Some Regularity Theorems for Operators in an Enveloping Algebra*

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INTRODUCTION

The regularity properties of linear elliptic differential operators, acting on various function spaces, have been intensively studied in recent years (cf. [2]). In this paper, we consider the corresponding problem for an elliptic operator $L$ in the enveloping algebra of a real Lie algebra $\mathfrak{g}$, acting in a Hilbert space $\mathcal{H}(\pi)$ via the differential $\partial\pi$ of a unitary representation $\pi$ of the Lie group $G$.

This problem was first studied by Nelson and Stinespring [16]. Later Langlands [14] extended their results to include representations on Banach spaces. In both cases the operator $\partial\pi(L)$ and its adjoint were studied primarily via the action of the right invariant differential operator $L$ on the representative functions on $G$, using the standard elliptic regularity theory. Although this method was sufficient, e.g., to show that $\partial\pi(L)$ is essentially self-adjoint provided $L$ is elliptic and symmetric (in the case of unitary $\pi$), it did not give the precise domain of the adjoint of $\partial\pi(L)$.

This additional information is very useful in the application of elliptic operators, e.g., to constructing analytic vectors for a representation, and in establishing regularity properties for the eigenfunction expansion associated with $\partial\pi(L)^*$ (in case of symmetric $L$ and unitary $\pi$).

In this paper we establish some domain results for a restricted class of elliptic operators in the enveloping algebra of $\mathfrak{g}$, relative to a unitary representation $\pi$. Namely, if $L$ is of order $2m$ and is associated with a Hermitian elliptic form (see Section 2 for definition), then the Hilbert-space adjoint of $\partial\pi(L)$ has as domain the space $\mathcal{H}^{2m}(\pi)$ of $2m-$times differentiable vectors for $\pi$.

The proof of this result involves two developments. The first (Section 1) consists of establishing regularity properties for the action of $\mathfrak{g}$ on the chain

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\{ \mathcal{H}^k(\pi) \} \) of spaces of differentiable vectors. The second (Sections 2–3) involves proving the analogue of Garding’s inequality for the sesquilinear form \((\partial \pi(L)u, u)\), where \(u\) is a \(C^\infty\) vector for the representation. These two elements are then combined (Section 4) via the “Hilbert-space approach” to elliptic regularity, as developed in [2], to study the operator \(\partial \pi(L)^*\). Applications of these results and examples are given in Sections 5 and 6.

1. Spaces of Differentiable and Generalized Vectors

Let \(\pi\) be a strongly continuous unitary representation of a Lie group \(G\) on a complex Hilbert space \(\mathcal{H}(\pi)\). In this section we assemble some (mostly known) facts about the spaces \(\mathcal{H}^k(\pi)\) of \(k\)-times differentiable vectors and their duals (cf. [7], [9], and [20]).

If \(v \in \mathcal{H}(\pi)\), let \(\hat{v}\) be the \(\mathcal{H}(\pi)\)-valued function on \(G\) given by \(\hat{v}(g) = \pi(g)v\). If we denote by \(C^k(G; \mathcal{H}(\pi))\), \(k\) a nonnegative integer, the space of \(k\)-times continuously differentiable functions from \(G\) to \(\mathcal{H}(\pi)\) (in the strong topology on \(\mathcal{H}(\pi)\)), then by definition

\[
\mathcal{S}^k(n) = \{ Z \in \mathcal{S}(n) \mid \hat{Z} \in C^k(G; \mathcal{H}(\pi)) \}.
\]

The space \(\mathcal{H}^{\infty}(\pi)\) of infinitely differentiable vectors for \(\pi\) is thus the intersection of the spaces \(\mathcal{H}^k(\pi)\), \(k \geq 0\).

Let \(g\) be the Lie algebra of \(G\), and for \(X \in g\) denote by \(d\pi(X)\) the infinitesimal generator (in the sense of operator theory) of the one-parameter unitary group \(t \mapsto \pi(\exp tX)\). If \(\partial \pi(X)\) is the restriction of \(d\pi(X)\) to the invariant subspace \(\mathcal{H}^{\infty}(\pi)\), then the mapping \(X \mapsto \partial \pi(X)\) is a Lie algebra homomorphism from \(g\) to skew-symmetric operators on \(\mathcal{H}^{\infty}(\pi)\). It extends uniquely to an associative algebra homomorphism, which we also denote by \(\partial \pi\), from the complex universal enveloping algebra \(u(g)\) to linear operators on \(\mathcal{H}^{\infty}(\pi)\). We write \(\mathcal{H}^k(\pi) = \mathcal{H}^k\), \(u(g) = u\), and \(\partial \pi(T)u = Tu\) when \(G\) and \(\pi\) are understood from the context.

The spaces \(\mathcal{H}^k\) are Hilbertian. Two equivalent Hilbert norms are the following: Pick a basis \(X_1, \ldots, X_d\) for \(g\), and set \(A = X_1^2 + \cdots + X_d^2\). Let \(A\) be the closure of the operator \(\partial \pi(1 - A)\); the operator \(A\) is self-adjoint and positive, and we let \(B = A^{1/2}\) (positive square root). Then one has [9, Propositions 1.1 and 1.3]:

\[
\mathcal{H}^k(\pi) = \mathcal{D}(B^k) = \bigcap_{|\alpha| = k} \mathcal{D}(d\pi(X_1)^{\alpha_1} \cdots d\pi(X_d)^{\alpha_d}). \tag{1.1}
\]
(\mathcal{D}(T)\) denotes the domain of definition of an unbounded operator \(T\) on \(\mathcal{H}(\pi)\), and \(|\alpha| = \alpha_1 + \cdots + \alpha_d\) if \(\alpha = (\alpha_1, \ldots, \alpha_d)\). Thus,

\[
\sum_{|\alpha| \leq k} \| d\pi(X_1)^{\alpha_1} \cdots d\pi(X_d)^{\alpha_d} \|_2^2
\]

and

\[
\| B^t v \|_2^2
\]

are both squares of Hilbert norms on \(\mathcal{H}(\pi)\). The inclusion \(\mathcal{H}^k(\pi) \subseteq \mathcal{H}^{k-1}(\pi)\) is continuous, and \(\mathcal{H}^{\infty}(\pi)\) is dense in \(\mathcal{H}^k(\pi)\) in the topology defined by these norms [9, Corollary I.2]. Thus \(\{\mathcal{H}^k(\pi)\}\) is a discrete chain of Hilbertian spaces, in the terminology of [18].

For any \(t \in \mathbb{R}\), define \(B^t\) by the spectral theorem, and let \(\mathcal{H}^t(\pi)\) be the completion of \(\mathcal{D}(B^t)\) with respect to the norm \(\| B^t u \|\). Since \(B\) is unitarily equivalent to a multiplication operator on \(L^2(\Omega)\) for some measure space \(\Omega\), it follows from [18, Chap. VIII, Section 3, Theorem 2] that \(\{\mathcal{H}^t(\pi)\}\) is a continuous scale of Hilbertian spaces, and is obtainable from the discrete chain \(\{\mathcal{H}^k(\pi)\}\) by quadratic interpolation and duality. In particular, this scale of spaces is independent of the choice of basis used to define \(\mathcal{D}\). Taking a fixed \(\mathcal{D}\), we set

\[
\| v \|_t = \| B^t v \|, \quad t \in \mathbb{R}, \quad v \in \mathcal{H}^t(\pi).
\]  

(Here, \(\| \cdot \|\) denotes the original Hilbert norm in \(\mathcal{H}\); when \(t \leq 0\) the operator \(B^t\) appearing in (1.2) is the extension of the operator defined by the spectral theorem to a continuous mapping from \(\mathcal{H}^t\) to \(\mathcal{H}\).) By realizing \(B\) as a multiplication operator on \(L^2(\Omega)\), one easily estimates that for any \(\varepsilon > 0\), and \(s < t\), there exists a constant \(C\) such that

\[
\| v \|_s \leq \varepsilon \| v \|_t + C \| v \|, \quad v \in \mathcal{H}^s(\pi).
\]  

Let \(u_n\) denote the subspace of \(u\) spanned by \(1\) and products \(Y_1 \cdots Y_m\), with \(m \leq n\) and \(Y \in \mathfrak{g}\). If \(T \in u_n\) and \(k\) is a nonnegative integer, then by equation (1.1) there exists a constant \(C_k\) such that

\[
\| \partial \pi(T) u \|_k \leq C_k \| u \|_{k+n}, \quad u \in \mathcal{H}^k(\pi).
\]  

Now since \(\mathcal{H}^\infty\) is dense in \(\mathcal{H}^t\) for any \(t\), it follows that

\[
\| v \|_t = \sup \{ (v, u) : u \in \mathcal{H}^\infty, \| u \|_{-t} = 1 \}.
\]  

