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Subgroups of Type A_1 Containing a Fixed Unipotent Element in an Algebraic Group

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INTRODUCTION

Let G be a simple algebraic group defined over an algebraically closed field of characteristic $p > 0$. In [28, 4.1], it was shown that any element $u \in G$ of order p is contained in a closed subgroup of type A_1, provided that p is good. The corresponding result was also obtained for finite simple groups of Lie type over finite fields of good characteristic. These results have found many applications in investigations of subgroup structure of these simple groups. In this paper we extend the results to cover also the cases where the characteristic is not good. We also extend some of the proofs in [28].

The main result of this paper is the following.

MAIN THEOREM. *Let G be a simple algebraic group defined over an algebraically closed field of characteristic $p > 0$. Let σ be either a Frobenius morphism of G, or $\sigma = 1$. Let $u \in G$ be a σ-invariant element of order p.*

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Then u is contained in a closed σ-invariant subgroup of G of type A_1, *except in the following cases*:

(i) $G = G_2$, $p = 3$, *and u is an element of order* 3 *of type* $A_1^{(3)}$,

(ii) $G = G_2$, $p = 3$, *and* σ *is a morphism involving the graph morphism of* G,

(iii) $G = B_2$ *or* $G = F_4$, $p = 2$, *and* σ *is a morphism involving the graph morphism of* G.

We also obtain a somewhat more general result for semisimple algebraic groups defined over an algebraically closed field of characteristic $p > 0$. Further, we prove a result for the corresponding finite groups.

Since this paper has been under preparation, an important paper of Seitz [18] has come to our attention. In it, assuming that the characteristic is good, Seitz shows that each element of order p is contained in a "canonical" subgroup of type A_1, unique up to conjugacy in $C_G(u)$.

1. q-FROBENIUS MORPHISMS

In this section we introduce a special type of Frobenius morphism of a semisimple algebraic group and consider how it behaves with respect to restriction to certain semisimple subgroups.

Let G be a semisimple algebraic group defined over an algebraically closed field K of characteristic $p > 0$. Let $\sigma : G \to G$ be a surjective morphism of algebraic groups with finite fixed point subgroup G_σ, that is, a *Frobenius morphism* of G. Note that σ will in fact be bijective; see [26, 7.1]. Fix a σ-invariant Borel subgroup B of G and a σ-invariant maximal torus T of B. Then the opposite Borel subgroup B^- is σ-invariant, as are $U = R_u(B)$ and $U^- = R_u(B^-)$. Let $\Phi = \Phi(G, T)$ denote the roots of G relative to T, and let Δ denote the simple roots. For each $\alpha \in \Phi$ we may choose isomorphisms (of algebraic groups) $x_\alpha : \mathbb{G}_a \to U_\alpha$ such that $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1) \in N_G(T)$. The family $(x_\alpha)_{\alpha\in\Phi}$ is called a *realization* of Φ in G (see [23, 11.2] for more details). There then exists a permutation ρ of Φ and powers $q(\alpha)$ of p, $\alpha \in \Phi$, such that $\sigma x_\alpha(k) = x_{\rho\alpha}(c_\alpha k^{q(\alpha)})$, for some $c_\alpha \in \mathbb{G}_m$ and all $k \in K$ (see [26, 11.2]). We have $q(\alpha) = q(-\alpha)$, $c_\alpha c_{-\alpha} = 1$, and may take $c_\alpha = 1$ for $\pm\alpha \in \Delta$, modifying the isomorphisms if necessary (see [8, 16.2C]). We will call σ a *q-Frobenius morphism* if $q(\alpha) = q$ for all $\alpha \in \Phi$. We can then write $\sigma = gq$, where $qx_\alpha(k) = x_\alpha(k^q)$ and $gx_\alpha(k) = x_{\rho\alpha}(c_\alpha k)$, for all $\alpha \in \Phi$ and $k \in K$. Note that g is an algebraic automorphism of G, since some power of g is the identity map. When G is simple, we remark that any morphism involving a twisted graph morphism in bad characteristic is not a q-Frobenius morphism (see Section 3).

Finally if $G_1, \dots, G_n$, $n \geq 1$, are the simple components of G, we shall call G *σ-simple* if, after possible renumbering, we have $\sigma G_i = G_{i+1}$, subscripts taken modulo n. For a more detailed account of the above, consult [26].

The following two lemmas are concerned with the restriction of a q-Frobenius morphism to semisimple subgroups of G, with simple components of type A_1. For the first one, cf. [12, Proposition 1.13].

LEMMA 1.1. *Let G be a semisimple algebraic group defined over an algebraically closed field K of characteristic $p > 0$. Let σ be a q-Frobenius morphism of G. Suppose X is a closed σ-invariant A_1-subgroup of G. Then $\sigma \mid X$ is a q-Frobenius morphism of X.*

Proof. Let $T \subseteq B$ be as above. Considering a suitable power of $\sigma = gq$, we may assume that $g = 1$. Let B_X be a σ-invariant Borel subgroup of X and T_X a σ-invariant maximal torus of B_X. As $C_G(T_X)$ is connected and σ-invariant, it contains a σ-invariant maximal torus ${}^xT$ of G, for some $x \in G$. Since T is σ-invariant, we have $n = x^{-1}(\sigma x) \in N_G(T)$. Moreover, $T_X \subseteq C_G({}^xT) = {}^xT$. For $t \in T$, we have $\sigma t = t^q$, so T_X^x is σ-invariant, and $n \in N_G(T_X^x)$. If $\Phi(X, T_X) = \{\pm\beta\}$, then $\beta(\sigma({}^xt)) = q\beta({}^{xn}t)$, for all ${}^xt \in T_X$. Since $T_X^x \cong \mathbb{G}_m$, it follows from [26, 11.2(b)] that $i_n \mid T_X^x = 1$. Now apply [26, 11.2(c)]. ∎

