On the Steinberg presentation for Lie-type groups of type $C_2$

C. Müller

Institut für Theoretische Physik, Universität Gießen, Heinrich-Buff-Ring 16, 35392 Gießen, Germany

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1. Introduction

Let $\Phi$ be an irreducible root-system of rank $\ell \geq 2$ satisfying the crystallographic condition. (That is, $\Phi$ is one of types $A_\ell$, $B_\ell$, $C_\ell$, $l \geq 2$, $D_\ell$, $\ell \geq 4$, $E_\ell$, $6 \leq \ell \leq 8$, $F_4$ or $G_2$.) Inspired by the Steinberg presentation of Chevalley groups, recently Timmesfeld considered the following situation (cf. [1–3]):

Let $G$ be an abstract group generated by subgroups $A_\alpha, \alpha \in \Phi$, satisfying the following hypothesis denoted by (H):

(i) If $\beta \neq -\alpha$ then $[A_\alpha, A_\beta] \leq \langle A_{m\alpha+n\beta} \mid m, n \in \mathbb{N} \text{ and } m\alpha+n\beta \in \Phi \rangle$. (In particular this implies $A'_\alpha = 1$)

(ii) $X_\alpha = \langle A_\alpha, A_{-\alpha} \rangle$ is a rank-one group with unipotent subgroups $A_\alpha$ and $A_{-\alpha}$. (For definition of a rank-one group see Section 2.)

Clearly, all Chevalley groups satisfy (H). Hence the question arises which possibilities exist in general for the structure of a group satisfying (H). For the case that in condition (i) equality always holds, Timmesfeld solved this problem in [1]:

1.1. Theorem. Suppose $G$ satisfies (H) with equality holding in condition (i). Then there exists a surjective homomorphism $\varphi : G \rightarrow \overline{G}$, where $\overline{G}$ is a group of Lie-type $B$, $B$ an irreducible, spherical Moufang building, which maps the $A_\alpha$.
\( \alpha \in \Phi \), onto the root subgroups of \( G \) corresponding to some apartment of \( B \). Further \( \ker(\varphi) \leq Z(G) \).

Moreover, with the help of this theorem Timmesfeld was able to classify the groups satisfying (H) and with the root-system having simple bonds only (cf. [2]):

**1.2. Theorem.** Suppose \( G \) satisfies (H) with \( \Phi \) of one of types \( A_\ell, \ell \geq 2 \), \( D_\ell, \ell \geq 4 \), or \( E_\ell, 6 \leq \ell \leq 8 \). Then \( \Phi \) is the disjoint union of irreducible subsystems \( \Psi_i \) (here a pair \( \{ \pm \alpha \} \) is regarded as a subsystem of type \( A_1 \)) such that the following hold:

(a) If \( \alpha \in \Psi_i, \beta \in \Psi_j \) with \( i \neq j \), then \( [A_{\alpha}, A_{\beta}] = 1 \).

(b) If \( \alpha, \beta \in \Psi_i \) with \( \alpha + \beta \in \Phi \), then \( \alpha + \beta \in \Psi_i \) and \( [A_{\alpha}, A_{\beta}] = A_{\alpha + \beta} \).

Hence, applying Theorem 1.1 to the \( G_i = \langle A_{\alpha} | \alpha \in \Psi_i \rangle \), one obtains that \( G \) is a central product of Lie-type groups and rank-one groups, respectively. As the first step in the proof of Theorem 1.2, Timmesfeld showed:

**1.3. Lemma.** Suppose \( G \) satisfies (H) with \( \Phi = \{ \pm r, \pm s, \pm (r + s) \} \) of type \( A_2 \). Then one of the following holds:

(a) \( G = X_r \ast X_s \ast X_{r+s} \).

(b) In condition (i) of (H) always equality holds.

Now one wishes to generalize Theorem 1.2 to root systems containing a double bond. Unfortunately, a direct generalization is not possible, as the example below \( (G = SL_2(k) \ast SL_3(k), \Phi = C_2) \) shows and since it is well known that the commutator relations degenerate for Chevalley groups of types \( C_\ell \) and \( F_4 \) in characteristic two. (That is, one has strict inequality in (i) of (H) for certain choices of \( \alpha \) and \( \beta \).) But Timmesfeld showed in [3] that if \( G \) satisfies (H) with \( \Phi \) of type \( B_\ell, C_\ell \) or \( F_4 \), then either equality holds in (i) or there is a central factor of Lie type or one has a characteristic-two situation. Now for the proof of this theorem, similarly, as Lemma 1.3 is needed for Theorem 1.2, one needs to treat the case \( \Phi = C_2 \) first, which is the purpose of the present note.

Before stating our proposition, which describes the \( C_2 \) case, we consider the following specific example, which we have already mentioned above.