If \(T \rightarrow T^*\) is the involutive conjugate-linear antiautomorphism of \(u\) such that \(X^* = -X\) when \(X \in \mathfrak{g}\), then \(u_n^* = u_n\). The relation \(\partial \pi(T) u, v \rangle = (u, \partial \pi(T^*) v)\) also holds for \(u\) and \(v\) in \(\mathcal{H}^\infty\), since \(\partial \pi(X)\) is skew-Hermitian.
when \( X \in \mathfrak{g} \). It follows from this and (1.5) that estimate (1.4) holds when \( k \) is a negative integer also. By quadratic interpolation we obtain estimate (1.4) for all real \( k \). Thus, \( \partial \pi(T) \) extends by continuity to a continuous linear map from \( \mathcal{H}^{t+n} \) to \( \mathcal{H}^t \), whenever \( T \in \mathcal{D} \) and \( t \in \mathbb{R} \). Letting

\[
\mathcal{H}^{-\infty}(\pi) = \bigcup_t \mathcal{H}^t(\pi),
\]

with the inductive limit topology, we obtain a representation \( \delta \pi \) of \( \mathcal{H}^{-\infty} \) by continuously extending the representation \( \partial \pi \) from the dense subspace \( \mathcal{H}^\infty \). The representations \( \partial \pi \) and \( \delta \pi \) are related by

\[
(\partial \pi(T)u, v) = (u, \delta \pi(T^*v)), \quad u \in \mathcal{H}^\infty, \quad v \in \mathcal{H}^{-\infty},
\]

(1.6)

where \((\cdot, \cdot)\) denotes the continuous extension of the original scalar product in \( \mathcal{H} \) to a sesquilinear form on \( \mathcal{H}^\infty \times \mathcal{H}^{-\infty} \). Indeed, relation (1.6) holds when \( v \) is also in \( \mathcal{H}^\infty \), and both sides of (1.6) are continuous conjugate-linear functions of \( v \in \mathcal{H}^{-\infty} \).

We shall need the following version of the quadratic interpolation theorem for estimating certain sesquilinear forms on \( \mathcal{H}^\tau \):

**Lemma 1.1.** Suppose \( S \) and \( S^\tau \) are continuous linear maps of \( \mathcal{H}^\infty \) into \( \mathcal{H}^\infty \) which satisfy \((Su, v) = (u, S^\tau v)\) for \( u, v \in \mathcal{H}^\infty \). Suppose there exist real numbers \( \alpha, \beta, \) and \( \tau \) and a positive constant \( C \) such that for all \( u \in \mathcal{H}^\infty \)

\[
\| Su \|_\alpha \leq C \| u \|_{\alpha+\tau}, \quad (1.7)_\alpha
\]

\[
\| S^\tau u \|_{\beta-\tau} \leq C \| u \|_\beta. \quad (1.8)_\beta
\]

Then,

\[
\| (B^\lambda S B^{-\lambda} u, u) \| \leq C \| u \|_{\tau+\frac{\tau}{2}} \quad (1.9)
\]

for \( \lambda = \frac{1}{2} \tau + t \alpha + (1 - t) \beta, \ 0 \leq t \leq 1 \).

**Remark.** Estimate \((1.7)_\alpha \) always implies estimate \((1.8)_{\alpha+\tau} \), by virtue of equation (1.5). The lemma yields new estimates only in case \( \alpha \neq \beta \).

**Proof of Lemma 1.1.** Consider the operator \( T = SB^{-\tau} \). By \((1.7)_\alpha \) we have

\[
\| Tu \|_\alpha \leq C \| u \|_\alpha. \quad (1.10)
\]

Using equation (1.5) we have

\[
\| Tu \|_\beta = \sup \{ (Tu, v) : \| v \|_{-\beta} = 1 \}
\]

\[
= \sup \{ (B^\tau u, B^{\beta-\tau} S^\tau v) : \| v \|_{-\beta} = 1 \}
\]

\[
\leq C \| u \|_\beta,
\]

for \( \lambda = \frac{1}{2} \tau + t \alpha + (1 - t) \beta, \ 0 \leq t \leq 1 \).
by Schwarz' inequality and (1.8)\textsuperscript{+}. Hence by the quadratic interpolation theorem,
\[ \|T u\|_\gamma \leq C\| u\|_\gamma, \]
when \( \gamma = t\alpha + (1-t)\beta, \) \( 0 \leq t \leq 1. \) Equivalently, \( B^\nu S B^{-\nu-r} \) is bounded in the 0-norm. Hence the corresponding sesquilinear form satisfies
\[ \|(B^\nu S B^{-\nu-r} u, u)\| \leq C\| u\|^2. \] (1.10)
Replacing \( u \) by \( B^{t/2}u \) in (1.10) (this is possible since \( B^t \) maps \( \mathcal{H}^\infty \) onto \( \mathcal{H}^\infty \) for all real \( t \)), we obtain (1.9).

Q.E.D.

The next lemma is the basic technical tool for handling "lower order terms" in \textit{a-priori} estimates for operators in the enveloping algebra.

**Lemma 1.2.** Let \( T \in \mathfrak{u}_n, a, b \in \mathbb{R}. \) Then there exists a constant \( C \) such that
\[ \| (B^a \partial \pi(T) B^b u, u) \| \leq C\| u\|^{(a+b+n)/2}. \] (1.11)

If \( |a - b| < n, \) then there exists a constant \( C \) such that
\[ \| (B^a[\partial \pi(T), B] B^b u, u) \| \leq C\| u\|^{(a+b+n)/2}. \] (1.12)

**Proof.** By replacing \( a \) with \( a - b \) and \( u \) with \( B^{-b}u, \) we see that it suffices to treat the case \( b = 0. \) The operators \( \partial \pi(T) \) and \( \partial \pi(T^*) \) satisfy (1.7)\textsubscript{a} for all \( \alpha, \) with \( r = n, \) as remarked earlier. Hence, the same is true of the operators \( S = B^a \partial \pi(T) \) and \( S^+ = \partial \pi(T^*) B^a, \) with \( r = n + a. \) In particular, we may take \( \alpha = 0, \beta = -\alpha - r, \) and \( t = \frac{1}{2} \) in Lemma 1.1 to get estimate (1.9) with \( \gamma = 0. \)

To obtain the commutator estimate, we start with the estimate \( (ad Y(T) \) denoting \( YT - TY)\)
\[ \|ad Y_{i_1} \cdots ad Y_{i_k}(B)u\|_{1-\gamma} \leq C_{k,\gamma}\|u\|_{1-\gamma}, \] (1.13)
where \( Y_j = \partial \pi(X_j), \) \( \{X_j\} \) being a basis for \( \mathfrak{g}, \) and \( 0 < \gamma < 1. \) (See [6, Lemma 6] and following remark. Note that the scale \( \mathcal{H}^2 \) in [6] is defined relative to the operator \( A = B^2. \) By the standard commutator formulas (cf. [15, Lemma 2.1]), \( [\partial \pi(T), B] \) is a linear combination of terms
\[ \{ad Y_{j_1} \cdots ad Y_{j_k}(B)\} Y_{j_{k+1}} \cdots Y_{j_n}, \]
with \( n \gg k \gg 1. \) Hence \( S = B^a[\partial \pi(T), B] \) satisfies estimate (1.7)\textsubscript{a-\gamma}, with \( r = a + n. \) Since \( S^+ = [\partial \pi(T^*), B] B^a, \) we have by the same argument applied to \( T^* \) the estimate (1.8)\textsubscript{a+}, where \( \beta = \gamma - a - n. \) Thus Lemma 1.1 gives estimate (1.9) for
\[ \lambda = \frac{1}{2}(a + n) - t(a + \gamma) + (1-t)(\gamma - a - n), \]
with \( 0 \leq t \leq 1 \). Writing

\[
\lambda = \frac{1}{2}(n - a) - \{ty + (1 - t)(n - y)\},
\]

we see that we can achieve \( \lambda = 0 \) by choosing \( y \) sufficiently small, provided \( 0 < \frac{1}{2}(n - a) < n \), i.e., provided \( |a| < n \).

**Q.E.D.**

2. Sesquilinear Forms and Hermitian Symbols

If \( \pi \) is a unitary representation of \( G \) and \( S_j, T_k \in u(\mathfrak{g}) \), \( c_{j,k} \in \mathbb{C} \), then

\[
\left( \sum c_{j,k}(\partial \pi(S_j)u, \partial \pi(T_k)v) \right)
\]

(finite sum) is a sesquilinear form on \( \mathcal{H}^{\infty}(\pi) \), which may also be represented as \( (\partial \pi(L)u, v) \), where

\[
L = \sum c_{j,k}T_k^*S_j.
\]

We can describe such forms in a somewhat more intrinsic manner as follows: Let \( *u \) be the complex vector space which as an additive group is simply \( u = u(\mathfrak{g}) \), but having scalar multiplication \( \lambda \cdot T = \lambda^*T \) (\( \lambda^* \) = complex conjugate of \( \lambda \in \mathbb{C} \)). Let \( u \otimes *u \) denote the tensor product of \( u \) with \( *u \) as complex vector spaces. Then \( u \otimes *u \) solves the universal mapping problem for sesquilinear maps of \( u \times u \) into complex vector spaces. In particular, for every unitary representation \( \pi \) of \( G \), there exists a unique linear map \( Q \rightarrow Q^\pi \) from \( u \otimes *u \) to the complex vector space of sesquilinear forms on \( \mathcal{H}^{\infty}(\pi) \), such that

\[
(S \otimes T)^\pi(u, v) = (\partial \pi(S)u, \partial \pi(T)v), \quad (2.1)
\]

for \( S, T \in u \), and \( u, v \in \mathcal{H}^{\infty}(\pi) \), \( \otimes \) denoting the tensor product in \( u \otimes *u \). There also exists a unique linear map \( \gamma : u \otimes *u \rightarrow u \) such that

\[
\gamma(S \otimes T) = T^*S. \quad (2.2)
\]

These two maps are related by

\[
Q^\pi(u, v) = (\partial \pi(\gamma(Q))u, v). \quad (2.3)
\]

Finally, there is a conjugate linear involution \( Q \rightarrow Q^* \) on \( u \otimes *u \) such that \( (S \otimes T)^* = T \otimes *S \). Clearly one has

\[
\gamma(Q^*) = \gamma(Q)^*. \quad (2.4)
\]

**Definition.** An element \( Q \in u \otimes *u \) is **Hermitian** if \( Q = Q^* \).
If $Q$ is Hermitian, then $Q^\ast$ is a Hermitian form on $\mathcal{H}(\pi)$ for any unitary representation $\pi$, by formulas (2.4), (2.3), and (1.6).

