leqslant K_1 \cap K_2 \cap K_3$, and the result follows. □

## 3. Further preliminary results

Our notation for conjugates and commutators in a group $G$ is as follows: we write $a^b = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$. We write $[G, G]$ for the derived subgroup of a group $G$ and we set $G^{(0)} = G$, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geqslant 0$. We write $X^G$ for the normal subgroup of a group $G$ generated by a subset $X$.

Let $G$ be a group, let $K \lhd G$ and let $V$ be a right $\mathbb{Z}(G/K)$-module. It is convenient to regard the split extension of $V$ by $G/K$ as the group of matrices

$$\begin{pmatrix} G/K & 0 \\ V & 1 \end{pmatrix} = \left\{ \begin{pmatrix} gK & 0 \\ v & 1 \end{pmatrix} \,\middle|\, g \in G,\ v \in V \right\},$$

with multiplication defined in the obvious manner. We may regard $V$ as a $\mathbb{Z}G$-module. If $\delta : G \to V$ is a derivation (that is, if $\delta(g_1g_2) = \delta(g_1)g_2 + \delta(g_2)$ for all $g_1, g_2 \in G$) then the map

$$g \mapsto \begin{pmatrix} gK & 0 \\ g\delta & 1 \end{pmatrix}$$

is a group homomorphism with kernel containing $[K, K]$; if $\delta$ is an inner derivation (that is, $\delta$ is the map $g \mapsto v_0(g-1)$ for some element $v_0$ of $V$) then the kernel of this map is $K$. To examine the kernels of certain such maps in more detail we shall use the Magnus representation, as extended by Shmelkin and the first author; all we require can be deduced from the following result from [7]:

**Lemma 4.** *Let $F = A_1 * \cdots * A_n$ be the free product of the groups $A_1, \ldots, A_n$ and let $H = F/R$, where $A_i \cap R = 1$ for $i = 1, \ldots, n$. Let $T$ be the free right $\mathbb{Z}H$-module with basis $\{t_1, \ldots, t_n\}$. Let*

$$\varphi : F \to \begin{pmatrix} H & 0 \\ T & 1 \end{pmatrix}$$

*be the homomorphism defined on the free factors $A_i$ of $F$ by*

$$a \mapsto \begin{pmatrix} aR & 0 \\ t_i(a-1) & 1 \end{pmatrix} \quad \text{for } a \in A_i.$$

*Then* $\ker\varphi = [R, R]$.

We may extend Lemma 4 as follows.

**Lemma 5.**

(a) *If the requirement that $R \cap A_i = 1$ for each $i$ is omitted in Lemma* 4 *then*

$$\ker\varphi = [R, R]\langle (A_i \cap R)^F \bigm| 1 \leqslant i \leqslant n\rangle.$$

(b) *The conclusion of Lemma* 4 *remains true if the hypothesis on $T$ is replaced by the requirement that $\{t_2, \ldots, t_n\}$ is a basis of $T$ and $t_1 = 0$.*

**Proof.** Assertion (a) follows from standard arguments and assertion (b) from the formula

$$\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} aR & 0 \\ t_i(a-1) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} = \begin{pmatrix} aR & 0 \\ (t_i - t_1)(a-1) & 1 \end{pmatrix}. \qquad \square$$

**Lemma 6.** *Let $F = D_1 * D_2$ and $G = F/R$, where $D_1 \cap R = D_2 \cap R = 1$. Suppose that $\mathbb{Z}G$ is contained in a skew-field $Q$ and let $V$ be a vector space over $Q$ and $v_1, v_2 \in V$. Let*

$$\varphi : F \to \begin{pmatrix} G & 0 \\ V & 1 \end{pmatrix}$$

*be the group homomorphism defined by*

$$\varphi(x_1) = \begin{pmatrix} x_1R & 0 \\ v_1(x_1-1) & 1 \end{pmatrix}, \qquad \varphi(x_2) = \begin{pmatrix} x_2R & 0 \\ v_2(x_2-1) & 1 \end{pmatrix}$$

*for all $x_1 \in D_1$, $x_2 \in D_2$. Then either* $\ker\varphi = [R, R]$ *or* $\ker\varphi = R$*, and in the latter case $v_1 = v_2$.*

**Proof.** The conclusion holds if $v_1 = v_2 = 0$. Therefore, interchanging $D_1$, $D_2$ if necessary, we may assume that $v_1 \neq 0$.