Suppose \( G \) satisfies (H) with \( \Phi = \{ \pm r, \pm s, \pm (r + s), \pm (2r + s) \} \) of type \( C_2 \). Then \( \Psi = \{ \pm r, \pm s, \pm (r + s) \} \) is a subset (but not a subsystem!) of \( \Phi \) representing a root system of type \( A_2 \). Suppose further, \( [A_{\alpha}, A_{\beta}] = A_{\alpha + \beta} \) for \( \alpha, \beta \in \Psi \) with \( \alpha + \beta \in \Psi \) and \( [A_{\alpha}, A_{\beta}] = 1 \) for \( \alpha \in \Psi \), \( \beta \in \Phi \setminus \Psi \). Then clearly \( G = X_{2r+s} \ast \langle A_{\alpha} | \alpha \in \Psi \rangle \). A concrete realization would be \( G = SL_2(k) \ast SL_3(k), k \) is a field.

This example and the exceptional case of characteristic two in mind, our following proposition might appear quite naturally:
1.4. Proposition. Suppose $G$ satisfies (H) with $\Phi$ of type $C_2$. Then one of the following holds:

(A) $X_\alpha \leq G$ and $G = X_\alpha * C_G(X_\alpha)$ for some long root $\alpha \in \Phi$. Further, $X_\beta \leq C_G(X_\alpha)$ for all $\beta \in \Phi \setminus \{\pm \alpha\}$.

(B) $A_\alpha$ is an elementary abelian $2$-group for each $\alpha \in \Phi$.

(C) In condition (i) of (H) always equality holds.

The proof of this proposition will be given in Section 3. It consists of extensive commutator calculations together with applications of the theory of rank-one groups. For the reader’s convenience, the needed properties of rank-one groups will be listed in the next section.

2. Some properties of rank-one groups

A group $X$ generated by two different nilpotent subgroups $A$ and $B$ satisfying

(*) for each $a \in A^\#$ there exists a $b \in B^\#$ with $A^b = B^a$ and vice versa

is called a rank-one group with unipotent subgroups $A$ and $B$. The elements of

$\Omega := A^X$ are called unipotent subgroups of $X$. $X$ is called a special rank-one group, if

(**) $a^b = b^{-a} \quad (= (b^{-1})^a)$ for $a \in A^\#$, $b \in B^\#$ with $A^b = B^a$.

Moreover, a faithful $\mathbb{Z}X$-module $V$ is called a quadratic $X$-module, if $[V, A, A] = 0$.

For later application we briefly discuss two examples:

(1) For $t \in k$, $k$ is a field, let

$$a(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad b(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

$A = \{a(t) \mid t \in k\}$, $B = \{b(t) \mid t \in k\}$, and $X = \langle A, B \rangle$. Then $X = SL_2(k)$ and is a special rank-one group with abelian unipotent subgroups $A$ and $B$.

(2) Denote by $St_2(\mathbb{Q})$ the universal perfect central extension of $PSL_2(\mathbb{Q})$ in the sense of [4]; i.e. $St_2(\mathbb{Q})$ is generated by symbols $a(t)$, $b(t)$, $t \in \mathbb{Q}$, subject to the relations:

(i) $a(s)a(t) = a(s+t)$, $b(s)b(t) = b(s+t)$ for $s, t \in \mathbb{Q}$;

(ii) $a(s)^n(t) = b(-t^{-2}s)$ for $s, 0 \neq t \in \mathbb{Q}$ with $n(t) := a(t)b(-t^{-1})a(t)$.

Then $X = St_2(\mathbb{Q})$ is a special rank-one group with abelian unipotent subgroups $A = \{a(t) \mid t \in \mathbb{Q}\}$ and $B = \{b(t) \mid t \in \mathbb{Q}\}$. 

Note that in both examples \( n(t) := a(t)b(-t^{-1})a(t) \) interchanges \( A \) and \( B \) and we have \( n(t)^2 \in Z(X) \) for each \( 0 \neq t \in k \).

In the following we will state some important properties of rank-one groups, which will be needed for the proof of Proposition 1.4. Proofs of these properties can be found in [5, Chapter I].

Let \( X \) be a rank-one group with unipotent subgroups \( A \) and \( B \). Then the following hold:

2.1. For each \( a \in A^\# \) there exists exactly one \( b \in B^\# \) with \( A^b = B^a \). This unique element in \( B^\# \) will be denoted by \( b(a) \).

2.2. \( X = \langle C, D \rangle = \langle c, D \rangle \) for all \( C \neq D \in \Omega \) and \( c \in C^\# \). In particular \( C \cap D = 1 \) for \( C \neq D \in \Omega \).