Let $\{X_j\}$, $1 \leq j \leq d$, be a basis for $\mathfrak{g}$, and set $X(t) = t_1X_1 + \cdots + t_dX_d$ for $t \in \mathbb{R}^d$. If $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a $d$-tuple of nonnegative integers with $|\alpha| = n$, define the element $X(\alpha)$ of $\mathfrak{g}_n$ by the identity

$$\frac{1}{n!} X(t)^n = \sum_{\alpha} \frac{t^n}{\alpha!} X(\alpha).$$

Then $X(\alpha) = X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ mod $\mathfrak{u}_{n-1}$, and the cosets $\{X(\alpha) + \mathfrak{u}_{n-1}\}$ form a basis for $\mathfrak{u}_n/\mathfrak{u}_{n-1}$ (cf. [11, Chap. 2, Section 2]). Thus, an element $Q \in \mathfrak{g} \otimes \mathfrak{u}^*$ can be written uniquely as a finite sum $Q = \sum c_{\alpha,\beta}X(\alpha) \otimes X(\beta)$, so that $Q = Q^\ast$ if and only if $c_{\alpha,\beta} = c_{\beta,\alpha}$.

Let $m$ be an integer $\geq 1$. By the Poincaré–Birkhoff–Witt theorem [3], there is a canonical linear isomorphism of $\mathfrak{u}_m/\mathfrak{u}_{m-1}$ with the space $S_m(\mathfrak{g}_c)$ of symmetric tensors of degree $m$ over $\mathfrak{g}_c$. Since $S_m(\mathfrak{g}_c)$ is naturally isomorphic to the space $\mathcal{P}_m(\mathfrak{g}_c)$ of homogeneous polynomial functions of degree $m$ on the dual space $\mathfrak{g}_c'$, it follows that one has a canonical linear map

$$\sigma_m : \mathfrak{u}_m(\mathfrak{g}_c) \to \mathcal{P}_m(\mathfrak{g}_c').$$

If $X(\xi) = \langle \xi, X \rangle$ for $\xi \in \mathfrak{g}_c'$, $X \in \mathfrak{g}_c$, then for a multiindex $\alpha$ with $|\alpha| = m$, the function $\sigma_m(X(\alpha))$ is easily seen to be given by the formula

$$\sigma_m(X(\alpha))(\xi) = X_1(\xi)^{\alpha_1} \cdots X_d(\xi)^{\alpha_d} \xi^\alpha. \quad (2.5)$$

Let $S, T \subset \mathfrak{u}_m$, and define $(S \otimes T)^\ast$ to be the function on $\mathfrak{g}_c'$ given by

$$(S \otimes T)^\ast(\xi) = \sigma_m(S(\xi))\sigma_m(T(\xi))^\ast. \quad (2.6)$$

The right-hand side of (2.6) is a sesquilinear function of the pair $(S, T)$, so depends only on $S \otimes T$ and $m$. (To be precise, we will write $^m$ in place of $^$ when necessary.) Thus we obtain a linear map $Q \to \hat{Q}$ from $\mathfrak{u}_m \otimes \mathfrak{u}_m$ to functions on $\mathfrak{g}_c'$. We shall refer to $\hat{Q}$ as the Hermitian symbol of $Q$ ($m$ being understood). If $Q = \sum c_{\alpha,\beta}X(\alpha) \otimes X(\beta)$, with $|\alpha| \leq m$, $|\beta| \leq m$, then

$$\hat{Q}(\xi) = \sum_{|\alpha| = |\beta| = m} c_{\alpha,\beta} \xi^\alpha(\beta)^\ast.$$

**Definition.** An element $Q$ of $\mathfrak{u}_m \otimes \mathfrak{u}_m$ is Hermitian elliptic (of degree $m$) if $\hat{Q}(\xi) > 0$ for all $\xi \neq 0$ in $\mathfrak{g}_c'$.

**Example.** If $A_j \in \mathfrak{u}_m$ and the polynomials $\{\sigma_m(A_j)\}$ have no common complex zero, except $\xi = 0$, then $Q = \sum A_j \otimes A_j$ is Hermitian elliptic.
Since \( u \) and \( \ast u \) are associative algebras over \( \mathbb{C} \), so is \( u \otimes \ast u \), where

\[(A \otimes \ast B) \cdot (C \otimes \ast D) = AC \otimes \ast BD.\]

If \( P \in u_k \otimes \ast u_k \) and \( Q \in u_m \otimes \ast u_m \), then \( PQ \in u_n \otimes \ast u_n \), \( n = k + m \).

The Hermitian symbols are related by

\[(PQ)^\ast = P^\ast Q^\ast \quad \text{(2.7)}\]

(pointwise product on the right side of (2.7)). Hence if \( P \) and \( Q \) are Hermitian elliptic of degrees \( k \) and \( m \), then \( PQ \) is Hermitian elliptic of degree \( k + m \).

The usefulness of the notion of Hermitian ellipticity in connection with a-priori estimates for forms \( Q^n \) comes from the following version of a result of D. Quillen:

**Lemma 2.1.** Suppose \( Q \in u_m \otimes \ast u_m \) is Hermitian elliptic of degree \( m \). For \( n \) a positive integer define the \( n \)-th prolongation \( Q_n \) (with respect to the basis \( \{X_j\} \) for \( g \)) by \( Q_n = Q \Gamma_n \), where

\[
\Gamma_n = \sum_{\gamma \vdash n} \frac{n!}{\gamma!} \: X(\gamma) \otimes \ast X(\gamma).
\]

Then if \( n \) is sufficiently large, there exist elements \( A_j \in u_{m+n} \) such that

\[
Q_n = \sum A_j \otimes \ast A_j \mod \left( \sum_{p+q < 2r} u_p \otimes \ast u_q \right), \quad r = m + n. \quad \text{(2.8)}
\]

Furthermore, the \( A_j \) form a linear basis for \( u_{m+n} \mod(u_{m+n-1}) \).

**Proof.** By (2.5) we have

\[
\Gamma'_n(\xi) = \sum_{\gamma \vdash n} \frac{n!}{\gamma!} \: \xi^\gamma(\xi^\gamma)^* = |\xi|^2n.
\]

Hence, by (2.7), \( Q_n(\xi) = |\xi|^2n Q(\xi) \). We can apply Corollary 1 of [19], which asserts that for \( n \) sufficiently large there exists a linear basis \( \{P_j\} \) for the space \( \mathcal{P}_{m+n}(g_{e'}) \) such that

\[
Q_n(\xi) = \sum |P_j(\xi)|^2.
\]

Choose \( A_j \in u_{m+n} \) such that \( \sigma_{m+n}(A_j) = P_j \). Then \( R = Q_n - \sum_j A_j \otimes \ast A_j \) satisfies \( R^r = 0, \: r = m + n \). Hence by the Poincaré–Birkhoff–Witt theorem, (2.8) and the last statement of the lemma hold. Q.E.D.

The Hermitian symbol of \( Q \in u_m \otimes \ast u_m \) can also be defined analytically, although we shall not need this alternate definition in the present paper.
For this purpose, we first obtain an expression for $Q^\pi$, a unitary representation, in terms of a sesquilinear form on germs of $\mathcal{H}(\pi)$-valued functions on $G$. Write $\mathcal{H} = \mathcal{H}(\pi)$, and let $C^\infty(e; \mathcal{H})$ denote the space of $\mathcal{H}$-valued $C^\infty$ functions defined on an open neighborhood of the identity element $e$ in $G$ (the neighborhood depending on the function). Let $\rho$ be the right regular representation of $g$ on this space:

$$\rho(X)f(g) = \left. \frac{d}{dt} f(g \cdot \exp tX) \right|_{t=0}.$$ 

Extend $\rho$ to a representation of $u$. If $f(g) = \pi(g) v = \tilde{v}(g)$, for $v \in \mathcal{H}^\infty(\pi)$, then $\rho(T)\tilde{v}(g) = (\tilde{\pi}(T)v)^\ast(g)$, when $T \in u$ (cf. [20]). In particular,

$$\rho(T)\tilde{v}(e) = \tilde{\pi}(T)v. \quad (2.9)$$

Given $C = A \otimes B \in u \otimes u$, define $\tilde{C}_{\mathcal{H}}$ to be the sesquilinear form on $C^\infty(e; \mathcal{H})$ given by

$$C_{\mathcal{H}}(f_1, f_2) = (\rho(A)f_1(e), \rho(B)f_2(e)) \quad (2.10)$$

($\langle \cdot, \cdot \rangle$ denoting the inner product in $\mathcal{H}$). The map $C \to \tilde{C}_{\mathcal{H}}$ extends to a linear map from $u \otimes u$ to sesquilinear forms on $C^\infty(e; \mathcal{H})$. By (2.9) and (2.10) we have

$$Q^\pi(u, v) = \tilde{Q}_{\mathcal{H}}(\tilde{u}, \tilde{v}), \quad (2.11)$$

for $Q \in u \otimes u$ and $u, v \in \mathcal{H}^\infty(\pi)$. When $\mathcal{H} = \mathbb{C}$ with $(z, w) = zw^\ast$, we simply write $\tilde{Q}_{\mathcal{H}} = \tilde{Q}$. If $\xi \in \mathfrak{g}^\prime$, define $f_s(x) = \exp\langle \xi, \log x \rangle$, for $x$ in a neighborhood of $e$ on which $\log: G \to \mathfrak{g}$ is defined.