Let $W$ be a vector space over $Q$ with basis $\{w_1, w_2\}$ and define

$$\sigma : F \to \begin{pmatrix} G & 0 \\ W & 1 \end{pmatrix}$$

by

$$x_1 \mapsto \begin{pmatrix} x_1 R & 0 \\ w_1(x_1 - 1) & 1 \end{pmatrix}, \qquad x_2 \mapsto \begin{pmatrix} x_2 R & 0 \\ w_2(x_2 - 1) & 1 \end{pmatrix}$$

for all $x_1 \in D_1$, $x_2 \in D_2$. From Lemma 4 we have $\ker \sigma = [R, R]$.

Let $\tau_1, \tau_2 : W \to V$ be the linear maps defined by

$$\tau_1(w_1) = \tau_1(w_2) = \tau_2(w_1) = v_1, \qquad \tau_2(w_2) = v_2.$$

Then $\dim \ker \tau_1 = 1$ and $\dim \ker \tau_2 \leqslant 1$. The maps $\tau_1$, $\tau_2$ and the identity map $G \to G$ induce two group homomorphisms

$$\lambda_1, \lambda_2 : \begin{pmatrix} G & 0 \\ W & 1 \end{pmatrix} \to \begin{pmatrix} G & 0 \\ V & 1 \end{pmatrix}.$$

Thus $\lambda_2 \sigma = \varphi$, and

$$\lambda_1 \sigma(x) = \begin{pmatrix} xR & 0 \\ v_1(x - 1) & 1 \end{pmatrix}$$

for all $x \in D_1 \cup D_2$, and hence for all $x \in F$. Thus $\ker \lambda_1 \sigma = R$.

Clearly $[R, R] \leqslant \ker \varphi \leqslant R$. Suppose that $\ker \varphi \neq [R, R]$ and choose $r \in \ker \varphi \setminus [R, R]$. Let

$$\sigma(r) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}.$$

Since $\lambda_1 \sigma(r) = 1$ and $\lambda_2 \sigma(r) = \varphi(r) = 1$ we have $w \in \ker \tau_1 \cap \ker \tau_2$. We conclude that $\ker \tau_1 = \ker \tau_2 = wQ$, and so $\lambda_1 = \lambda_2$ and $\varphi = \lambda_1 \sigma$, as required. □

**Lemma 7.** *Let $G$ be a group and $\mathcal{L}$ be a filter of normal subgroups such that $\bigcap_{L \in \mathcal{L}} L = 1$. Let $\mathcal{B}$ be a finite set of subgroups of $G$. Suppose that for each $L \in \mathcal{L}$ there are $n$ subgroups from $\mathcal{B}$ whose images in $G/L$ generate their free product in $G/L$. Then there are $n$ subgroups from $\mathcal{B}$ which generate their free product in $G$.*

**Proof.** Let $\mathcal{B}_1, \ldots, \mathcal{B}_q$ be the subfamilies of $\mathcal{B}$ with $n$ elements. For each $i$ write $F_i$ for the free product of the groups in $\mathcal{B}_i$ and $F_{i,L}$ (for $L \in \mathcal{L}$) for the free product of the family $\{BL/L \mid B \in \mathcal{B}_i\}$. If the conclusion of the lemma is false then for each $i$ there is an element $w_i \in F_i \setminus \{1\}$ whose image in $G$ is trivial. The word $w_i$ is a product of a sequence of non-trivial elements of $\bigcup_{B \in \mathcal{B}_i} B$ with no two consecutive terms from the same member of $\mathcal{B}_i$; choose $L_i \in \mathcal{L}$ such that the distinct elements in this sequence are non-trivial and non-congruent modulo $L_i$, and choose $M \in \mathcal{L}$ with $M \leqslant \bigcap_{i=1}^{q} L_i$. Then for each $i$ the image of $w_i$ in $F_{i,M}$ is non-trivial, and since the image of $w_i$ in $G/M$ is trivial, the map from $F_{i,M}$ to $G/M$ is not injective. The result follows from this contradiction. □