LEMMA 1.2. *Let G be a semisimple algebraic group defined over an algebraically closed field K of characteristic $p > 0$, with simple components $G_1, \dots, G_n, n \geq 1$. Let $\sigma : G \to G$ be a q-Frobenius morphism of G for which G is σ-simple. Suppose X_1 is a closed σ^n-invariant A_1-subgroup of G_1. Then $Y = X_1(\sigma X_1)\cdots(\sigma^{n-1}X_1)$ is a closed semisimple σ-invariant σ-simple subgroup of G, and $\sigma \mid Y$ is a q-Frobenius morphism of Y.*

Proof. Let $T \subseteq B$ be as above. Clearly Y is σ-invariant, so it contains a σ-invariant Borel subgroup B_Y and σ-invariant maximal torus T_Y of B_Y. Moreover, $T_Y \subseteq {}^xT$, for some $x \in G$. Let $(x_{\pm\beta_i})$, $i = 1, \dots, n$, be a realization of $\Phi(Y, T_Y)$ in Y, where $\pm\beta_i$ corresponds to $X_i = \sigma^{i-1}X_1$. By [26, 11.2], we have $\sigma \mid X_i : X_i \to X_{i+1}$, where $\sigma x_{\pm\beta_i}(k) = x_{\pm\beta_{i+1}}(k^{q_i})$, for all $k \in K$, with q_i a power of p (modifying isomorphisms if necessary). Define $g_i : X_i \to X_{i+1}$ by $g_i x_{\pm\beta_i}(k) = x_{\pm\beta_{i+1}}(k)$. Then g_i is a bijective morphism of algebraic groups (abstractly, $g_i = \sigma q_i^{-1}$, and use the big cell [27, 5.7] for variety morphism). Moreover, $g_i^{-1} = g_{i-1}\cdots g_1 g_n \cdots g_{i+1}$, so g_i is an isomorphism of algebraic groups. Now $T_i = T_Y \cap X_i$ is a maximal torus of X_i. Let ${}^xt \in T_i$. Then $\sigma({}^xt) = {}^{(\sigma x)}(gt^q) = g_i(({}^xt)^{q_i})$, where $\sigma = gq$ on G. Let $\phi = i_{x^{-1}} g_i^{-1} i_{(\sigma x)} g \mid T_i^x$. Then ϕ is an algebraic automorphism of T_i^x, satisfying $\phi(t^q) = t^{q_i}$, for all $t \in T_i^x$. Since $T_i^x \cong \mathbb{G}_m$, we must have $q = q_i$, as required. ∎

2. PRODUCTS OF A_1S

In this section we handle the basic step of our induction. In the process, we provide a clarification of [28, 4.1], together with some details of its proof, which were absent there. In fact, [28, 4.1] is not quite correct as stated; however, it is valid for simple groups, as well as for semisimple groups G, under minor restrictions which are satisfied in applications. For example, they hold if G is a subsystem subgroup of a simple group in good characteristic, with σ the restriction to G of a Frobenius morphism of the simple group.

Let $G = X_1X_2 \cdots X_n$, $n > 1$, be a semisimple algebraic group defined over an algebraically closed field K of characteristic $p > 0$, with simple components X_i, $i = 1, \ldots, n$, of type A_1. Let $\sigma : G \to G$ be a Frobenius morphism of G. Choose a σ-invariant Borel subgroup B of G, and a σ-invariant maximal torus T of B. Let $\Phi = \{\pm\alpha_i\}$ be the corresponding set of roots, with $\pm\alpha_i$ corresponding to the component X_i. Let $(x_{\pm\alpha_i})$ be a realization of Φ in G. By [26, 11.2] there is a permutation ρ of Φ, and powers $q_i = p^{e_i}$ of p, such that $\sigma x_{\pm\alpha_i}(k) = x_{\pm\rho\alpha_i}(c_{\pm i}k^{q_i})$, for some $c_{\pm i} \in \mathbb{G}_m$ with $c_ic_{-i} = 1$, and all $k \in K$. We can modify the realization so that $c_i = 1$. The following two lemmas are special cases of the above situation and are essentially the only ones that arise in our problem.

LEMMA 2.1. *Suppose $\rho\alpha_i = \alpha_{i+1}$, where $\alpha_{n+1} = \alpha_1$. Let $u \in G_\sigma$ with $o(u) = p$.*

(i) *If $u \in X$, a closed σ-invariant A_1-subgroup of G, then $n \mid (\sum_{i=1}^n e_i)$.*

(ii) *If σ is a q-Frobenius morphism (i.e., each $q_i = q$), then $u \in X$, a closed σ-invariant A_1-subgroup of G, with $\sigma \mid X$ a q-Frobenius morphism of X.*

Proof. (i) Suppose X is such a subgroup. Choose a σ-invariant Borel subgroup B_X of X, and a σ-invariant maximal torus T_X of B_X. Let $(x_{\pm\beta})$ be a realization of $\Phi_X = \Phi(X, T_X)$ in X. Now X is σ^n-invariant, and $\sigma^n = q_0 = \prod_{i=1}^n q_i$ on G relative to the realization $(x_{\pm\alpha_i})$. By Lemma 1.1 we have $\sigma^n x_{\pm\beta}(k) = x_{\pm\beta}(k^{q_0})$, for all $k \in K$, modifying the realization if necessary. But by [26, 11.2], $\sigma x_{\pm\beta}(k) = x_{\pm\beta}(c_{\pm\beta}k^{q_\beta})$, with $q_\beta = p^e$, for some $e \geq 1$, and $c_{\pm\beta} \in \mathbb{G}_m$. Thus $ne = \sum_{i=1}^n e_i$, as required.