**Proposition 2.1.** Let $Q \in u_m \otimes u_m$, and $\xi \in \mathfrak{g}_e$. Then the Hermitian symbol $\tilde{Q}$ (of degree $m$) satisfies

$$\tilde{Q}(\xi) = \lim_{s \to \infty} s^{-2m} \tilde{Q}(f_{s\xi}, f_{s\xi}). \quad (2.12)$$

**Proof.** If $f$ is a $C^\infty$ function on a neighborhood of $e$, then defining $X(t)$ as before, $t \in \mathbb{R}^d$, we have

$$\left( \frac{\partial}{\partial t} \right)^3 f(\exp X(t)) \bigg|_{t=0} = \rho(X(\alpha))f(e)$$

([11, p. 98]. We have incorporated the factor $\alpha!$ into our definition of $X(\alpha)$).
Thus $\rho(X(\alpha))f_{e}(e) = \xi^\alpha$. If $Q = \sum c_{\alpha,\beta}X(\alpha) \otimes X(\beta)$, with $|\alpha| \leq m$ and $|\beta| \leq m$, then it follows that

$$s^{-2m}Q(f_{s\alpha}, f_{s\beta}) = \sum_{|\alpha| = m} c_{\alpha,\beta} \xi^\alpha(\xi^\beta)^* + O(s^{-1}),$$

which gives (2.12).

**Q.E.D.**

### 3. Coercive Forms

Let $\pi$ be a unitary representation of $G$, and suppose $Q \in \mathfrak{u}_m \otimes \star \mathfrak{u}_m$, $m > 0$. Then by the Schwarz inequality one has an estimate

$$|Q^\pi(u, u)| \leq C \|u\|^2_m$$

holding for some constant $C$ and all $u \in \mathcal{H}^\pi(\pi)$.

**Definition.** $Q$ is $\pi$-coercive (of order $m$) if there exist constants $C_0 > 0$ and $\lambda_0$ such that

$$\Re Q^\pi(u, u) \geq C_0 \|u\|^2_m - \lambda_0 \|u\|^2_0$$

for all $u \in \mathcal{H}^\pi(\pi)$ (\(\Re\) denoting real part).

By estimate (1.3), an equivalent definition of coerciveness is that an estimate

$$\Re Q^\pi(u, u) \geq C_0 \|u\|^2_m - C_1 \|u\|^2_t$$

should hold for some $C_0 > 0$ and $t < m$.

If $W = \frac{1}{2}(Q + Q^\star)$, then $W$ is Hermitian-symmetric and $W^\pi = \Re Q^\pi$. Thus, to obtain criteria for coerciveness, it suffices to consider Hermitian $Q$.

**Our principal result** is the following theorem:

**Theorem 3.1.** Suppose $Q \in \mathfrak{u}_m \otimes \star \mathfrak{u}_m$ is Hermitian elliptic (of degree $m > 0$). Then $Q^\pi$ is $\pi$-coercive (of order $m$) for all unitary representations $\pi$.

Let $\{X\}$ be a basis for $\mathfrak{g}$, and let $Q_n$ be the $n$-th prolongation of $Q$ (with respect to this basis), as defined in Section 2. Then the form $Q_n^\pi$ is given by

$$Q_n^\pi(u, v) = \sum_{|\gamma| = n} \frac{n!}{\gamma!} \bar{Q}(X(\gamma)u, X(\gamma)v).$$
(When π is understood, we write \( \dot{\pi}(T)u = Tu \), for \( T \in \mathcal{H} \) and \( u \in \mathcal{H}^\pi(\pi) \). Indeed, if \( Q = \sum_{\alpha, \beta} c_{\alpha, \beta} X(\alpha) \otimes X(\beta) \), then

\[
Q_n = \sum_{|\gamma| = n} \sum_{\alpha, \beta} \frac{n!}{\gamma!} c_{\alpha, \beta} X(\alpha) X(\gamma) \otimes X(\beta) X(\gamma).
\]

By the definition of \( Q^n \) and \( Q_n^n \) we thus obtain relation (3.2). Notice that if \( Q \) is Hermitian then so is \( Q_n \), and the sesquilinear forms \( Q_n^n \) are then all Hermitian symmetric. Before proving theorem 3.1 we first show that coerciveness is stable under prolongation.

**Lemma 3.1.** \( Q \) is \( \pi \)-coercive (of order \( m > 0 \)) if and only if \( Q_n \) is \( \pi \)-coercive (of order \( m + n \)) for all positive integers \( n \).

**Proof of Lemma 3.1.** Since \( (Q^n)_n = (Q^n)^* \), it suffices to consider Hermitian \( Q \). (We do not need Hermitian ellipticity of \( Q \) for this lemma, however.) Let \( L = γ(Q) \in \mathcal{H}_m \) be the operator associated with \( Q \). Then \( Q^n(u, u) = (Lu, u) \), and hence by (3.2) we have

\[
Q^n_n(u, u) = (-1)^n \sum_{|γ| = n} \frac{n!}{γ!} (X(γ)LX(γ)u, u)
\]

where \( A \) is the closure of the operator \( \partial π(1 - A) \) and \( R \in \mathcal{H}_{2m+2n-1} \). Thus we can write

\[
Q^n_n(u, u) = (-1)^n \sum_{|γ| = n} \frac{n!}{γ!} (X(γ)LX(γ)u, u) = (La^n u, u) + (Ru, u),
\]

with \( B = A^{1/2} \) as in Section 1. Now if \( n \) is even, then \( [L, B^n]u = Tu, T \in \mathcal{H}_{2m+2n-1} \); so by Lemma 1.2 the last two terms in (3.3) are bounded by \( C(\|u\|_{m+n-(1/2)})^2 \). If \( n = 2k + 1 \) is odd, then

\[
[L, B^n]B^n = [L, A^k]B^{n+1} + B^{2k}[L, B]B^{2k-1}.
\]

Since \( \deg(L) = 2m \geq 2 \), Lemma 1.2 is again applicable, so that for any \( n \)

\[
|Q^n_n(u, u) - Q^n(B^n u, B^n u)| \leq C(\|u\|_{m+n-(1/2)})^2.
\]

Suppose now that \( Q \) is \( \pi \)-coercive. Then

\[
Q^n(B^n u, B^n u) \geq C_0 \|B^n u\|_m^2 - \lambda_0 \|B^n u\|^2_0 \geq C_0 \|u\|_{m+n}^2 - \lambda_0 \|u\|^2_n.
\]
Since $m > 0$, it follows from (3.1)' and (3.4) that $Q_n$ is $\pi$-coercive. Conversely, if $Q_n$ is $\pi$-coercive, then by (3.4) we have

$$Q^*(B^n u, B^n u) \geq C_0 \| u \|^2_{m+n} - C_1 \| u \|^2_{l+n},$$

with $t = m - \frac{1}{2}$. Replacing $u$ by $B^{-n} u$ and using (3.1)', we obtain the coerciveness of $Q$. Q.E.D.

Proof of Theorem 3.1. By Lemma 2.1 we can write, for sufficiently large $n$,

$$Q_n^*(u, u) = \sum_j \| A_j u \|^2 + (S u, u),$$

where $\{A_j\}$ is a basis for $u_{m+n} \bmod u_{m+n-1}$, and $S \in u_{2m+2n-1}$. By Lemmas 1.2 and 3.1, it suffices to show that the form $\sum_{\alpha} \| A_\alpha u \|^2$ is coercive (of order $m + n$). But if $\alpha$ is a multi-index with $|\alpha| = m + n$, then we can write $X(\alpha) = \sum a_j A_j + R$, where $a_j \in \mathbb{C}$ and $R \in u_{m+n-1}$. Thus by Schwarz' inequality,

$$|(X(\alpha) u, v)| \leq C \left( \sum \| A_\alpha u \|^2 + \| u \|^2 \right)^{1/2} \cdot \| v \|,$$

for any $u \in \mathcal{H}^\infty$, $v \in \mathcal{H}$, where $t = m + n - 1$. It follows that

$$\| X(\alpha) u \|^2 \leq C^2 \left( \sum \| A_\alpha u \|^2 + \| u \|^2 \right).$$

Q.E.D.