2.3. \( X \) acts doubly transitively on \( \Omega \). In particular, there exist \( x \in X \) with \( A^x = B \), \( B^x = A \). (In this situation we will use the shorthand notation \( A \xleftarrow{x} B \).)

2.4. Let \( A \leq Y < X \) and \( a^x \in Y \) for some \( a \in A \), \( x \in X \). Then \( a^x \in A \).

2.5. \( \langle a^X \rangle \) is not nilpotent for each \( a \in A^\# \). In particular, \( X \) is not nilpotent.

2.6. Suppose \( X \) is special with abelian unipotent subgroups. Then one of the following holds:

(a) \( A \) is an elementary abelian \( p \)-group and for each \( a \in A^\# \) we have \( X(a) := \langle a, b(a) \rangle \cong (P)SL_2(p) \).

(b) \( A \) is torsion-free and divisible. For each \( a \in A^\# \), let \( A(a) = \{ a^{m/n} \mid m, 0 \neq n \in \mathbb{Z} \} \), \( B(a) = \{ b(a)^{m/n} \mid m, 0 \neq n \in \mathbb{Z} \} \). Then \( X(a) := \langle A(a), B(a) \rangle \) is a quotient of \( SL_2(\mathbb{Q}) \).

Let \( N \trianglelefteq X \) and \( \overline{X} = X/N \). Then the following hold:

2.7. Either \( N \leq Z(X) \) or \( X = NA \).

2.8. \( \overline{X} \) is a rank-one group if and only if \( N \leq Z(X) \).

Suppose there exists a quadratic \( X \)-module \( V \). Then the following hold:

2.9. \( X \) is special.
2.10. Suppose either \( X \cong (P)\text{SL}_2(p) \) or \( X \) is perfect with \( X/Z(X) \cong \text{PSL}_2(\mathbb{Q}) \); i.e., in the latter case \( X \) is a quotient of \( \text{St}_2(\mathbb{Q}) \). Suppose further that \( CV(X) = 0 \) and \( V = [V, X] \). Then
\[
\tau = -\text{id}_V,
\]
where \( \tau \in X \) is the image of \( n(1)^2 \in \text{SL}_2(p) \) respectively \( \text{St}_2(\mathbb{Q}) \) under the natural homomorphism and \( n(1) \) is defined as in the examples above. (Cf. the proof of (3.5) in [5, Chapter I].)

2.11. Let \( X \) and \( \tau \) be as in Lemma 2.10, but \( X \neq (P)\text{SL}_2(2) \). Then \( \tau \neq \text{id}_V \).

3. Proof of the proposition

In this section we assume that \( G = \langle A_\alpha \mid \alpha \in \Phi \rangle \) satisfies (H) with \( \Phi \) of type \( C_2 \):

\[
\begin{array}{c}
s \quad r+s \\
\quad 2r+s \\
-r \\
-2r-s \quad -r-s \quad -s \\
\end{array}
\]

In addition to the notation of the preceding sections we fix the following notation for \( \alpha \in \Phi \).

Firstly, Lemma 2.3 allows us to pick an \( n_\alpha \in X_\alpha \) with \( A_\alpha \xleftarrow{n_\alpha} A_{-\alpha} \). (\( n_\alpha \) is not necessarily uniquely determined.) Secondly, let
\[
U_\alpha = \langle A_\beta \mid \beta \in \Phi \text{ is between } \alpha \text{ and } -\alpha \text{ in clockwise sense} \rangle
\]
(cf. the figure above). For example, \( U_s = \langle A_{r+s}, A_{2r+s}, A_r \rangle = A_{r+s}A_{2r+s}A_r \), where the latter identity follows from (i) of (H).

Timmesfeld showed in [1, (2.1) and (2.4)] the following:

3.1. Lemma. The following hold for \( \alpha, \beta \in \Phi \):

(a) \( X_\alpha \) normalizes \( U_\alpha \) and \( U_{-\alpha} \).
(b) \( A_\alpha U_\alpha \) and \( A_{-\alpha} U_\alpha \) are nilpotent.
(c) \( A_\alpha \cap U_\alpha = 1 = A_{-\alpha} \cap U_\alpha \). In particular, \( A_\alpha \cap A_\beta = 1 \) for \( \beta \neq \alpha \).

3.2. Lemma. One of the following holds:

(a) \( X_s \trianglelefteq G \) and \( G = X_s \ast C_G(X_s) \) with \( X_\alpha \trianglelefteq C_G(X_s) \) for all \( \alpha \in \Phi \setminus \{\pm s\} \).
(b) Let $\overline{U}_s := U_s/A_{2r+s}$. Then $C_{\overline{U}_s}(A_s) = \overline{A}_{r+s} = [\overline{A}_r, A_s]$, $C_{\overline{U}_s}(A_{-s}) = \overline{A}_r = [\overline{A}_{r+s}, A_{-s}]$ and $\overline{U}_s = \overline{A}_{r+s} \times \overline{A}_r$. The corresponding holds for the action of $X_s$ on $\overline{U}_{-s} := U_{-s}/A_{-2r-s}$.