(ii) We have $u = u_1(\sigma u_1) \cdots (\sigma^{n-1}u_1)$, where $u_1 \in X_1$ satisfies $o(u_1) = p$ and $\sigma^n u_1 = u_1$. There exists $x \in (X_1)_{\sigma^n}$ with ${}^x u_1 = x_{\alpha_1}(\lambda)$, for suitable $0 \neq \lambda \in F_{q^n}$. Moreover $x(\sigma x) \cdots (\sigma^{n-1}x) \in G_\sigma$, so it will be enough to assume $u_1 = x_{\alpha_1}(\lambda)$. Let $X = \{x(gx) \cdots (g^{n-1}x) : x \in X_1\}$, where $\sigma = gq$ (see Section 1). Consider the isogeny $\theta : X_1 \to X$ where $\theta(x) = x(gx) \cdots (g^{n-1}x)$, $x \in X_1$. Then X is a closed A_1-subgroup of G.

Write $T_1 = T \cap X_1$ and $T_X = \theta T_1$. Then $(x_{\pm\beta})$, where $x_{\pm\beta} = \theta x_{\pm\alpha_1}$, is a realization of $\Phi(X, T_X)$ in X. Moreover $\sigma x_{\pm\beta}(k) = x_{\pm\beta}(k^q)$, for all $k \in K$. Now there exists $t \in T_1$ with ${}^t x_{\alpha_1}(k) = x_{\alpha_1}(\lambda k)$, $k \in K$. Since $\lambda^{q^n} = \lambda$, we have $\sigma^n t = tz$, $z \in Z(X_1)$. Set $a = t(\sigma t) \cdots (\sigma^{n-1} t) \in G$ and $y = \theta x_{\alpha_1}(1) \in X$. Then ${}^a y = u$ and $\sigma a = az$. It follows that ${}^a X$ is the required subgroup.

Note that Lemma 2.1 (ii) is a special case of the converse of (i). In fact, we remark that the converse is true in general, but the proof is more involved, and we do not give it here. In the next lemma, we consider the other extreme, where ρ fixes each α_i.

LEMMA 2.2. *Suppose $\rho = 1$. Let $u = \prod_{i=1}^n u_i$, where $u_i \in (X_i)_\sigma$ and $o(u_i) = p$. Then $u \in X$, a closed diagonal σ-invariant A_1-subgroup of G, if and only if $q_1 = q_2 = \cdots = q_n = q$. Moreover if such an X exists, then $\sigma \mid X$ is a q-Frobenius morphism of X.*

Proof. Suppose X is such a subgroup. Let $n = 2$, the general case being essentially the same. We may assume G is of adjoint type (cf. paragraph 4 of Proof of Theorem 5.1), so $G = X_1 X_2$ is a direct product of algebraic groups. Choose a σ-invariant Borel subgroup B_X of X, and a σ-invariant maximal torus T_X of B_X. Let $(x_{\pm\beta})$ be a realization of $\Phi_X = \Phi(X, T_X)$ in X. By [26, 11.2] we have $\sigma x_{\pm\beta}(k) = x_{\pm\beta}(c_{\pm\beta} k^q)$, for all $k \in K$, with q some power of p, and we may take $c_\beta = 1$. Now $T_X \subseteq {}^x T$, for some $x \in G$, with $n = x^{-1}(\sigma x) \in N_G(T)$. Let ${}^x t \in T_X$, $t \in T$. Then $\sigma({}^x t) = ({}^x t)^q$. We have $T = T_1 T_2$, where $T_i = T \cap X_i$ is a maximal torus of X_i, so can write (uniquely) $t = t_1 t_2$, $t_i \in T_i$. Since $\sigma t_i = t_i^{q^i}$, we obtain $t_1^q t_2^q = {}^n(t_1^{q_1} t_2^{q_2})$, and so ${}^n(t_i^{q_i}) = t_i^q$. The projection $\mathrm{pr}_i : X^x \to X_i$ is an isogeny, so $\mathrm{pr}_i(T_X^x)$ is a maximal torus of X_i contained in T_i, hence equal to T_i. Since $T_i \cong \mathbb{G}_m$, we obtain $q_i = q$ (cf. Proof of Lemma 1.2).

Conversely, assume $n = 2$ by induction. There exists $x_i \in (X_i)_\sigma$ with $u_i = {}^{x_i} x_{\alpha_i}(\lambda_i)$, for suitable $\lambda_i \in F_q$, $i = 1, 2$. Set $a = x_1 x_2 \in G_\sigma$. Then $u = {}^a(x_{\alpha_1}(\lambda_1) x_{\alpha_2}(\lambda_2))$, so it is enough to consider $x_{\alpha_1}(\lambda_1) x_{\alpha_2}(\lambda_2)$. Without loss of generality, there is a central isogeny $\psi : X_1 \to X_2$ (for definition, see [2, 22]). Modifying ψ by an inner automorphism, if necessary, we may assume $\psi B_1 = B_2$ and $\psi T_1 = T_2$, where $B_i = B \cap X_i$, $T_i = T \cap X_i$. (Then $\psi B_1^- = B_2^-$ by the uniqueness of opposite Borel subgroups.) Then $(\psi x_{\pm\alpha_1})$ is another realization of $\Phi(X_2, T_2)$ in X_2, so there exists $\mu \in \mathbb{G}_m$ with $\psi x_{\pm\alpha_1}(k) = x_{\pm\alpha_2}(\mu^{\pm 1} k)$ for all $k \in K$ (see [23, 11.2.1]). Replacing ψ by $i_t \psi$ for suitable $t \in T_2$, we may further assume that $\mu = \lambda_1^{-1} \lambda_2$. Define $X = \{x(\psi x) : x \in X_1\}$, a closed A_1-subgroup of G. Consider the isogeny $\theta : X_1 \to X$ where $\theta(x) = x(\psi x)$, $x \in X_1$. Define $T_X = \theta T_1 \subseteq \theta B_1$. Then $(x_{\pm\beta})$, where $x_{\pm\beta} = \theta x_{\pm\alpha_1}$, is a realization of $\Phi(X, T_X)$ in X. Since $\lambda_i^q = \lambda_i$, we have $\sigma x_{\pm\beta}(k) = x_{\pm\beta}(k^q)$ for all $k \in K$. Moreover $u = x_\beta(\lambda_1) \in X$, and so we are done. ∎

3. TWISTED CASES AND $A_1^{(3)}$ ELEMENTS IN G_2

In this section we consider the exceptional cases stated in the Introduction.