Remarks. If $Q$ is coercive for the regular representation of $G$ on $L^2(G)$, then by the converse to Gårding's inequality [1, Theorem 7.12] the symbol $\hat{Q}$ must be strongly elliptic, i.e., $\hat{Q}(\xi) > 0$ for $\xi$ real on $\mathfrak{g}$. Conversely, if $\hat{Q}$ is strongly elliptic and $G$ is compact or abelian, then Gårding's inequality [1, Theorem 7.6] and standard facts from representation theory can be used to show that $Q$ is $\pi$-coercive for all unitary representations $\pi$. In the non-compact, nonabelian case we do not know if strong ellipticity of $\hat{Q}$ suffices to force $\pi$-coerciveness of $Q$.

For irreducible $\pi$, $Q$ may be $\pi$-coercive without having $\hat{Q}$ strongly elliptic. If $Q = Q_0 + Q_1$, where $Q_0$ is Hermitian elliptic of degree $m$ and

$$| Q^*(u, u) | \leq C \| u \|_t$$

for some $t < m$, then $Q^*$ will be $\pi$-coercive. For example, let $G$ be simple, noncompact, and $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ a Cartan decomposition. If $\{X_j\}$, $\{Y_j\}$ are orthonormal bases for $\mathfrak{t}$ and $\mathfrak{p}$ respectively (with respect to the Killing form), then $Q = \sum X_j \otimes * X_j$ is not strongly elliptic ($\hat{Q}$ vanishes on $\mathfrak{t}^\perp$). However, $\omega = \sum Y_j \otimes * Y_j - Q$ is the Casimir element, while $Q_0 = \frac{1}{2} \omega + Q$ is Hermitian elliptic. In an irreducible representation $\pi$, $\omega^*(u, u) = \lambda \| u \|^2$.  


so \( Q \) is \( \pi \)-coercive (cf. [16, Section 2]). In Section 6 we will give another example of this phenomenon, based on the following criterion for \( \pi \)-coerciveness:

**Corollary 3.1.** Let \( Q \in \mathfrak{u}_m \otimes \mathfrak{u}_m^* \) and suppose that for some basis \( \{Z_j\}, 1 \leq j \leq d, \) of \( g_c \) and \( C_0 > 0 \), the form \( Q^\pi \) satisfies

\[
\mathcal{H}Q^\pi(u, u) \geq C_0 \sum_{j=1}^{d} \|Z_j^\pi u\|^2 - C_1 \|u\|^2,
\]

where \( t < m \). Then \( Q \) is \( \pi \)-coercive.

**Proof.** Let \( Q_0 = \sum Z_j^m \otimes Z_j^m \). Then \( Q_0(\xi) = \sum |Z_j(\xi)|^{2m} > 0 \) if \( \xi \neq 0 \), since \( \{Z_j\} \) is a basis for \( g_c \). Hence, \( Q_0 \) is \( \pi \)-coercive by Theorem 3.1, so that (3.5) is equivalent to (3.1)’. Q.E.D.

**Remark.** As a particular case of Corollary 3.1, we have an estimate

\[
\|X_{j_1} \cdots X_{j_m} u\| \leq C \left( \sum_{k=1}^{d} \|X_k^\pi u\| + \|u\| \right)
\]

holding for \( \{X_j\} \) a basis of \( g, u \in \mathcal{H}^{\infty}(\pi) \). If \( \pi \) is a continuous representation on a nonreflexive Banach space, then (3.6) need not hold. For a counterexample, with \( G = \mathbb{R}^2 \) and \( \pi \) the regular representation on \( L^1(G) \), see [17]. We do not know if (3.6) holds for representations on reflexive spaces (e.g., for the regular representation on \( L^p(G), 1 < p < \infty, p \neq 2 \)).

## 4. Operators Defined By Coercive Forms

Let \( Q \in \mathfrak{u}_m \otimes \mathfrak{u}_m^* \) and suppose \( \pi \) is a unitary representation of \( G \). We want to study the operators \( \partial \pi(L) \) and \( \partial \pi(L^*), L = \gamma(Q) \), under the assumption that \( Q \) is \( \pi \)-coercive. (Recall that \( Q^\pi(u, v) = (\partial \pi(L)u, v) \); so the \( \pi \)-coerciveness of \( Q \) means that Gårding’s inequality holds for \( \partial \pi(L) \).) This class of operators is stable under \( L \to L^* \) and perturbation by elements \( R \) of \( \mathfrak{u}_{m-1} \), by virtue of Lemma 1.2 and the fact that \( R = \gamma(Q_1) \) for some \( Q_1 \in \mathfrak{u}_m \otimes \mathfrak{u}_m^* \). We shall use the standard “Hilbert space approach” to elliptic regularity as developed, e.g., in [2]. We first prove three lemmas, adapted from the arguments in [2, pp. 173–175]. In these lemmas we assume \( L = \gamma(Q) \) as above, with \( Q \) \( \pi \)-coercive of order \( m \).

**Lemma 4.1.** For every \( t \in \mathbb{R} \) there exists a \( \lambda_t > 0 \) and \( C_t > 0 \) such that

\[
C_t \|u\|_{l+m} \leq \|\partial \pi(L + \lambda)u\|_{l-m}
\]

whenever \( \mathcal{H}(\lambda) > \lambda_t \) and \( u \in \mathcal{H}^{\infty}(\pi) \).
Proof. By the quadratic interpolation theorem it suffices to prove (4.1) for $t \geq 0$. From the proof of Lemma 3.1 we have

$$\mathcal{H}(B^t \partial \pi(L + \lambda)u, B^t u) \geq C_\|u\|_{L^t+m}^2 + \mathcal{H}(\lambda)\|u\|_t^2 - \lambda_t\|u\|_0$$

(4.2)

for some $C_\| > 0, \lambda_t \geq 0$. If $\mathcal{H}(\lambda) \geq \lambda_t$, the right-hand side of (4.2) is bounded from below by $C_\|u\|_{L^t+m}^2$. By the Schwarz inequality, the left-hand side of (4.2) is bounded from above by $\| \partial \pi(L + \lambda)u \|_t \|u\|_{L^t+m}$. Dividing by $\|u\|_{L^t+m}$, we obtain (4.1) for $t$. The proof of Lemma 3.1 also gives an estimate

$$\mathcal{H}(\partial \pi(L + \lambda)B^{-2t}u, u) \geq C_\|\|u\|_{L^t+m}$$

for some $C_\| > 0$, provided $\mathcal{H}(\lambda) \geq \lambda_t$. Replacing $u$ by $B^{-2t}u$, we have

$$\mathcal{H}(B^{-m-t} \partial \pi(L + \lambda)u, B^{-m-t}u) \geq C_\|\|u\|_{L^t+m}$$

which yields (4.1) for $-t$ by virtue of Schwarz' inequality again. Q.E.D.

**Lemma 4.2.** Let $t \in \mathbb{R}$. Then if $\mathcal{H}(\lambda)$ is sufficiently large, $\partial \pi(L + \lambda)|_{\mathcal{H}^t}$ is an isomorphism onto $\mathcal{H}^{t-2m}$.

Proof. Since $L \in \mathcal{H}_{2m}$, we have $\| \partial \pi(L + \lambda)u \|_{t-2m} \leq C_\|u\|_t$. By Lemma 4.1 and the density of $\mathcal{H}^\alpha$ in $\mathcal{H}^t$, we obtain the estimates

$$C_\|u\|_t \leq \| \partial \pi(L + \lambda)u \|_{t-2m} \leq C_\|u\|_t$$

(4.3)

when $\mathcal{H}(\lambda)$ is sufficiently large, for all $u \in \mathcal{H}^t$. This implies that

$$\partial \pi(L + \lambda) : \mathcal{H}^t \rightarrow \mathcal{H}^{t-2m}$$

injectively with closed range. If $v \in \mathcal{H}^{2m-t}$ is orthogonal to $\partial \pi(L + \lambda)(\mathcal{H}^t)$, then $(\partial \pi(L^* + \lambda^*)v, u) = (v, \partial \pi(L + \lambda)u) = 0$ for all $u \in \mathcal{H}^{2m-t}$. Hence $\partial \pi(L^* + \lambda^*)v = 0$. If $\mathcal{H}(\lambda)$ is so large that (4.3) holds for $L^* + \lambda^*$, then $v = 0$. Thus, $\partial \pi(L + \lambda)\mathcal{H}^t = \mathcal{H}^{t-2m}$ in this case. Q.E.D.

**Lemma 4.3.** If $u \in \mathcal{H}^{-\alpha}$ and $\partial \pi(L)u \in \mathcal{H}^t$, then $u \in \mathcal{H}^{t+2m}$.

Proof. Let $v = \partial \pi(L)u$. Assume that $u \in \mathcal{H}^s$ for some $s \leq t$. Then $v + \lambda u = \partial \pi(L + \lambda)u \in \mathcal{H}^s$. Hence taking $\lambda \geq 0$ sufficiently large, we have by Lemma 4.2 that $u \in \mathcal{H}^{s_1}$, $s_1 = s + 2m$. If $s_1 < t$, we may repeat this argument until we get $u \in \mathcal{H}^s$ for some $s \geq t$. A final application of lemma 4.2 then gives $u \in \mathcal{H}^{t+2m}$. Q.E.D.

We now state and prove the regularity theorem for operators defined by coercive forms. If $T$ is an operator on a Banach space $\mathcal{H}$, then $\mathcal{D}(T)$ denotes the domain of $T$ as a normed space with the graph norm $\| x \| + \| Tx \|$. 