By symmetry, an analogous lemma also holds for $X_{2r+s}$ acting on $\overline{U}_{2r+s} := U_{2r+s}/A_{-s}$ and $\overline{U}_{-2r-s} := U_{-2r-s}/A_{s}$, respectively.

To prove Proposition 1.4 we will assume throughout that possibility (A) does not hold; i.e. neither $X_s$ nor $X_{2r+s}$ is a central factor in $G$. By Lemma 3.2 this assumption implies that Lemma 3.2(b) and its corresponding version for $X_{2r+s}$ hold. In the following, we will state and prove assertions for $X_s$, only. The reader is supposed to keep in mind that analogous versions also hold for $X_{2r+s}$.

3.3. Lemma. The following hold for the action of $X_s$ on $\overline{U}_s = \overline{A}_{r+s} \times \overline{A}_r$:

(a) $\overline{A}_{r+s} = [\overline{A}_r, a_s]$ and $\overline{A}_r = [\overline{A}_{r+s}, a_{-s}]$ for each $a_s \in A_s^\#$, $a_{-s} \in A_{-s}^\#$.
(b) $C_{\overline{A}_s}(\overline{a}_r) = 1 = C_{\overline{A}_{-s}}(\overline{a}_{r+s})$ for each $\overline{a}_r \in \overline{A}_r^\#$, $\overline{a}_{r+s} \in \overline{A}_{r+s}^\#$.

Proof. (a) Suppose $\overline{A}_0 := [\overline{A}_r, a_s] < \overline{A}_{r+s}$. Then $\overline{A}_r \times \overline{A}_0 < \overline{A}_r \times \overline{A}_{r+s}$. But $X_s = \langle a_s, A_{-s} \rangle$ normalizes $\overline{A}_r \overline{A}_0$. Hence, $\overline{A}_{r+s} = \overline{A}_r^{\overline{a}_s} \subseteq \overline{A}_r \overline{A}_0$ by Lemma 3.2(b), a contradiction to the above.

(b) Let $1 \neq a_s \in C_{\overline{A}_s}(\overline{a})$ for some $\overline{a} \in \overline{A}_r$. Then $X_s = \langle a_s, A_{-s} \rangle \leq C(\overline{a})$. Hence, $\overline{a} \in \overline{A}_r \cap \overline{A}_r^{\overline{a}_s} = \overline{A}_r \cap \overline{A}_{r+s} = 1$ by Lemma 3.2(b).

The other relations follow by the same argument. □

3.4. Lemma. For $x \in X_s$ with $A_s \leftrightarrow^x A_{-s}$ we have $A_r \leftrightarrow^x A_{r+s}$.

Proof. By Lemma 3.2(b), $x$ interchanges $A_r A_{2r+s}$ and $A_{r+s} A_{2r+s}$. Hence we have $A_r^x \leq A_{r+s} A_{2r+s}$. Suppose there exists a $y \in A_r^x - A_{r+s}$. Then $\emptyset \neq y A_{r+s} \cap A_{2r+s} \subseteq y U_{-2r-s} \cap A_{2r+s}$.

Hence, $\langle y^{X_{2r+s}} \rangle$ is not nilpotent, since by Lemma 2.5 it is not nilpotent modulo $U_{-2r-s}$. But on the other hand, since $[X_s, X_{2r+s}] = 1$ by (i) of (H), we have

$$\langle y^{X_{2r+s}} \rangle \leq \langle (A_r^x)^{X_{2r+s}} \rangle \leq U_{2r+s}^x$$

by Lemma 3.1(a), which is nilpotent by Lemma 3.1(b), a contradiction to the above. We obtain $A_r^x \leq A_{r+s}$ and, with symmetry, also $A_{r+s}^x \leq A_r$. These relations hold for each $x' \in X_s$ with $A_s \leftrightarrow^{x'} A_{-s}$. Choosing $x' = x^{-1}$, we finally get $A_r^x = A_{r+s}$ and $A_{r+s}^x = A_r$. □

3.5. Lemma. We have $[A_r, A_{r+s}] \leq [A_r, A_s]$ and $[A_{-r}, A_{-r-s}] \leq [A_{-r}, A_{-s}]$. In particular, $[A_r, A_s] = A_{r+s} A_{2r+s}$ if $[A_r, A_{r+s}] = A_{2r+s}$, and $[A_{-r}, A_{-s}] = A_{-r-s} A_{-2r-s}$ if $[A_{-r}, A_{-r-s}] = A_{-2r-s}$.
**Proof.** By Lemma 3.2(b) we have \([A_r,A_s]A_{2r+s} = A_{r+s}A_{2r+s}\). Since \([A_r,A_s]\) is \(A_r\)-invariant, this yields

\([A_{r+s},A_r] = [A_r,A_s],A_r \leq [A_r,A_s]\).