Let G be a simple algebraic group of type B_2, G_2, or F_4 defined over an algebraically closed field K of characteristic $p = 2, 3$, or 2, respectively. Fix a Borel subgroup B of G, a maximal torus T of B, and a realization $(x_\alpha)_{\alpha\in\Phi}$ of $\Phi = \Phi(G, T)$ in G. Let $g : G \to G$ be the twisted graph morphism corresponding to the symmetry of the Dynkin diagram of G. The symmetry extends to a permutation ρ of Φ which interchanges long and short roots. Then $gx_\alpha(k) = x_{\rho\alpha}(c_\alpha k^{p(\alpha)})$, for all $k \in K$, where $p(\alpha) = 1$ if α is long, $p(\alpha) = p$ if α is short. Moreover $c_\alpha = \pm 1$, with $c_\alpha = 1$ for $\pm\alpha$ simple. Let $\sigma = gq : G \to G$, where $qx_\alpha(k) = x_\alpha(k^q)$, $q = p^e$, $e \geq 0$. Note that σ is *not* a q-Frobenius morphism of G. See [27, 10] for more details.

LEMMA 3.1. *There are no closed σ-invariant A_1-subgroups of G.*

Proof. We have $\sigma^2 = pq^2 = p^{2e+1}$ on G relative to the given realization. Suppose X is a closed σ-invariant A_1-subgroup of G. Choose a σ-invariant Borel subgroup B_X of X and a σ-invariant maximal torus T_X of B_X. Let $(x_{\pm\beta})$ be a realization of $\Phi_X = \Phi(X, T_X)$ in X. Since X is σ^2-invariant, Lemma 1.1 gives $\sigma^2 x_{\pm\beta}(k) = x_{\pm\beta}(k^{p^{2e+1}})$, for all $k \in K$ (modifying if necessary). But [26, 11.2] gives $\sigma x_{\pm\beta}(k) = x_{\pm\beta}(c_{\pm\beta} k^{q_\beta})$, for some $q_\beta = p^f$, and so $2f = 2e + 1$, a contradiction. ∎

However it is still possible that G_σ contains (finite) A_1-subgroups, over fields of characteristic p, overlying unipotent elements. We now consider the various possibilities. If G is of type B_2 and $\sigma : G \to G$ is defined as above, then $G_\sigma = {}^2B_2(2^{2e+1})$ of order $r^2(r^2 + 1)(r - 1)$, $r = 2^{2e+1}$. Since 3 does not divide this order, it follows that G_σ does not contain any subgroups isomorphic to $SL_2(2) \cong \mathrm{Sym}_3$. Now let G be of type G_2, so that $G_\sigma = {}^2G_2(3^{2e+1})$. By [17, 8.5] the Sylow 2-subgroups of G_σ are elementary abelian of order 8. It follows that G_σ does not contain any subgroups isomorphic to $SL_2(3^i)$ or $PGL_2(3^i)$, $i \geq 1$. Indeed $SL_2(3)$ has one Sylow 2-subgroup which is isomorphic to Q_8, and $PGL_2(3) \cong \mathrm{Sym}_4$ has three Sylow 2-subgroups which are isomorphic to D_8. However, we have the following result, where $G_2(a_1)$ denotes the class of subregular unipotent elements of G.

LEMMA 3.2. *Let G be a simple algebraic group of type G_2 defined over an algebraically closed field of characteristic $p = 3$, and let $\sigma : G \to G$ be* *the morphism defined above. Let $u \in G_2(a_1) \cap G_\sigma$. Then u lies in a subgroup of G_σ isomorphic to $PSL_2(3^{2e+1})$. Moreover, such a subgroup contains representatives of the two G_σ-classes in $G_2(a_1) \cap G_\sigma$.*

Proof. Let $u \in G_2(a_1) \cap G_\sigma$. Then $C_G(u) = V\langle t\rangle$, where $V = C_G(u)^\circ$ is unipotent and t is an involution (see [10, 3.1], for example). Let $y = vt \in C_G(u)$, $v \in V$, be an involution. Then $\langle y, t\rangle$ is dihedral of order $2o(yt)$, and so y is conjugate to t by an element of $\langle v\rangle$. Thus by Lang's theorem there exists an involution $s \in (C_G(u))_\sigma$. Now by [29], s is a representative of the unique class of involutions in G_σ, with centralizer isomorphic to $\langle s\rangle \times PSL_2(3^{2e+1})$. Thus $u \in (C_G(s))_\sigma = C_{G_\sigma}(s) = \langle s\rangle \times PSL_2(3^{2e+1})$, and so $u \in PSL_2(3^{2e+1})$ by the Jordan decomposition. ∎

For G of type G_2, we are left with the class $A_1^{(3)}$. A representative of this class is $u = x_{2a+b}(1)x_{3a+2b}(1)$, with a, b simple and a short. We have $C_G(u) = U$ and $u \in G_\sigma$ (see [7, Table 2; 11, Table B], for example). In particular, u is semiregular and $A_1^{(3)} \cap G_\sigma$ gives rise to a single class in G_σ with representative u.