Theorem 4.1. Let \( \pi \) be a unitary representation of \( G \), and suppose \( Q \in u_m \otimes u_m^* \) is \( \pi \)-coercive. Let \( L = \gamma(Q) \in u_{2m} \), and let \( A \) be the closure of the operator \( \partial \pi(L) \) in \( \mathcal{H}(\pi) \). Then

(i) \( A = \partial \pi(L^*)^* \) (Hilbert space adjoint);

(ii) \( (A + \lambda)^{-1} \) exists as a bounded operator on \( \mathcal{H}(\pi) \) for \( \lambda \) in a half-plane \( \mathcal{H}(\lambda) > \lambda_0 \), and satisfies

\[
\|(A + \lambda)^{-1}\| \leq [\mathcal{H}(\lambda - \lambda_0)]^{-1};
\]

(iii) \( \mathcal{Q}(A^n) = \mathcal{H}^{2m}(\pi) \) with equivalent norms.

Corollary 4.1. Suppose \( L = L^* \) (as an element of \( u \)). Then \( A \) is self-adjoint and bounded below. Pick \( \lambda \geq 0 \) so that \( A + \lambda \geq I \), and set \( \Gamma = (A + \lambda)^{1/2m} \). Then for all \( t \geq 0 \),

\[
\mathcal{Q}(\Gamma^t) = \mathcal{H}^t(\pi)
\]

with equivalent norms.

Proof of Theorem 4.1. Since \( \partial \pi(L) \subseteq \partial \pi(L^*)^* \) by relation (1.6), we have \( A \subseteq \partial \pi(L^*)^* \). Suppose \( \{u, v\} \in \mathcal{H} \oplus \mathcal{H} \) is orthogonal to the graph of \( A \). Then for all \( w \in \mathcal{H} \),

\[
0 = \langle u, w \rangle + \langle v, \partial \pi(L)w \rangle = \langle u, w \rangle + \langle \partial \pi(L^*)v, w \rangle.
\]

Hence, \( \partial \pi(L^*)v = -u \). Since \( u \in \mathcal{H} \), we have by Lemma 4.3 that \( v \in \mathcal{H}^{2m} \).

Since \( L^* \) is of degree \( 2m \), \( v \) is thus in the domain of the closure, call it \( \Lambda^* \), of \( \partial \pi(L^*) \), and \( \Lambda v = -u \). If now \( w \in \mathcal{Q}(\partial \pi(L^*)) \), we have

\[
\langle u, w \rangle + \langle v, \partial \pi(L^*)^*w \rangle = \langle u, w \rangle + \langle A^*v, w \rangle = 0.
\]

Hence \( \{u, v\} \) is orthogonal to the graph of \( \partial \pi(L^*)^* \). This proves (i).

As a consequence of (i), we obtain that \( A \) is the restriction of \( \partial \pi(L) \) to \( \mathcal{Q} = \{u \in \mathcal{H}, \partial \pi(L)u \in \mathcal{H} \} \). Indeed, by the definition of Hilbert-space adjoint, \( \partial \pi(L)|_{\mathcal{Q}} = \partial \pi(L^*)^* \). But by Lemma 4.3, \( \mathcal{Q} = \mathcal{H}^{2m}(\pi) \). Statement (ii) is now an immediate consequence of Lemma 4.2 and Schwarz' inequality applied to the estimate

\[
\mathcal{H}(\partial \pi(L + \lambda)u, u) \geq \mathcal{H}(\lambda - \lambda_0)\|u\|^2,
\]

which holds for some \( \lambda_0 \geq 0 \) by the coerciveness assumption on \( Q \).
Since $L^n$ is of degree $2mn$, the inclusion $\mathcal{H}^{2mn}(\pi) \subseteq \mathcal{D}(A^n)$ holds and is continuous. In case $n = 1$, we obtained the opposite inclusion in the preceding paragraph. Suppose inductively that $\mathcal{D}(A^n) \subseteq \mathcal{H}^{2mn}(\pi)$. If $u \in \mathcal{D}(A^{n+1})$, then $Au \in \mathcal{H}^{2mn}(\pi)$. Since $Au = \delta\pi(L)u$, we obtain from Lemma 4.3 that $u \in \mathcal{H}^{2mn+1}(\pi)$. By the closed graph theorem (or by Lemma 4.1), the norm on the Banach space $[\mathcal{D}(A^n)]$ is equivalent to $\| \cdot \|_{mn}$.

Proof of Corollary 4.1. The only point to verify is Eq. (4.4). Since $\mathcal{H}^q(\pi) = [\mathcal{D}(B')]$, and (4.4) holds for $t = 2mn$, we obtain (4.4) for all $t \geq 0$ by the monotonicity theorem of Loewner–Heinz [10, Satz 3].

Q.E.D.

5. Applications

Our first application of the results of Sections 3 and 4 is the following sharpening of a result of Nelson and Stinespring [16, Corollary 2.4]:

**Theorem 5.1.** Let $\pi$ be a unitary representation of $G$, and suppose $Q \in u_m \otimes \ast u_m$ is Hermitian symmetric and $\pi$-coercive (e.g., assume $Q$ is Hermitian elliptic). Let $L = \gamma(Q)$, and assume $T \in \mathfrak{u}(g)$, commutes with $L$. Then,

(i) $\mathcal{D}(T)$ has a dense set of bounded vectors;

(ii) $\mathcal{D}(T^*)^\ast$ is the closure in $\mathcal{H}$ of $\mathcal{D}(T)$.

**Remark.** A bounded vector for an unbounded operator $S$ is a vector $v \in \cap_n \mathcal{D}(S^n)$ for which there exists a constant $C$ with $\| S^n v \| \leq C^n$.

**Proof of Theorem 5.1.** Replace $L$ by $L + \lambda$ to achieve $\mathcal{D}(L) \supseteq 1$. Let $\Lambda = \mathcal{D}(L)^\ast$. Then $\Lambda$ is a self-adjoint operator and $\mathcal{D}(\Lambda^n) = \mathcal{H}^{2mn}(\pi)$ by Theorem 4.1. If $n_0$ is chosen so that $2mn_0 \geq \text{order}(T)$, one has an estimate

$$\| Tu \| \leq M \| \Lambda^{n_0} u \|, \quad u \in \mathcal{H}^\infty(\pi).$$

Since $\Lambda$ is self-adjoint and $\cap_n \mathcal{D}(\Lambda^n) = \mathcal{H}^\infty(\pi)$, it follows by spectral theory that $\Lambda$ has a dense set of bounded vectors in $\mathcal{H}^\infty(\pi)$. But if $u \in \mathcal{H}^\infty$ and $\| \Lambda^n u \| \leq C^n$, then $\| T^n u \| \leq M \| \Lambda^{n_0} T^{n-1} u \| = M \| T^{n-1} \Lambda^{n_0} u \|$. Iterating this estimate, we obtain $\| T^n u \| \leq M \| \Lambda^{n_0} \| \| M^n \| \| \Lambda^{n_0} \| = (MC^n)^n$, giving (i).

To prove (ii), we observe that $T^* T$ commutes with $L$, since $L = L^*$. Hence by (i) and a theorem of Nelson [15], the operator $\mathcal{D}(T^* T)$ is essentially self-adjoint. By Lemma 2.3 of [16], this implies (ii).

**Remark.** For operators $L$ that are elliptic on $G$ (i.e., $\hat{Q}(\xi) \neq 0$ for $\xi \in \mathfrak{g}$'), Theorem 5.1 can in fact be deduced from the results of [16]. Indeed, elliptic regularity gives $\Lambda$ self-adjoint and $\mathcal{H}^\infty(\pi) = \cap_n \mathcal{D}(\Lambda^n)$ in this case, and the argument just given may be used.
As the next application of our coerciveness estimates we obtain a "filtered" version of Theorem 1.1 of [7] in the case of unitary representations of $G$. This result can be used to facilitate giving an explicit description of elements of $\mathcal{H}^m(\pi)$ in concrete representations, just as Theorem 1.1 of [7] was used for the space $\mathcal{H}^\infty(\pi)$.

**Theorem 5.2.** Let $\{X_j\}, 1 \leq j \leq d$, be a basis for $\mathfrak{g}$, and let $G_j$ be the one-parameter subgroup of $G$ generated by $X_j$. Suppose that $\pi$ is a unitary representation of $G$, and set $\tau_j = \pi|_{G_j}$. Then, for every positive integer $m$,

$$\mathcal{H}^m(\pi) = \bigcap_{j=1}^d \mathcal{H}^m(\tau_j).$$

**Proof.** The left-hand side of (5.1) is of course contained in the right-hand side. For the converse we consider the Hermitian element $Q = \sum_j X_j^m \otimes X_j^m$ of $\mathfrak{u}_m \otimes \mathfrak{u}_m$, and the associated operator $L = \sum (-1)^m X_j^{2m}$. $Q$ is $\pi$-coercive by Corollary 3.1; so by Corollary 4.1 the self-adjoint operator $\Lambda = \partial \pi(L)^* \pi$ has domain $\mathcal{H}^{2m}(\pi)$. Since $\Lambda \geq 0$, the operator $(\Lambda + 1)^{1/2}$ has domain $\mathcal{H}^m(\pi)$. On the other hand, $\mathcal{H}^m(\pi_j) = \mathcal{D}(d\pi(X_j)^m)$; so it suffices to show that if $u \in \bigcap_{j=1}^d \mathcal{H}^m(\tau_j)$ then $u \in \mathcal{D}((\Lambda + 1)^{1/2})$.