Together these equations imply \([A_r,A_s] = A_{r+s}A_{2r+s}\) if \([A_r,A_{r+s}] = A_{2r+s}\). By the same argument the other relations can be shown. \(\square\)

**3.6. Lemma.** If \([A_r,A_{r+s}] = A_{2r+s}\) and \([A_{-r},A_{r+s}] = A_s\) then in condition (i) of (H) always equality holds.

**Proof.** From \([A_r,A_{r+s}] = A_{2r+s}\) we get \([A_r,A_s] = A_{r+s}A_{2r+s}\) by Lemma 3.5, which implies \([A_{r+s},A_{-s}] = A_rA_{2r+s}\) by Lemma 3.4. Furthermore, conjugation with \(n_{2r+s} \in X_{2r+s}\) yields \(A_{-2r-s} = [A_r,A_{r+s}]^{2r+s} = [A_{-r-s},A_{-r}]\) by Lemma 3.4. Hence, \([A_{-r},A_{-s}] = A_{-r-s}A_{-2r-s}\) by Lemma 3.5, which implies \([A_{-r-s},A_s] = A_{-r}A_{-2r-s}\) by Lemma 3.4. In the same way, \([A_{-r},A_{r+s}] = A_s\) yields the remaining non-trivial commutator-identities. \(\square\)

**3.7. Lemma.** \(X_s\) is a special rank-one group.

**Proof.** Let \(K\) be the kernel of the action of \(X_s\) on \(\overline{U}_s = U_s/A_{2r+s} = \overline{A}_{r+s} \times \overline{A}_r\). Then \(K \leq Z(X_s)\), since otherwise \(X_s = KA_s\) by Lemma 2.7, a contradiction to Lemma 3.2(b). We have \(\overline{U}_s = 1\) and \([\overline{U}_s,A_s,A_s] \leq [\overline{A}_{r+s},A_s] = 1\). Hence, by Lemma 2.8 \(\overline{U}_s\) is a quadratic module for the rank-one group \(\overline{X}_s = X_s/K\), which is special by Lemma 2.9. For \(a \in A^#, b \in A_{-s}^#\) with \(A^b_s = A^a_{-s}\) this means \(a^bK = b^{-a}K\), which implies \(b^{-ab^{-1}} \in aK\). Hence Lemma 2.4 yields \(b^{-ab^{-1}} = a\) and thus \(X_s\) is special. \(\square\)

**3.8. Lemma.** One of the following holds:

(a) \(A_\alpha\) is an elementary abelian \(p\)-group for each \(\alpha \in \Phi\).

(b) \(A_\alpha\) is torsion-free and divisible for each \(\alpha \in \Phi\).

**Proof.** It suffices to consider \(A_\alpha\) with positive \(\alpha \in \Phi\). Since \(X_s\) is by Lemma 3.7 special, by Lemma 2.6 either \(A_s\) is an elementary abelian \(p\)-group or \(A_s\) is torsion-free and divisible.

(a) First suppose, \(A_s\) is an elementary abelian \(p\)-group. By Lemma 3.3(a) we have \(\overline{A}_{r+s} = [\overline{A}_r,a]\) for \(a \in A^#_s\). Since \(A_s\) acts quadratically on \(A_r\), we have

\[1 = [\overline{a}_r,a^p] = [\overline{a}_r,a]^p\]

for each \(\overline{a}_r \in \overline{A}_r\). Hence, \(\overline{A}_{r+s}\) is an elementary abelian \(p\)-group. By the same argument this also holds for \(\overline{A}_r\). Now, \(\overline{A}_{r+s} \simeq A_{r+s}\) and \(\overline{A}_r \simeq A_r\) by Lemma 3.1(c). Consider the action of the special rank-one group \(X_{2r+s}\) on \(\overline{U}_{2r+s} = U_{2r+s}/A_{-s} = \overline{A}_r \times \overline{A}_{-r-s}\). By Lemma 2.6, \(A_{2r+s}\) either is an
elementary abelian $q$-group or torsion-free and divisible. As we will show in part (b), in the latter case also $A_r$ is torsion-free, a contradiction to the above. Hence, $A_{2r+s}$ is an elementary abelian $q$-group and $q = p$.