LEMMA 3.3. *Let G be a simple algebraic group of type G_2 defined over an algebraically closed field K of characteristic 3. Let $u = x_{2a+b}(1)x_{3a+2b}(1) \in A_1^{(3)}$. Then u does not lie in any subgroup of G isomorphic to $PSL_2(3)$.*

Proof. Suppose $u \in H \subseteq G$, where $H \cong PSL_2(3) \cong \mathrm{Alt}_4$. Then $H = V : \langle u\rangle$, where $V \trianglelefteq H$ is the Klein 4-group. An element of H either centralizes V, and hence lies in V, or via conjugation permutes the involutions of V transitively. Let $1 \neq t_1 \in V$. Since $u \notin V$, we have ${}^u t_1 = t_2 \neq 1, t_1$, and $V = \langle t_1, t_2\rangle$. Now t_1 and t_2 are commuting semisimple elements of G, which is simply connected, and so $V \subseteq T$, a maximal torus of G (see [9, 2.11; 24, II, 5.1]). We claim that

$$N_G(V) = N_G(T). \tag{1}$$

Since $\dim(T) = 2$, we have $V = \Omega_1(O_2(T))$, the subgroup of T generated by its involutions, and so $N_G(T) \subseteq N_G(V)$. In the notation of [3, 7.1], we have $V = \langle h(\chi_1), h(\chi_2)\rangle$, where $\chi_1(a) = -1$, $\chi_1(b) = 1$, $\chi_2(a) = 1$, and $\chi_2(b) = -1$ (the remaining involution is $h(\chi_1\chi_2)$). Suppose $x_r(k) \in C_G(V)$, $r \in \Phi^+$, $k \in K$. Then $\chi_1(r) = \chi_2(r) = 1$, since $h(\chi_i)x_r(k)h(\chi_i)^{-1} = x_r((\chi_i(r))k)$ by [3, 7.1]. Writing $r = \lambda_1 a + \lambda_2 b$, gives $1 = (-1)^{\lambda_1} = (-1)^{\lambda_2}$, and so $\lambda_i \in \{0, 2, 4, \ldots\}$, a contradiction as there are no roots of this form. It follows that $C_G(V)^\circ = T$ (see [24, II, 4.1(b)]). Since V is a diagonalizable group, we have $N_G(V)^\circ = C_G(V)^\circ = T$, by rigidity [8, 16.3]. Thus if $x \in N_G(V)$, then $xTx^{-1} \subseteq N_G(V)^\circ = T$, and so (1) holds. In particular, $u \in N_G(T)$. We next show

$$N_G(T) \text{ has a unique class of elements of order 3.} \tag{2}$$

First $W = N_G(T)/T \cong \mathrm{Dih}_{12}$. So if $x \in N_G(T)$ has order 3, then xT is uT or $u^{-1}T$. Suppose $x = u^{-1}s$, $s \in T$ (the former case is simpler and is omitted) and consider the map $\phi : T \to T$, where $\phi t = utu^{-1}t^{-1}$, $t \in T$. Clearly ϕ is a morphism of varieties and is injective since u is semiregular. By dimensions we have ϕ onto as well. Thus $s = utu^{-1}t^{-1}$, for some $t \in T$, and so $x = tu^{-1}t^{-1}$. But u^{-1} is conjugate to u by $h(\chi_1\chi_2) \in T$, and so (2) holds. Finally, using [6, 2], we have elements $n_2, n_3 \in N_G(T)$ satisfying

$$n_2 x_r(k) n_2^{-1} = x_{-r}(-k) \quad \text{and}$$
$$n_3 x_r(k) n_3^{-1} = x_{w_3(r)}(k), \quad \text{for all} \quad r \in \Phi, k \in K, \tag{3}$$

where $w_3 \in W$ has order 3. Since the $x_r(k)$ generate G, and $Z(G) = 1$, Eqs. (3) imply $o(n_2) = 2$, $o(n_3) = 3$, and $[n_2, n_3] = 1$. But u is conjugate to n_3 in G by (2), contradicting the fact that u is semiregular, thus proving the lemma. ∎

We now consider the implications of this result. Let $\tau = q$ or $\tau = gq = \sigma$, as defined above. Then $G_\tau = G_2(q)$ or $G_\tau = {}^2G_2(3q^2)$, respectively.

COROLLARY 3.4. (i) *If $u \in A_1^{(3)}$ then u does not lie in any closed* (*algebraic*) *A_1-subgroup of G.*

(ii) *Let $\tau = q$ or $\tau = gq$ and let $u \in A_1^{(3)} \cap G_\tau$. Then u does not lie in any* (*finite*) *A_1-subgroup of G_τ.*

Proof. (i) Suppose $u \in X$, a closed A_1-subgroup of G. Since u is semiregular, $X \cong PSL_2(K)$, so X contains a subgroup $Y \cong PSL_2(3)$. Choose $v \in Y$ with $o(v) = 3$. Then u is conjugate to v in X, contradicting Lemma 3.3.

(ii) Suppose u lies in a (finite) A_1-subgroup of G_τ. Since u is semiregular, it must lie in a subgroup isomorphic to $PSL_2(q_1)$, $q_1 = 3^i$, $i \geq 1$. This group has two conjugacy classes of elements of order 3, which are fused by an outer automorphism. Therefore, every element of order 3 in $PSL_2(q_1)$ lies in a $PSL_2(3)$-subgroup. This contradicts Lemma 3.3. ∎

Finally let G be of type F_4 and let $\sigma : G \to G$ be as above. Then $G_\sigma = {}^2F_4(2^{2e+1})$. By [19, 2, Corollary 2], G_σ has two classes of involutions with representatives $u_1 = \alpha_{10}(1)$ and $u_2 = \alpha_{12}(1)$ in the notation of [20, Table II]. Moreover u_1 is non-central and u_2 is central; that is, u_2 lies in the centre of a Sylow 2-subgroup of G_σ, while u_1 does not (also see [21, VII, 7.1]).