Now $(\Lambda + 1)^{1/2}$ is essentially self-adjoint on $\mathcal{H}^\infty(\pi)$, since this is the space $C^\infty$-vectors for this operator by Corollary 4.1. Thus, $u \in \mathcal{D}((\Lambda + 1)^{1/2})$ if and only if there is a constant $C > 0$ such that

$$|\langle (\Lambda + 1)^{1/2} v, u \rangle| \leq C \| v \|$$

for all $v \in \mathcal{H}^\infty(\pi)$. Replacing $v$ by $(\Lambda + 1)^{1/2} v$ and using Corollary 4.1, we see that (5.2) is equivalent to

$$\langle Lv, u \rangle \leq C \| v \|_m$$

holding for all $v \in \mathcal{H}^\infty(\pi)$. But now if $u \in \bigcap_{j} \mathcal{H}^m(\tau_j)$, then

$$(Lv, u) = \sum (\partial \pi(X_j)^m) v, d\pi(X_j)^m u)$$

for all $v \in \mathcal{H}^\infty(\pi)$. By Schwarz' inequality we obtain (5.3). Q.E.D.

We can use Theorem 5.2 to obtain regularity properties for a vector from regularity properties of the associated representative functions on $G$:

**Corollary 5.1.** (Hypotheses of Theorem 5.2). Suppose $u \in \mathcal{H}(\pi)$ is such that for every $v \in \mathcal{H}(\pi)$ and $1 \leq j \leq d$ the function

$$\varphi(t) = (\pi(\exp tX_j)u, v), \quad t \in \mathbb{R},$$

is of class $C^{m-1}$, with $\varphi^{(m-1)}(t)$ Lipschitz-continuous at $t = 0$. Then $u \in \mathcal{H}^m(\pi)$. 

Proof. Consider first the case \( m = 1 \). By the uniform boundedness principle, the \( \mathcal{H} \)-valued function \( t \to \pi(\exp tX_j)u \) satisfies a Lipschitz condition at \( t = 0 \). It follows that \( u \in \mathcal{D}(d\pi(X_j)) \). Indeed, \( i\pi(X_j) \) is a self-adjoint operator, and if we take it as multiplication by a function \( \xi \) on \( L^2(\Omega) \), then by the Lipschitz condition and Fatou’s lemma we obtain

\[
\int \lim_{t \to 0} t^{-2} | e^{it\xi} - 1 |^2 | u |^2 < \infty.
\]

Since \( \lim_{t \to 0} t^{-2} | e^{it\xi} - 1 |^2 = \xi^2 \), this implies that \( u \in \mathcal{D}(d\pi(X_j)) \). (We could have also invoked a general theorem of Butzer to obtain this conclusion, cf. [4].) Iteration of this argument and Theorem 5.2 complete the proof.

As the last application we have the following generalization of Theorem 2 of [6]:

**Theorem 5.3.** Let \( Q \in u_m \otimes u_m^\ast \) be Hermitian symmetric and \( L = \gamma(Q) \) the associated Hermitian element of \( u_{2m}^\prime \). Suppose \( \pi \) is a unitary representation of \( G \), such that \( Q \) is \( \pi \)-coercive. Set \( \Lambda = \text{closure in } \mathcal{H}(\pi) \) of \( d\pi(L) \), and let \( \mathcal{H}^\omega(\pi) \) be the space of analytic vectors for \( \pi \). Then,

\[
v \in \mathcal{H}^\omega(\pi) \iff v \in \bigcap_n \mathcal{D}(\Lambda^n) \quad \text{and} \quad \| \Lambda^nv \| \leq C^n(2mn)!
\]

for some constant \( C \) (depending on \( v \)) and all \( n \).

We may also state this theorem in the following form, which displays it as a generalization of a theorem of Paley and Wiener (cf. Introduction to [7]):

**Corollary 5.2.** (Hypotheses of Theorem 5.3.) Let \( T : \mathcal{H}(\pi) \to L^2(\Omega, d\mu) \) be a unitary map such that \( T \Lambda T^{-1} \) is multiplication by a measurable function \( \varphi \) on \( \Omega \). Write \( Tu = \hat{u} \). Then \( u \in \mathcal{H}^\omega(\pi) \iff \hat{u} \) satisfies

\[
\int_\Omega | \hat{u}(\omega)|^2 \exp(\| \varphi(\omega) \|^{1/2m}) \, d\mu(\omega) < \infty \tag{5.4}
\]

for some \( r > 0 \).

**Proof of Theorem 5.3.** By Theorem 4.1 the space \( \bigcap_n \mathcal{D}(\Lambda^n) \) coincides with \( \mathcal{H}^\omega(\pi) \). Pick \( \lambda > 0 \) sufficiently large so that \( \Lambda + \lambda I \), and set \( \Gamma = (\Lambda + \lambda)^{1/2m} \). By Corollary 4.1 we have

\[
\| u \|_1 \leq C \| \Gamma u \|
\]

\[
\| u \|_{2m} \leq C \| \Gamma^{2m} u \|
\]
if \( u \in \mathcal{H}^{\omega}(\pi) \). On the other hand, since \( L \in \mathfrak{u}_{2n} \), we have an estimate

\[
\|[\text{ad} X_{i_1}, \ldots, \text{ad} X_{i_n}(L)]u\| < C^{n} \| u \|_{2n}
\]

for \( u \in \mathcal{H}^{\omega}(\pi) \) and \( \{X_i\} \) a basis for \( \mathfrak{g} \). It follows from Theorem 1’ and the proof of Theorem 2 of [6] that \( u \in \mathcal{H}^{\omega}(\pi) \Leftrightarrow u \) is an analytic vector for the operator \( \Gamma \). By elementary estimates this is equivalent to \( \|A^nu\| \leq C^n(2mn)! \)

\( \Box \).

Proof of Corollary 5.2. Such a unitary map \( \mathcal{F} \) exists by the spectral theorem. By Proposition 4.1 of [7] and elementary estimates, condition (5.4) is equivalent to \( u \) being an analytic vector for \( \Gamma \) (\( \Gamma \) as in preceding proof), which in turn has been shown equivalent to \( u \in \mathcal{H}^{\omega}(\pi) \).

Q.E.D.

6. An Example

To illustrate the kind of concrete analytical information that can be obtained from our estimates, we consider the following situation (generalizing Section 6 of [7]):

Let \( g_d \), \( d \geq 1 \), be the \((d + 2)\)-dimensional real nilpotent Lie algebra with basis \( X, Y_j \), \( j = 0, 1, \ldots, d \), satisfying the commutation relations

\[
[X, Y_j] = Y_{j-1}, \quad [Y_j, Y_k] = 0.
\]

(Set \( Y_{-1} = 0 \).) Let \( G_d \) be the corresponding simply-connected group (\( G_1 \) is the so-called "Heisenberg group"). An infinite-dimensional irreducible representation \( \pi \) of \( G_d \) may be realized on \( L^2(\mathbb{R}^d) \), with \( \partial \pi(X) : = d/dx \), \( \partial \pi(Y_j) \) = multiplication by \( iP_j(x) \), all acting on the Schwartz space \( \mathcal{S} \).

Here \( P_j \) is a real polynomial satisfying \( P_j' = P_{j-1} \), \( P_0(x) = \sigma \), with some \( P_j \neq 0 \) [13, Section 9]. We may restrict our attention to representations for which \( \sigma \neq 0 \), since otherwise we can pass to a representation of \( g_d/(\mathfrak{y}_0) \) \( \simeq g_{d-1} \).

We shall call such a representation nondegenerate. In this case \( P_d(x) = \sigma x^d/d! + \cdots \) is of degree exactly \( d \). Let \( S_{a, \beta}^{\delta} \subset \mathcal{S} \) be the space defined by Gel’fand and Shilov in [5], and as before let \( \mathcal{H}^{\omega}(\pi) \) be the space of analytic vectors for \( \pi \).

Lemma 6.1. Let \( \pi \) be a nondegenerate irreducible unitary representation of \( G_d \), realized as above. Then

\[
\mathcal{H}^{\omega}(\pi) = S_{1/d}^{\delta}. \quad (6.1)
\]

Remarks. 1. A function \( f \in S_{1/d}^{\delta} \) if and only if \( f \) is the restriction to \( \mathbb{R} \) of
a function holomorphic in some strip $|\mathcal{I}(z)| < r$ which satisfies the growth estimate

$$|f(x + iy)| < C \exp(-a|x|^d)$$

(6.2)

for some $r > 0, a > 0$ ($r, a, C$ depending on $f$), and $|y| < r$ (cf. [5]).