(b) Now suppose, $A_s$ is torsion-free and divisible. Let $n \in \mathbb{N}$, $a \in A_s^\#$, $\bar{a}_{r+s} \in \overline{A}_{r+s} = [\overline{A}_r, a]^\#$, and $\bar{a}_r \in \overline{A}_r$ with $\bar{a}_{r+s} = [\bar{a}_r, a]$. Suppose $\bar{a}_{r+s}^n = 1$. Then $1 = \bar{a}_{r+s} = [\bar{a}_r, a^n]$, a contradiction to Lemma 3.3(b). Thus, $\overline{A}_{r+s}$ is torsion-
free. Since $A_s$ is divisible, there exists an $a' \in A_s$ with $(a')^n = a$. Hence, $\bar{a}_{r+s} = [\bar{a}_r, a'']$ and $\overline{A}_{r+s}$ is divisible. By the same argument this also holds for $\overline{A}_r$. As in part (a), considering the action of $X_{2r+s}$ on $\overline{U}_{2r+s}$, one finally gets that also $A_{2r+s}$ is torsion-free and divisible. □

Note, that by Lemma 3.8 each $A_\alpha$, $\alpha \in \Phi$, can be considered as a vector space over some prime field $k$.

In the following we will investigate the action of $X_s$ on the $\mathbb{Z}X_s$-modules $\overline{U}_s$, $\overline{U}_s := U_s/U'_s$ and $Z_s := Z(U_s)$. It is easy to see that $\overline{U}_s = \overline{A}_{2r+s} \times (\overline{A}_{r+s} \overline{A}_r)$ and $Z_s = A_{2r+s} \times ((A_{r+s} \cap Z_s)(A_r \cap Z_s))$. By the comment above, all these $\mathbb{Z}X_s$-modules are $k$-vector spaces, for which we temporarily prefer to use an additive notation, e.g. $\overline{U}_s = \overline{A}_{2r+s} \oplus (\overline{A}_{r+s} + \overline{A}_r)$.

If $k = \mathbb{Z}_2$, then by Lemma 3.8 Proposition 1.4 (B) holds. From now on, we will assume in addition that $\text{char}(k) \neq 2$ and show that in this situation Proposition 1.4 (C) holds.

Since $X_s$ is special, Lemma 2.6 implies that for each $a \in A_s^\#$ there exists a subgroup $X_s(a) = \langle A_s(a), A_{-s}(a) \rangle$ of $X_s$ which is isomorphic either to $(P)\text{SL}_2(p)$ with $p \neq 2$ or to a quotient of $\text{St}_2(\mathbb{Q})$. Hence $X_s(a)$ is generated by the images of the generators of $\text{SL}_2(p)$, respectively $\text{St}_2(\mathbb{Q})$, under the natural homomorphism (cf. examples (1) and (2) in Section 2). Slightly changing notation, in this section we will denote these generators of $X_s(a)$ by $a(t)$, $b(t)$, $t \in k$.

3.9. Lemma. Let $a \in A_s^\#$ and $\tau := n(1)^2 \in X_s(a)$ with $n(1) = a(1)b(-1)a(1)$ (i.e., $\tau$ is as in Lemma 2.10). Then the following hold:

(a) $C_{\overline{U}_s}(X_s(a)) = 0$ and $[\overline{U}_s, X_s(a)] = \overline{U}_s$.
(b) $\tau |_{\overline{U}_s} = -\text{id}_{\overline{U}_s}$.
(c) $[A_r, \tau] \leq A_r$ and $[A_{r+s}, \tau] \leq A_{r+s}$.

Proof. (a) By Lemma 3.3(a), $[\overline{A}_r, A_s(a)] = \overline{A}_{r+s}$ and $[\overline{A}_{r+s}, A_{-s}(a)] = \overline{A}_r$; whence $[\overline{U}_s, X_s(a)] = \overline{U}_s$. Suppose, $1 \neq a_r \in C_{A_s}(A_s(a))$. Then

$X_r = \langle a_r, A_{-r} \rangle \leq C(A_s(a))$,

a contradiction to $[\overline{A}_r, A_s(a)] \neq 1$. This yields $C_{\overline{U}_s}(A_s(a)) = \overline{A}_{r+s}$ and, in the same way, $C_{\overline{U}_s}(A_{-s}(a)) = \overline{A}_r$. Hence, $C_{\overline{U}_s}(X_s(a)) \leq \overline{A}_{r+s} \cap \overline{A}_r = 0.$
(b) Let $K \leq Z(X_s(a))$ be the kernel of the action of $X_s(a)$ on $\overline{U}_s$ and $\overline{X}_s(a) := X_s(a)/K = \langle \bar{a}(t), \bar{b}(t) \mid t \in k \rangle$. Then either $\overline{X}_s(a) \simeq (P)SL_2(p)$ with $p \neq 2$ or $\overline{X}_s(a)$ is perfect with $\overline{X}_s(a)/Z(\overline{X}_s(a)) \simeq PSL_2(\mathbb{Q})$. Hence, $\bar{n}(1)^2 = -\text{id}_{\overline{U}_s}$ by Lemma 2.10 and (a). For the coimage $\tau$ of $\bar{n}(1)^2$ this implies $\tau = -\text{id}_{\overline{U}_s}$, as well.