LEMMA 3.5. *In the notation above, u_1 lies in a subgroup of G_σ isomorphic to $SL_2(2^{2e+1})$. However, u_2 does not lie in any subgroup of G_σ isomorphic to $SL_2(2)$.*

Proof. For u_1 see [14, 2.2; 21, II, (2.2)]. In particular, u_1 inverts an element $t \in G_\sigma$ of order 3. So $N = N_{G_\sigma}(\langle t \rangle) = C_{G_\sigma}(t) : \langle u_1 \rangle \cong SU_3(2^{2e+1}) : 2$ by [14, 1.2(1); 20, 3.2]. All involutions in N inverting t are conjugate in N to u_1 by [1, 19.8(i)]. Since G_σ has a unique class of elements of order 3 (see [20, Table IV]), the statement about u_2 follows. ∎

4. DISTINGUISHED CLASSES IN BAD CHARACTERISTIC

In this section, for a simple algebraic group G, we determine the distinguished classes of elements of order p, where p is a bad prime for G.

LEMMA 4.1. *Let G be a simple classical algebraic group of type B_l, C_l, or D_l defined over an algebraically closed field of characteristic 2. Suppose G contains a distinguished involution. Then G is of type C_2 and has a unique class $\mathscr{C}$ of distinguished involutions. Moreover, if σ is a q-Frobenius morphism of G, or $\sigma = 1$, and $u \in \mathscr{C} \cap G_\sigma$, then $u \in X$, a closed σ-invariant A_1-subgroup of G.*

Proof. For each type, it is enough to consider one group in the isogeny class (see Proof of Theorem 5.1, for example). Moreover, by [2, 23.6], there exists a (purely inseparable) isogeny $\psi : O_{2l+1}(K) \to Sp_{2l}(K)$, so we need not consider type B_l at all. Let $G = Sp(V)$ be of type C_l, with $\dim(V) = 2l$, $l \geq 2$. By [1, Sect. 7] we have $l = 2$, and G has a unique class of distinguished involutions with representative

$$v = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \quad \text{relative to} \quad A = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix},$$

where A is the matrix of the symplectic form on V. Moreover, $C_G(v)$ is a maximal connected unipotent subgroup of G, so no splitting occurs in G_σ. Thus by Lang's theorem, we may take σ to be the qth-power map on matrix elements. Then $\sigma v = v$ and v lies in a closed σ-invariant subgroup of G of type $A_1 A_1$, so we are done by Lemma 2.2. Finally, if G is of type D_l, there are no such classes by [1, Sect. 8]. ∎

LEMMA 4.2. *Let G be a simple exceptional algebraic group defined over an algebraically closed field of bad characteristic p. If $u \in G$ is distinguished of order p, then G is of type G_2, $p = 3$, and $u \in G_2(a_1)$ or $u \in A_1^{(3)}$. Moreover, if σ is a q-Frobenius morphism of G, or $\sigma = 1$, and $u \in G_2(a_1) \cap G_\sigma$, then $u \in X$, a closed σ-invariant A_1-subgroup of G.*

Proof. For type G_2, use [7, 11] to obtain the given classes. For the subregular class $G_2(a_1)$, we may take G_σ representatives $x_{a+b}(1)x_{3a+b}(1)$ and $x_{a+b}(1)x_{3a+b}(\mu)$, where $\mu \in F_q$ is a non-square (with a, b simple and a short). Both lie in the closed σ-invariant semisimple subgroup $H = \langle U_{\pm(a+b)}, U_{\pm(3a+b)} \rangle$ of G of type $\tilde{A}_1 A_1$, so we may apply Lemma 2.2. For type F_4, use [11, 19, 22], and for types E_n, $n = 6, 7, 8$, use [11, 15, 16], to see that there are no distinguished classes of order p. ∎

5. MAIN THEOREM

We are now in a position to prove the main result of this paper, as stated in the Introduction. In the process we obtain a more general result for semisimple algebraic groups.

THEOREM 5.1. *Let G be a semisimple algebraic group defined over an algebraically closed field K of characteristic $p > 0$. Let $\sigma : G \to G$ be a q-Frobenius morphism of G, or $\sigma = 1$. Let $u \in G_\sigma$ with $\mathrm{o}(u) = p$. If $p = 3$ and G has a simple component of type G_2, assume that the projection of u into this component does not lie in the class $A_1^{(3)}$. Then $u \in X$, a closed σ-invariant A_1-subgroup of G, with $\sigma \mid X$ a q-Frobenius morphism of X.*

Proof. We use induction on $\dim(G) \geq 3$, the case where $\dim(G) = 3$, where G is of type A_1, being immediate. If $\sigma = 1$, the result follows from [28, 4.1], Lemmas 4.1 and 4.2, so assume σ is a q-Frobenius morphism of G. Let $G_1, G_2, \ldots, G_n$ be the simple components of G.

First suppose G is σ-simple, with $n \geq 2$. Then $u = u_1(\sigma u_1)\cdots(\sigma^{n-1}u_1)$, where $u_1 \in G_1$ satisfies $\mathrm{o}(u_1) = p$ and $\sigma^n u_1 = u_1$. Moreover, $\sigma^n \mid G_1$ is a q^n-Frobenius morphism of G_1. By induction, $u_1 \in X_1$, a closed σ^n-invariant A_1-subgroup of G_1. Thus $u \in Y = X_1(\sigma X_1)\cdots(\sigma^{n-1}X_1)$, and so we are done in this case by Lemmas 1.2 and 2.1(ii). Next consider the general case $G = H_1 H_2 \cdots H_t$, where H_i is the product of components in an orbit of $\langle \sigma \rangle$ on the simple components of G, and so is a closed σ-simple semisimple (and possibly simple) subgroup of G. Then $u = u_1 u_2 \cdots u_t \in G$, where for $1 \leq i \leq t$, $u_i \in (H_i)_\sigma$ and $\mathrm{o}(u_i) = p$ (removing any superfluous H_i if necessary). By above, *assuming* the result holds for simple groups, we have $u_i \in X_i$, a closed σ-invariant A_1-subgroup of H_i, and so we are done by Lemmas 1.1 and 2.2.