**Lemma 8.** *Let $R_1 \geqslant R_2 \geqslant \cdots$ be a series of normal subgroups of a free product $E = A * B$ and suppose that*

$$\bigcap_{i=1}^{\infty}(A \cap R_i) = \bigcap_{i=1}^{\infty}(B \cap R_i) = 1.$$

*Suppose that for infinitely many indices $i$ we have*

$$R_{i+1} \leqslant [R_i, R_i](R_i \cap A)^E (R_i \cap B)^E.$$

*Then $\bigcap_{i=1}^{\infty} R_i = 1$.*

**Proof.** Fix some index $k$ and set $A_k = A/(A \cap R_k)$, $B_k = B/(B \cap R_k)$. Write $E_k = A_k * B_k$ and write $\overline{R}_i$ for the image of $R_i$ in $E_k$. We have $A_k \cap \overline{R}_k = B_k \cap \overline{R}_k = 1$ and so $\overline{R}_k$ is free by the Kurosh subgroup theorem (see [3, IV.1.10]). It follows that $\bigcap_{k=1}^{\infty}(\ker E \to E_k) = 1$ (see [3, I.3.4]). By hypothesis we have $\overline{R}_{i+1} \leqslant [\overline{R}_i, \overline{R}_i]$ for infinitely many indices $i \geqslant k$, and hence

$$\bigcap_{i=1}^{\infty} \overline{R}_i \leqslant \bigcap_{n=1}^{\infty} \overline{R}_k^{(n)} = 1.$$

Therefore

$$\bigcap_{i=1}^{\infty} R_i \leqslant \bigcap_{k=1}^{\infty}(\ker E \to E_k) = 1. \qquad \square$$

Now we discuss the subgroup $s(K)$ which occurs in property (iii) of $s$-filters, and place it in the context in which we shall use it. Let $F$ and $R$ be as in the Hypothesis from the Introduction. Let $K$ be a normal subgroup of $F$ such that $R \leqslant K$ and such that $\mathbb{Z}(F/K)$ can be embedded in a skew-field $Q_K$. Consider the right vector space $V$ over $Q_K$ with basis $\{t_A \mid A \in \mathcal{A}\}$ and the homomorphism

$$\varphi : F \to \begin{pmatrix} F/K & 0 \\ V & 1 \end{pmatrix},$$

which is defined on the free factors $A$ of $F$ by

$$a \mapsto \begin{pmatrix} aK & 0 \\ t_A(a-1) & 1 \end{pmatrix} \quad \text{for } a \in A.$$

By Lemma 5(a) we have $\ker\varphi = [K, K]\langle (A \cap K)^F \mid A \in \mathcal{A}\rangle$. For each $r \in R$ the element $\varphi(r)$ has the form

$$\begin{pmatrix} 1 & 0 \\ u(r) & 1 \end{pmatrix}.$$

Let $U$ be the subspace of $V$ spanned by $\{u(r) \mid r \in R\}$, let

$$\psi : F \to \begin{pmatrix} F/K & 0 \\ V/U & 1 \end{pmatrix}$$

be the map induced by $\varphi$, and define $s(K) = \ker\psi$. It is clear that $R\ker\varphi \leqslant s(K) \leqslant K$ and that $K/s(K)$ is a torsion-free abelian group. We note that $s(K)$ depends not only on $K$ but also on the choice of $Q_K$.

Let $\mathcal{F}$ be an $s$-filter and $K \in \mathcal{F}$. We define $K_0 = K$, $K_{i+1} = s(K_i)$ for $i \geqslant 0$ and $L_K = \bigcap_{i=0}^{\infty} K_i$. The family $\mathcal{C}_K = \{K_0, K_1, \ldots\}$ is an $s$-subfilter of $\mathcal{F}$; we call it the *chain generated by* $K$. Note that by construction $K \cap A = K_i \cap A$ for all $A \in \mathcal{A}$ and for $i = 0, 1, \ldots$.

## 4. Proof of the Main Theorem

Assume the hypotheses of the Main Theorem. We begin with some reductions.

**Lemma 9.** *It suffices to prove the Main Theorem under the following additional hypotheses*:

(a) *all groups in $\mathcal{B}$ are finitely generated linked groups*;
(b) *if $R \leqslant L \lhd F$ and $F/L$ is torsion-free, and if $L \leqslant K$ for some $K \in \mathcal{F}$, then*
  (i) *the image in $F/L$ of each group $B \in \mathcal{B}$ is linked, and*
  (ii) *for each group $A \in \mathcal{A}$ with non-trivial image in $G$, the image in $F/L$ of $A$ is non-trivial and moreover the images in $F/L$ of $A$ and $J$ do not generate their free product in $F/L$*;
(c) *the $s$-filter $\mathcal{F}$ is the chain $\{K = K_0, K_1, \ldots\}$ generated by some normal subgroup $K$.*