(c) We have $A_s(a) \xleftrightarrow{n(1)} A_{-s}(a)$, which yields $A_s \xleftrightarrow{n(1)} A_{-s}$, by Lemma 2.2. Hence, $A_r \xleftrightarrow{n(1)} A_{r+s}$ by Lemma 3.4. Thus, $\tau = n(1)^2$ normalizes $A_r$ and $A_{r+s}$. □

3.10. Lemma. Let $a \in A_s^\#$ and $V = \tilde{U}_s$ or $V = Z_s$. Then

$$V = C_V(X_s(a)) \oplus [V, X_s(a)]$$

with $C_V(X_s(a)) = C_V(\tau)$ and $[V, X_s(a)] = [V, \tau]$, where $\tau$ is as in Lemma 3.9.

Proof. We only consider the case $V = \tilde{U}_s$. By Lemma 3.9(b), $\tau = -\text{id}_{\tilde{U}_s}$. Hence, $V = C_V(\tau) \oplus [V, \tau]$, since $\text{char}(k) \neq 2$. Clearly, $\tau$ acts trivially on $W := V/[V, \tau] \simeq C_V(\tau)$. Suppose $X_s(a)$ acts non-trivially on $W$. Then $W$ is a quadratic module for $\overline{X}_s(a) := X_s(a)/N$ with $N \leq Z(X_s(a))$ denoting the kernel of the action. But $\bar{\tau} = \text{id}_W$, a contradiction to Lemma 2.11. Hence, $[W, X_s(a)] = 0$ and $[V, X_s(a)] = [V, \tau]$. For $v \in C_V(\tau), x \in X_s(a)$ this implies $[v, x] \in [V, \tau] \cap C_V(\tau) = 0$; whence $C_V(\tau) = C_V(X_s(a))$. □

Now we return to the consideration of the commutator relations between the $A_r$. For this purpose we switch to the multiplicative notation for the $\mathbb{Z}X_s$-modules $\overline{U}_s$, $\tilde{U}_s$, and $Z_s$ again.

3.11. Lemma. $[A_r, A_s] \leq A_{r+s}[A_r, A_{r+s}]$.

Proof. Considering the action of $X_s$ on $\tilde{U}_s$ we show $[\tilde{A}_r, A_s] \leq \tilde{A}_{r+s}$, which by Lemma 3.5 implies the assertion. Let $a \in A_s^\#$ and $\tau \in X_s(a)$ be as in Lemma 3.9. By Lemma 3.9(b) we have $[\tilde{A}_r, \tau] \times \tilde{A}_{2r+s} = \tilde{A}_r \times \tilde{A}_{2r+s}$. This implies $\tilde{A}_r = [\tilde{A}_r, \tau] \leq [\tilde{U}_s, \tau]$ and, in the same way, $\tilde{A}_{r+s} \leq [\tilde{U}_s, \tau]$. Since $\tilde{A}_{2r+s} \leq C_{\tilde{U}_s}(\tau)$, this yields $\tilde{A}_r \tilde{A}_{r+s} = [\tilde{U}_s, \tau]$, which is $X_s(a)$-invariant by Lemma 3.10. Hence,

$$[\tilde{A}_r, A_s(a)] \leq \tilde{A}_r \tilde{A}_{r+s} \cap \tilde{A}_{r+s} \tilde{A}_{2r+s} = (\tilde{A}_r \cap \tilde{A}_{r+s} \tilde{A}_{2r+s}) \tilde{A}_{r+s} = \tilde{A}_{r+s}$$

by (i) of (H) and Lemma 3.1(c). This relation holds for each $a \in A_s^\#$.

3.12. Lemma. The following hold:

(a) $C_{U_{-r}}(A_r) = A_{2r+s}$ or $[A_r, A_{r+s}] = 1$.