We are thus reduced to the case where G is simple. Relative to some maximal torus T of G, and realization $(x_\alpha)_{\alpha \in \Phi}$ of $\Phi = \Phi(G, T)$, we have $\sigma = gq : G \to G$. If u is not distinguished, there exists a semisimple element $1 \neq s \in C_G(u)^\circ$. Since $C_G(u)^\circ$ is connected and σ-invariant, it contains a σ-invariant torus S. Now $u \in C_G(S)$, which is connected and σ-invariant, so contains a σ-invariant maximal torus ${}^xT$ of G, for some

$x \in G$ (and $S \subseteq {}^xT$, since $C_G({}^xT) = {}^xT$). Since T is also σ-invariant, we have $n = x^{-1}(\sigma x) \in N_G(T)$. Now $C_G(S) = {}^x\langle T, U_\alpha : \alpha \in \Phi_1\rangle$, for some closed subsystem Φ_1 of Φ. Define $G_1 = \langle T, U_\alpha : \alpha \in \Phi_1\rangle$ and $H = G_1' = \langle U_\alpha : \alpha \in \Phi_1\rangle$. Let ρ be the symmetry of the Dynkin diagram of G corresponding to the graph automorphism g (possibly trivial); see [26, 11.6]. Now $nT = w \in W = N_G(T)/T$, and $w\rho\Phi_1 = \Phi_1$. Let Δ_1 be a base of Φ_1. Then $w\rho\Delta_1$ is another one, so there is $w_1 \in W_1 = N_{G_1}(T)/T = (N_G(T) \cap G_1)/T$ with $w\rho\Delta_1 = w_1\Delta_1$. Thus $w_1^{-1}w\rho\Delta_1 = \Delta_1$, so $w_1^{-1}w\rho$ corresponds to a symmetry ρ_1 of the Dynkin diagram of H. Let $g_1 : H \to H$ be the graph automorphism of H corresponding to ρ_1. Choose $n_1 \in N_{G_1}(T)$ such that $n_1T = w_1$. Then for $\alpha \in \Delta_1$, we have $(i_{n_1^{-1}}i_ng)(U_\alpha) = U_{\rho_1\alpha} = g_1(U_\alpha)$. Thus $i_{n_1^{-1}}i_ngx_\alpha(1) = x_{\rho_1\alpha}(c_\alpha)$, for some $c_\alpha \in \mathbb{G}_m$, whereas $g_1x_\alpha(1) = x_{\rho_1\alpha}(1)$. Since Δ_1 is a linearly independent subset of $X(T)$, there exists $t \in T$ with $i_ti_{n_1^{-1}}i_ngx_\alpha(1) = g_1x_\alpha(1)$, for all $\alpha \in \Delta_1$. Set $\psi = g_1^{-1}i_ti_{n_1^{-1}}i_ng \in \mathrm{Aut}_{\mathrm{alg}}(H)$. Then $\psi x_\alpha(1) = x_\alpha(1)$, for all $\pm\alpha \in \Delta_1$ (see [25, 5.2]). Then ψ is a field automorphism of H by [25, 5.7]; also see [4, Proposition 6]. Thus ψ is the identity map, and so $i_ng = i_yg_1$, where $y = n_1t^{-1} \in G_1$. By Lang's theorem, there is $a \in G_1$ with ${}^xy = \sigma({}^xa)^{-1}({}^xa)$. For $\alpha \in \Phi_1$, define $y_\alpha = i_{xa}x_\alpha : \mathbb{G}_a \overset{\text{iso}}{\to} {}^{xa}U_\alpha \subseteq {}^xH$. Relative to the realization $(y_\alpha)_{\alpha\in\Phi_1}$, we have $\sigma y_\alpha(k) = i_{\sigma(xa)n^{-1}}(i_ng)x_\alpha(k^q) = i_{xa}g_1x_\alpha(k^q) = y_{\rho_1\alpha}(d_\alpha k^q)$, for some $d_\alpha \in \mathbb{G}_m$, and all $k \in K$. Thus $\sigma \mid {}^xH$ is a q-Frobenius morphism, and we are done by induction in this case.

Now let u be distinguished. If p is bad, we are done by Lemmas 4.1 and 4.2, so assume p is good. We may assume G is of adjoint type. Indeed, let $\psi = \mathrm{ad} : G \to \overline{G} = G_{\mathrm{ad}}$ be the adjoint representation of G. Set $\overline{T} = \psi T$. For $\alpha \in \Phi$, define $\overline{x}_\alpha = \psi x_\alpha : \mathbb{G}_a \to \overline{U}_\alpha = \psi U_\alpha$. Then $(\overline{x}_\alpha)_{\alpha\in\Phi}$ is a realization of $\Phi(\overline{G}, \overline{T})$ in $\overline{G}$. Define $\overline{\sigma} : \overline{G} \to \overline{G}$ by $\overline{\sigma}(\psi x) = \psi(\sigma x)$, $x \in G$. Now $\overline{u} = \psi u \in \overline{G}_{\overline{\sigma}}$ and $\mathrm{o}(\overline{u}) = p$, so by assumption $\overline{u} \in \overline{X}$, a closed $\overline{\sigma}$-invariant A_1-subgroup of $\overline{G}$. Define $X = (\psi^{-1}(\overline{X}))^\circ$. Then X is simple of dimension $\dim(\overline{X})$, so is of type A_1. Moreover, $u \in X$ (since $\psi^{-1}(\overline{X}) = XZ(G)$) and X is σ-invariant, as required.