2. The case $d = 1$ was treated in [7, Theorem 6.2] in more detail. (The statement of this theorem in [7] has a misprint. It should read:

$$\mathcal{H}^a_1(\pi^\lambda) \supseteq \mathcal{H}^a_1 \supseteq \mathcal{H}^a_2(\pi^\lambda).$$

Proof of Lemma 6.1. Let $P = d/dx, Q = $ multiplication by $x$, acting on $\mathcal{S}$. By Lemma 5.1 of [8], the space $S^{1,\lambda}_{1,0}$ consists of all $f \in \mathcal{S}$ for which there exists a constant $C$ with

$$\|Q^r P^s f\| \leq C^{r+s} (d/d)^{s} f$$

(6.3)

$r, s = 1, 2, 3, \ldots$ ($C$ depending on $f$).

On the other hand, since $\pi$ is nondegenerate we can choose a basis $\{X_j\}$ for $g_d$ such that $\partial \pi(X_j) = iQ^j, 0 \leq j \leq d,$ and $\partial \pi(X_{d+1}) = i\partial$. By using the coordinate system $\exp t_1X_1 \cdots \exp t_{d+1}X_{d+1}$ for $G_d$, we see that $f \in \mathcal{H}^a(\pi)$ if and only if $f \in \mathcal{S}$ and there exists a constant $C$ with

$$\|Q^r P^s f\| \leq C^{r+s} r_1 \cdots r_{d+1} s!$$

(6.4)

Here $r = \sum_{k=0}^{d} kr_k, n = \sum_{k=0}^{d} r_k,$, and (6.4) is to hold for $r_k, s = 0, 1, 2, \ldots$. Elementary estimates show that (6.3) and (6.4) are equivalent, modulo a change in $C$.

We now apply the foregoing theory to obtain the following analysis of the spectra and eigenfunction expansions for certain ordinary differential operators with polynomial coefficients:

**Theorem 6.1.** Let $L$ be a formally self-adjoint differential operator with polynomial coefficients on $L^2(\mathbb{R})$ of the form

$$L = (-1)^m (d/dx)^{2m} + c_0 x^{2m\alpha} + R,$$

where $c_0 > 0$, $m$ is a positive integer, and

$$R = \sum c_{jk} x^j (d/dx)^k \quad ((j/d) + k < 2m).$$

Then, $L$ is essentially self-adjoint on the Schwartz space $\mathcal{S}$, and the closure $\Lambda$ of $L$ has these properties:

(i) $\cap_n \mathcal{D}(\Lambda^n) = \mathcal{S}$;

(ii) $\Lambda$ is bounded from below and the resolvent of $\Lambda$ is of Hilbert–Schmidt class for $m = 1$, and of trace class for $m \geq 2$. 


(iii) Let \( \{ \varphi_k \} \) be an orthonormal basis of eigenfunctions for \( \Lambda \), with corresponding eigenvalues \( \{ \lambda_k \} \). If \( f \in L^2(\mathbb{R}) \) and \( \hat{f}(k) = (f, \varphi_k) \), then

\[
\hat{f} \in S^1_{1/2} \Rightarrow \hat{f}(k) = O(\exp -r|\lambda_k|^{1/2m})
\]

for some \( r > 0 \).

Remarks. 1. For the case \( m = d = c_0 = 1, R = 0 \), the functions \( \{ \varphi_k \} \) are the Hermite functions, and (iii) was first proved by Hille (cf. Section 6 of [7]).

2. The eigenfunctions \( \{ \varphi_k \} \) are of course entire functions, since \( L \) has no finite complex singularities. It would be of interest to know if \( \varphi_k \) is an entire vector for the representation \( \pi \) of \( G_d \) (in the sense of [7]). By Lemma 4.3 of [8], this will be the case, e.g., if \( \varphi_k \) is in the space \( S^{0}_{a, d} \) of Gel'fand–Shilov for some \( a < 1 \) (This space is nontrivial provided \( a > d/(d + 1) \)). In case \( m = d = 1 \) and \( R = 0 \), \( \varphi_k \in S^{1/2}_{1/2} \) by the behavior in the complex domain of the Hermite functions. In the case \( m = 1 \) and \( d > 1 \), the asymptotic estimates of [12] might be useful for establishing a similar result.

Proof of Theorem 6.1. Let the Lie algebra \( \mathcal{G}_d \) and representation \( \pi \) be as above. We first observe that \( L \) is of order \( 2m \) relative to \( \mathcal{G}_d \). Indeed, if \( j, k \) are non-negative integers with \( (j/d) + k < 2m \), write \( j = ld + r, 0 \leq r < d \).

Then \( l + k \leq 2m - 1 \), and \( Q^jP_k = Q^r(Q^{d})^lP^k \) is hence of order \( 2m \) relative to \( \mathcal{G}_d \), since \( Q^r, Q^d, P \in \partial \pi(\mathcal{G}_d) \). The "leading term"

\[
L_0 = (-1)^{m}(d/\mathcal{d})^{2m} + c_0 x^{2md}
\]

is obviously of degree \( 2m \) relative to \( \mathcal{G}_d \). We have

\[
(L_0f, f) = \|P^m f\|^2 + c_0 \|Q^{md} f\|^2 \tag{6.5}
\]

if \( f \in \mathcal{F} \). By Corollary 3.1 the form (6.5) is \( \pi \)-coercive of degree \( m \). To show that \( L \) itself is associated with a \( \pi \)-coercive form of degree \( m \), it thus suffices to have an estimate

\[
|\langle R, f, f \rangle | \leq C \|f\|_s^2, \quad f \in \mathcal{F}, \tag{6.6}
\]

for some \( s < m \), where \( \| \cdot \|_s \) is the norm on \( \mathcal{H}^s(\pi) \).

To establish (6.6) we first prove that

\[
\|Q^lP^k f\| \leq C \|f\|_r, \tag{6.7}
\]

where \( r = (j/d) + k \). It suffices to consider the case \( 0 < j < d \), since \( Q^{d}P^k \) is of order \( l + k \) relative to \( \mathcal{G}_d \), for integers \( l, k \).

Now if \( f \in \mathcal{F} \), then \( \|Q^l\| = \|Q^l f\| \), where \( |Q| \) is the operator of multiplication by \( |x| \). Furthermore, \( \mathcal{D}(|Q|) = \mathcal{D}(d\pi(X_d)) \supseteq \mathcal{D}(B) \), where
$B$ is the operator defining the scale $\mathcal{H}^\ell(\pi)$. Hence by the monotonicity theorem of Loewner–Heinz [10, Satz 3], one has
\[
\| Q^j f \| = \| Q^d |Q^j|^{1/d} \| \leq C \| B^{i/d} f \|,
\]
$0 \leq j \leq d$. Together with (1.4) this yields (6.7).

Next observe that the commutation relation $[P, Q] = -i$ gives, by the usual commutator formula, the relation
\[
P^k Q^j = \sum_{l} \binom{k}{l} (-i)^l Q^{j-l} P^{k-l}, \tag{6.8}
\]
with $\binom{k}{l} = k! j! ((k - l)! (j - l)! l!)$. Hence if $S = Q^j P^k$ and $S' = P^k Q^j$, then by (6.7) and (6.8) the hypotheses of Lemma 1.1 are satisfied, with $\alpha = 0$, $\beta = (j/d) + k$, and $\gamma = -r$. Taking $t = \frac{1}{2}$ in this lemma, we obtain the estimate
\[
|(S f, f)| < C \| f \|_{1/2}^2,
\]
for $f \in \mathcal{F}$. Since $R$ is a sum of operators of the form $cS$, with $r < 2m$, we obtain estimate (6.6).

We may now apply the conclusions of Corollary 4.1 and Theorem 5.3 to the operator $A$.

Since $\mathcal{F} = \mathcal{H}^\infty(\pi)$, statement (i) and the first part of statement (ii) follow immediately. To obtain the rest of (ii), we compare $A$ with a “known” operator, namely the closure $H$ of the operator $P^2 + Q^2 + 1$. This operator is self-adjoint, and $H^{-1}$ is of Hilbert–Schmidt class ($H$ has simple point spectrum with eigenvalues $\lambda_n = 2n + 2$, $n = 0, 1, ...$). On the other hand, $H$ is of order 2 relative to the scale $\mathcal{H}^\ell(\pi)$, and $\mathcal{D}(H^m) \supseteq \mathcal{H}^{2m}(\pi)$. If $\lambda > 0$ is so large that $(A + \lambda)^{-1}$ exists, then it follows from Corollary 4.1 that $H^m(A + \lambda)^{-1} \equiv T$ is a bounded operator on $L^2(\mathbb{R})$. Hence we can factor $(A + \lambda)^{-1}$ as $H^{-m} T$. Together with the resolvent equation $(A + \mu)^{-1} = (A + \lambda)^{-1} + (\lambda - \mu)(A + \lambda)^{-1}(A + \mu)^{-1}$ this establishes (ii), since $H^{-m}$ is trace class for $m > 1$, and the Hilbert–Schmidt and trace classes are ideals in $\mathcal{B}(\mathcal{H})$.

To establish (iii), it suffices by Corollary 5.2 and Lemma 6.1 to show that
\[
\sum_k |f(k)|^2 \exp(a|\lambda_k|^{1/2m}) < \infty \tag{6.9}
\]
for some $a > 0$ provided that $f(k) = O(\exp - r|\lambda_k|^{1/2m})$ for some $r > 0$. (The converse is trivially true, with $a = 2r$.) But by (ii), the series
\[
\sum_k (1 + |\lambda_k|^{1/2m})^{-4m} < \infty.
\]

Hence (6.9) holds for any $a < 2r$. Q.E.D.
REFERENCES