**Proof.** For each $A \in \mathcal{A}$ write $\overline{A}$ for the image of $A$ in $G$.

Suppose that the hypotheses of the Main Theorem hold, and that the conclusion is false. For each group $A \in \mathcal{A}$ with $\overline{A} \neq 1$ and each $B \in \mathcal{B}$ we choose a finitely generated subgroup $M_{B,A}$ of $B$ such that some non-trivial element of $\langle M_{B,A} \mid B \in \mathcal{B}\rangle * \overline{A}$ maps to 1 in $G$. If $|\mathcal{B}| \geqslant n - m$ then for each subset $\mathcal{X}$ of $\mathcal{B}$ with $|\mathcal{X}| \geqslant n - m$ we also choose a non-trivial element $w_{\mathcal{X}}$ of the free product of the groups in $\mathcal{X}$ such that the image of $w_{\mathcal{X}}$ in $G$ is trivial.

Now choose for each $B \in \mathcal{B}$ a finitely generated linked subgroup $H_B$ that contains $\langle M_{B,A} \mid A \in \mathcal{A},\ \overline{A} \neq 1\rangle$ and such that the elements $w_{\mathcal{X}}$ (if any) lie in the free product of the groups $H_B$ with $B \in \mathcal{X}$. The hypotheses but not the conclusion of the theorem hold with $\mathcal{B}$ replaced by $\{H_B \mid B \in \mathcal{B}\}$.

Therefore we may assume in proving the Main Theorem that (a) holds. By Lemma 3 we can choose a subgroup $K_0 \in \mathcal{F}$ such that the images in $F/L$ of the groups of $\mathcal{B}$ are linked whenever $R \leqslant L \lhd F$, $F/L$ is torsion-free and $L \leqslant K_0$. For each $A \in \mathcal{A}$ with $\overline{A} \neq 1$ choose an element $w_A \in \overline{A} * J$ whose image in $G$ is non-trivial; then $w_A$ is a product of a sequence $Y_A$ of non-trivial elements of $\overline{A} \cup J$. Choose $K_1 \in \mathcal{F}$ such that all distinct elements in all of these sequences are non-trivial and non-congruent modulo $K_1$. If we replace the $s$-system $\mathcal{F}$ by $\{K \in \mathcal{F} \mid K \leqslant K_0 \cap K_1\}$, then (a), (b) hold.

Now we assume that (a), (b) hold. For each $K \in \mathcal{F}$ let $L_K$ be the intersection of the chain $\mathcal{C}_K = \{K = K_0, K_1, \ldots\}$ generated by $K$. The set $\mathcal{L} = \{L_K \mid K \in \mathcal{F}\}$ is a filter of normal subgroups of $F$ and $\bigcap_{K \in \mathcal{F}} L_K = R_{\mathcal{F}}$. Since conditions (a), (b) hold for $\mathcal{F}$, it follows that the hypotheses of the Main Theorem and conditions (a), (b) also hold for each chain $\mathcal{C}_K$. Thus if the theorem holds for $s$-filters that are chains, then we can conclude that for each $K \in \mathcal{F}$ there are $n - m$ subgroups of $\mathcal{B}$ whose images in $F/L_K$ generate their free product in $F/L_K$. It follows by Lemma 7 that there are $n-m$ subgroups of $\mathcal{B}$ whose images in $F/F_{\mathcal{F}}$ generate their free product in $F/F_{\mathcal{F}}$. $\square$

From now on *we assume that the additional hypotheses* (a)–(c) *of Lemma* 9 *hold*. Write $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ contains all subgroups $A$ with non-trivial images in $G$ and $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$ and write $G_i = G/K_i$, $Q_i = Q_{K_i}$ for each $i$. As $K \cap A = K_i \cap A$ for all $A \in \mathcal{A}$ and all $i \geqslant 0$, we can replace all groups $A$ from $\mathcal{A}_1$ by their images in $G$ and also identify them with their images in $G_i$. By hypothesis, the abelianization of each group $A \in \mathcal{A}_2$ is not a torsion group and so for each $A \in \mathcal{A}_2$ there is a non-zero group homomorphism $\nu_A : A \to \mathbb{Q} \subset Q_i$. Let $V_i$ be the right vector space over $Q_i$ with basis $\{t_{A,i} \mid A \in \mathcal{A}\}$ and define a group homomorphism

$$\theta_{i+1} : F \mapsto \begin{pmatrix} G_i & 0 \\ V_i & 1 \end{pmatrix}$$

by specifying its restrictions $\theta_{i+1}|_A$ to the free factors as follows:

$$a \mapsto \begin{pmatrix} a & 0 \\ t_{A,i}(a-1) & 1 \end{pmatrix} \quad \text{for } a \in A \in \mathcal{A}_1,$$
$$a \mapsto \begin{pmatrix} 1 & 0 \\ t_{A,i}(\nu_A(a)) & 1 \end{pmatrix} \quad \text{for } a \in A \in \mathcal{A}_2.$$

We note that the subspace of $V_i$ spanned by the bottom left-hand entries of the images of the elements of $F$ contains all elements $t_{A,i}$, and so is equal to $V_i$ and has dimension $n$.