We may assume $C_G(u)$ is connected. Given that u is distinguished, we have $C_G(u)$ disconnected if and only if u is not semiregular. (Here u is *semiregular* if $C_G(u)$ is unipotent.) For this we need the structure of the component group $A(u) = C_G(u)/C_G(u)^\circ$. See [5, 13.1], for example; the possibilities are Z_2^i, Sym_3, Sym_4, or Sym_5. When G is exceptional, we have $i = 1$. For the cases Z_2 and Sym_3, argue as in claim 7 of [28, 4.1]. The cases Sym_i, $i = 4, 5$, only occur for a single class in types F_4 and E_8, respectively. Moreover, in both cases, there exists $v \in G_\sigma$ where $u = gvg^{-1}$, $g \in G$, with σ acting trivially on the component group $A(v) = C_G(v)/C_G(v)^\circ$; see [16, Lemma 70; 22, (2.3)]. (These cases are also dealt with in [28, 4.1], but we provide here an alternative proof modelled on [22, (2.3)].) Thus we can write $C_G(v) = \dot{\bigcup}\, a_i C_G(v)^\circ$, for some $a_i \in (C_G(v))_\sigma$ by Lang's theorem. Now $C_G(u) = \dot{\bigcup}\, (ga_ig^{-1})C_G(u)^\circ$, and $h = g^{-1}\sigma(g) \in C_G(v)$. The coset $(ga_ig^{-1})C_G(u)^\circ$ is σ-invariant if and only if $[\overline{h}, \overline{a}_i] = \overline{1}$, bars denoting images in $A(v)$. Note that a non-σ-invariant coset cannot contain any σ-invariant elements, whereas each σ-invariant coset has a σ-invariant representative b_i by Lang's theorem. Moreover, distinct elements $\overline{a}_i, \overline{a}_j$ of $C_{A(v)}(\overline{h})$ give rise to distinct cosets. Therefore

$$C_{G_\sigma}(u) = \dot{\bigcup}_{\overline{a}_i \in C_{A(v)}(\overline{h})}\, b_i(C_G(u)^\circ)_\sigma,$$

where $b_i \in (ga_ig^{-1})C_G(u)^\circ$. Thus if r denotes the order of $C_{A(v)}(\overline{h})$, then r divides $|C_{G_\sigma}(u)|$. But $r \neq 1$ and r divides $|A(v)| = 2^3 \cdot 3$ or $2^3 \cdot 3 \cdot 5$, respectively, and so u commutes with a non-identity σ-invariant semisimple element of G, since p is good. We are then done by induction (cf. the non-distinguished case above). Finally we consider the case Z_2^i, $i > 1$, which only occurs for G classical of type B, C, or D. We claim there exists $v \in G_\sigma$ where $u = gvg^{-1}$, $g \in G$, with σ acting trivially on $A(v)$, so we can then argue as in the cases above. Let V be a finite dimensional vector space over K, endowed with either a non-degenerate symplectic or quadratic form. It will be convenient to work first in $\tilde{G} = I(V)$, the full classical group of V, so that $\tilde{G} = Sp(V)$ or $O(V)$, respectively. The claim will then follow on applying the adjoint representation to $G_1 = \tilde{G}^\circ$, on noting that every element of G which is distinguished but not semiregular is obtained from a like element in G_1. Let $\tilde{\sigma}$ be a q-Frobenius morphism of G_1 and extend naturally to $\tilde{G}$; so $\tilde{\sigma}$ acts trivially on $Z(O(V))$ in case $\tilde{G} = O(V)$. Note that the case of triality in D_4 does not arise here, as the component groups in that case are either trivial or Z_2 (see [5, Chap. 13], for example). Therefore $\tilde{\sigma}$ is induced by an element of $\Gamma L(V)$, the group of semilinear maps of V, as described in [13, Sect. 2], for example. Write $V = V_1 \perp \ldots \perp V_r$, an orthogonal direct sum of non-degenerate subspaces of different dimensions, all even dimensional for $\tilde{G} = Sp(V)$, and all odd dimensional for $\tilde{G} = O(V)$. From the definition of $\tilde{\sigma}$ in [13], it is clear that we may take each V_i to be $\tilde{\sigma}$-invariant. Let $H = \{g \in \tilde{G} : gV_i = V_i\} \cong I(V_1) \times \cdots \times I(V_r)$, where each $I(V_i) = Sp(V_i)$ or $O(V_i)$, depending on whether $\tilde{G} = Sp(V)$ or $O(V)$, respectively. Note that each $I(V_i) = \{g \in H : g \mid V_j = 1, j \neq i\}$ (and H) is $\tilde{\sigma}$-invariant. Let $v_i \in I(V_i)$ be a $\tilde{\sigma}$-invariant regular unipotent element. Then $v = v_1 \cdots v_r$ is a $\tilde{\sigma}$-invariant regular unipotent of H. By [28, 3.1, 3.2], every distinguished unipotent element of G_1 is conjugate in G_1 to an element of this form. Write $L = \prod_i Z(I(V_i)) \subseteq C_{\tilde{G}}(v)$. Then it follows from [24, IV, 2.26(ii)] that $C_{\tilde{G}}(v) = C_{\tilde{G}}(v)^\circ : L$. Note that $\tilde{\sigma}$ acts trivially on L. Since $A(v) = C_{G_1}(v)/C_{G_1}(v)^\circ$ is a subgroup of $C_{\tilde{G}}(v)/C_{\tilde{G}}(v)^\circ$, it follows that $\tilde{\sigma}$ acts trivially on $A(v)$, as required.

This leaves the semiregular classes, which are dealt with in [28], specifically, [28, Propositions 2.4, 3.5, and Claims 8, 9 of 4.1]. ∎

In particular, the main result stated in the Introduction follows from Theorem 5.1, combined with Lemma 3.1 and Corollary 3.4(i).

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