(b) $C_{U_{-r}}(A_{-r}) = A_s$ or $[A_{-r}, A_{r+s}] = 1$. 
Proof. (a) We consider the action of $X_r$ on $U_{-r}$. Since $A_r \cap A_{2r+s} = 1$, we have $C_{A_r}(A_r) = 1$ by Lemma 3.3(b). Hence, $C_{U_{-r}}(A_r) \leq A_{r+s}A_{2r+s}$. Suppose $C_{U_{-r}}(A_r) > A_{2r+s}$. Then there exists $1 \neq y \in C_{A_{r+s}}(A_r)$. This implies $y \in A_{r+s} \cap Z_s \neq 1$ and, by conjugation with $n_s$, also $A_r \cap Z_s \neq 1$. We show:

\[ (*) \quad (A_r \cap Z_s)(A_{r+s} \cap Z_s) \text{ is } X_s\text{-invariant.} \]

Let $a \in A_s^+$ and $\tau \in X_s(a)$ be as in Lemma 3.9. Then $[A_r \cap Z_s, \tau] \leq A_r \cap Z_s$ by Lemma 3.9(c). Hence, $[A_r \cap Z_s, \tau] \cdot A_{2r+s} = (A_r \cap Z_s) \cdot A_{2r+s}$ by Lemma 3.9(b). This implies $A_r \cap Z_s = [A_r \cap Z_s, \tau] \leq Z_s$, and, in the same way, $A_{r+s} \cap Z_s \leq Z_s$. Thus we have $(A_r \cap Z_s)(A_{r+s} \cap Z_s) = [Z_s, \tau]$, which is $X_s(a)$-invariant by Lemma 3.10. This holds for all $a \in A_s^+$, which shows $(*)$.

By $(*)$ and (i) of (H) we have

\[ [A_r \cap Z_s, A_s] \leq (A_r \cap Z_s)(A_{r+s} \cap Z_s) \cap A_{r+s}A_{2r+s} \leq A_{r+s}. \]

Hence, $X_r = \langle A_r \cap Z_s, A_{-r} \rangle$ normalizes $A_sA_{r+s}$. Thus,

$A_sA_{r+s} \cap U_s = (A_s \cap U_s)A_{r+s} \leq A_{r+s}$

is $A_r$-invariant by (i) of (H). This yields $[A_{r+s}, A_r] \leq A_{r+s} \cap A_{2r+s} = 1$. The same argument also shows (b). \( \Box \)

3.13. Lemma. $[A_r, A_{r+s}] = 1$ if and only if $[A_{-r}, A_{r+s}] = 1$.

Proof. First, suppose $[A_r, A_{r+s}] = 1$. Then $[A_r, A_s] \leq A_{r+s}[A_r, A_{r+s}] = A_{r+s}$ by Lemma 3.11. Hence, $X_r$ normalizes $M := A_sA_{r+s}$. Suppose $C_M(A_{-r}) = A_s$. Since $A_r$ acts quadratically on $U_{-r}$ by assumption, $A_n^{n_{-r}} = A_{-r}$ acts quadratically on $U_{-r} = U_{-r}^{n_{-r}}$. This yields $[A_{2r+s}, A_{-r}] \leq C_M(A_{-r}) = A_s$, a contradiction to Lemma 3.2(b) applied to the action of $X_{2r+s}$ on $\overline{U}_{-2r-s} = A_{-r} \times A_{r+s}$. Hence there exists $1 \neq z \in C_{A_{r+s}}(A_{-r})$. Since $[A_{-r-s}, A_{-r}] = [A_r, A_{r+s}]^{n_{-r-s}} = 1$ by Lemma 3.4, this implies $X_r \leq \langle z, A_{-r-s} \rangle \leq C(A_{-r})$.

The same argument shows the opposite direction. \( \Box \)

3.14. Lemma. $[A_r, A_{r+s}] = A_{2r+s}$ and $[A_{-r}, A_{r+s}] = A_s$.

Proof. Suppose $[A_r, A_{r+s}] = 1$. Then $[A_{-r}, A_{r+s}] = 1$ by Lemma 3.13. Let $M := A_sA_{r+s}$. Then we have $A_r \leq N(M)$ and $A_{-r} \leq C(M)$. This implies $X_r \leq C(M)$, a contradiction to Lemma 3.2(b). Consequently, $C_{U_{-r}}(A_r) = A_{2r+s}$ and $C_{U_{-r}}(A_{-r}) = A_s$ by Lemma 3.12. Now suppose $[A_r, A_{r+s}] < A_{2r+s}$. This implies $P := A_sA_{r+s}[A_r, A_{r+s}] \leq U_{-r}$, since

$A_{2r+s} \cap A_sA_{r+s} \leq A_{2r+s} \cap U_{-2r-s} = 1$. 
But $P$ is $X_r$-invariant by Lemma 3.11, which yields

$$A_{2r+s} = C_{U_{-r}}(A_r) = C_{U_{-r}}(A_{-r})^{nr} = A_{-s}^{nr} \leq P,$$

a contradiction to the above.

By the same argument the other commutator identity can be shown. $\square$

In view of Lemma 3.6, Lemma 3.14 implies that Proposition 1.4(C) holds. This completes the proof of our proposition.

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References