Let $R$ be generated as a normal subgroup of $F$ by the set $\{r_1, \ldots, r_m\}$. The elements $\theta_{i+1}(r_j)$ have the form

$$\begin{pmatrix} 1 & 0 \\ u_j & 1 \end{pmatrix}$$

and so they all lie in the subgroup

$$\begin{pmatrix} 1 & 0 \\ U_i & 1 \end{pmatrix},$$

where $U_i$ is the subspace of $V_i$ spanned by all elements $u_j$. Write $W_i = V_i/U_i$. Then the kernel of the map

$$\psi_{i+1} : F \to \begin{pmatrix} G_i & 0 \\ W_i & 1 \end{pmatrix}$$

induced by $\theta_{i+1}$ contains $R$, and (as $t_{A,i} \in U_i$ for $A \in \mathcal{A}_2$) it is equal to $K_{i+1}$. Therefore we may identify $G_{i+1}$ with $\psi_{i+1}(F)$. The map $\psi_{i+1}$ induces a map

$$\sigma_{i+1} : G \to \begin{pmatrix} G_i & 0 \\ W_i & 1 \end{pmatrix}.$$

Denote by $\sigma_0$ the canonical homomorphism $G \to G_0$. By construction we have

$$\sigma_{i+1}(g) = \begin{pmatrix} \sigma_i(g) & 0 \\ \beta_i(g) & 1 \end{pmatrix},$$

where $\beta_i : G \to V_i$ is a derivation.

**Lemma 10.** *For any finitely generated linked subgroup $D$ of $G$ there is an index $i(D)$ such that $\sigma_i|_D$ is injective and $\beta_i|_D$ is an inner derivation for all $i \geqslant i(D)$.*

**Proof.** We argue by induction on the height of $D$. If $D = \langle d_0 \rangle$ is cyclic then choose $i_0 = i(D)$ such that $\sigma_{i_0}(d_0) \neq 1$. For $i \geqslant i_0$ the map $\sigma_i$ is injective on $D$ and defining

$$w_{D,i} = \beta_i(d_0)\big(\sigma_i(d_0) - 1\big)^{-1}$$

we have $\beta_i(d) = w_{D,i}(\sigma_i(d) - 1)$ for all $d \in D$.

Suppose then that $D = \langle D_1, D_2 \rangle \neq D_1 * D_2$, where $D_1, D_2$ are finitely generated linked subgroups of smaller height, and that for $i \geqslant i_0 = \max(i(D_1), i(D_2))$ the maps $\sigma_i|_{D_1}$ and $\sigma_i|_{D_2}$ are injective and there are $y_1, y_2 \in V_i$ such that $\beta_i(d_1) = y_i(d_1 - 1)$, $\beta_i(d_2) = z_i(d_2 - 1)$ for all $d_1 \in D_1, d_2 \in D_2$. Let $S_i$ be the kernel of the homomorphism $D_1 * D_2 \to \langle \sigma_1(D_1), \sigma_2(D_2) \rangle \leqslant G_i$ for each $i \geqslant i_0$. As $\sigma(D_1) \cap S_i = \sigma_2(D_2) \cap S_i = 1$, by Lemma 6 either we have $S_{i+1} = [S_i, S_i]$ or we have $S_{i+1} = S_i$ and $y_i = z_i$. If the former holds for infinitely many values of $i$ then

$$\ker(D_1 * D_2 \to G) \leqslant \bigcap_{j=1}^{\infty} S_{i_0}^{(j)};$$

since $S_{i_0}$ is free by the Kurosh subgroup theorem we conclude that the map from $D_1 * D_2$ to $G$ is injective, and this is a contradiction. Therefore there is an index $i(D) \geqslant i_0$ such that $S_{i+1} = S_i$ and $y_i = z_i$ for all $i \geqslant i(D)$, as required. □

Using Lemma 10 and replacing $K$ by some $K_i$ if necessary, we may suppose the following additional property:

(d) $\sigma_i|_B$ is injective and $\beta_i|_B$ is an inner derivation for each $i \geqslant 0$ and each $B \in \mathcal{B}$.

We identify $B \in \mathcal{B}$ with its image in $G_i$ and let $\beta_i(b) = w_{B,i}(b - 1)$ for $b \in B$. Denote by $Y_i$ the subspace of $W_i$ spanned by $\{w_{B,i} \mid B \in \mathcal{B}\}$.

**Lemma 11.** *$Y_i = W_i$ for all sufficiently large indices $i$.*

**Proof.** It is enough to prove that for each $A \in \mathcal{A}_1$ we have $t_{A,i} \in Y_i$ for all sufficiently large $i$. Fix $A \in \mathcal{A}_1$. Set $\overline{W}_i = W_i / Y_i$ and consider the homomorphism

$$\overline{\sigma}_{i+1} : G \to \begin{pmatrix} G_i & 0 \\ \overline{W}_i & 1 \end{pmatrix}$$

induced by $\sigma_{i+1}$. Consider also the homomorphism $\tau : C = A * J \to \langle A, J\rangle \leqslant G$. Set $S_{i+1} = \ker \sigma_{i+1}\tau$, $R_{i+1} = \ker \bar{\sigma}_{i+1}\tau$. By construction $S_{i+1} \leqslant R_{i+1} \leqslant S_i$. If $t_{A,i} \in Y_i$ then $\sigma_i\tau = \bar{\sigma}_{i+1}\tau$ and $R_{i+1} = S_i$. If $t_{A,i} \notin Y_i$ and $\bar{t}_{A,i}$ is the image of $t_{A,i}$ in $\overline{W}_i$ then

$$\bar{\sigma}_{i+1}\bigl(\tau(a)\bigr) = \begin{pmatrix} a & 0 \\ \bar{t}_{A,i}(a-1) & 1 \end{pmatrix} \quad \text{for all } a \in A,$$

and

$$\bar{\sigma}_{i+1}\bigl(\tau(b)\bigr) = \begin{pmatrix} \sigma_i(b) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all } b \in J,$$

so that by Lemma 6 we have

$$R_{i+1} = [S_i, S_i](S_i \cap J)^C \leqslant [R_i, R_i](R_i \cap J)^C.$$

If the latter holds for infinitely many indices, then by Lemma 9 we have $\bigcap_{i=1}^{\infty} R_i = 1$ and $\tau$ is injective, contradicting the hypothesis of the theorem. Therefore $t_{A,i} \in Y_i$ for all sufficiently large $i$, as required. □

Using Lemma 11 we may suppose in addition to (a)–(d) that we have

(e) $Y_i = W_i$ for all $i \geqslant 0$.

Now we resume the proof of the theorem. We recall that $W_i = V_i/U_i$, $\dim V_i = |\mathcal{A}| = n$ and $\dim U_i \leqslant m$. Therefore $\dim Y_i = \dim W_i \geqslant n - m$ and, in particular, $|\mathcal{B}| \geqslant n - m$. So there is a subset $\mathcal{B}_0$ of $\mathcal{B}$ such that $|\mathcal{B}_0| = n - m$ and for an infinite set $\Omega$ of indices $i$ the family $\{w_{B,i} \mid B \in \mathcal{B}_0\}$ is linearly independent over $Q_i$. We claim that the subgroups $\mathcal{B}_0$ generate their free product in $G$. Write $H$ for the free product of the groups $B \in \mathcal{B}_0$ and consider the homomorphism $\tau : H \to \langle B \mid B \in \mathcal{B}_0\rangle \leqslant G$. Let $S_{i+1} = \ker \sigma_{i+1}\tau$. Note that $S_1$ is a free group. We have $B \cap S_{i+1} = 1$ and

$$\sigma_{i+1}\bigl(\tau(b)\bigr) = \begin{pmatrix} b & 0 \\ t_{B,i}(b-1) & 1 \end{pmatrix}, \quad b \in B,\ B \in \mathcal{B}_0.$$

By Lemma 4 we have $S_{i+1} = [S_i, S_i]$ for all $i \in \Omega$. Therefore

$$\bigcap_{i=1}^{\infty} S_i \leqslant \bigcap_{j=1}^{\infty} S_1^{(j)} = 1$$

and $\tau$ is injective. This concludes the proof of the Main Theorem.

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