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An extension problem for discrete-time almost periodically correlated stochastic processes

Daniel Alpay [a,*], Boris Freydin [a], Philippe Loubaton [b]

[a] *Department of Mathematics, Ben–Gurion University of the Negev, P.O. Box 653, 84105 Beer-Sheva, Israel*

[b] *Laboratoire Système de Communication, Université de Marne la Vallée, Cité Descartes, 5 Boulevard Descartes, Champs sur Marne, 77454 Marne la Vallée Cedex 2, France*

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Abstract

In (D. Alpay, A. Chevreuil, Ph. Loubaton, J. Time Ser. Anal., 2000, to appear) an extension problem for covariance matrix of discrete-time periodically correlated stochastic processes introduced by Gladyshev was treated. In this paper we study the same problem for discrete-time almost periodically correlated stochastic processes. This problem can be reformulated and solved within the framework of interpolation for upper triangular operators. More precisely one can reduce the problem to an interpolation problem in the class of upper triangular operators of the Schur class. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

A centered discrete-time stochastic process $(y(n))_{n\in\mathbb{Z}}$ is said to be almost periodically correlated (periodically correlated with period $T \in \mathbb{N}$) if for each n, the sequence $m \to E(y(m+n)y(m)^*)$ is almost periodic (periodic with period $T \in \mathbb{N}$). These classes were introduced by Gladishev [13]. In both cases the auto-covariance matrix may be written as

$$E(y(m+n)y(m)^*) = \sum_{k=0}^{\infty} R_k(n)\,\mathrm{e}^{2\mathrm{i}\pi\mu_k m},$$

* Corresponding author.

E-mail address: dany@math.bgu.ac.il (D. Alpay).

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for each $(m, n) \in \mathbb{Z}^2$, where in the periodic case the latter is reduced to

$$E(y(m+n)y(m)^*) = \sum_{k=0}^{T-1} R_k(n)\, \mathrm{e}^{2\mathrm{i}\pi km/T}.$$

The sequences $R_k(n)$ are sometimes called the cyclocorrelation sequences of y and they are of some importance in many signal processing problems.

The following twofold problem was considered and solved in [1].

Problem 1.1. Given T-periodic sequences $R_k(n)$, $k \in \mathbb{Z}$ and $n = 1, \ldots, N$, to find out whether or not they coincide with the first N values of the cyclocorrelation sequences of a periodically correlated process of period T and in the first case to characterize the set of all extended cyclocorrelation sequences.

By using the well-known fact that a process $y(n)$ is periodically correlated with period T if and only if the T-variate time-series

$$Y(n) = \begin{pmatrix} y(nT) \\ \vdots \\ y(nT+T-1) \end{pmatrix}$$

is stationary, the problem was reduced to the classical matrix-covariance extension problem with some additional symmetry constraints. In this paper we solve a problem of the same kind for discrete-time almost periodically correlated stochastic processes.

Problem 1.2. Given N almost periodic sequences $A_i(m)$, $i = 0, 1, \ldots, N$, with their spectra in a module $\mathscr{K}$, i.e., for every two elements x_1 and x_2 that belong to $\mathscr{K}$ and $k_1, k_2 \in \mathbb{Z}$ follows that $k_1x_1 + k_2x_2 \in \mathscr{K}$, to find out whether or not there exists an auto-covariance matrix of some almost periodically correlated stochastic process $(y(n))_{n\in\mathbb{Z}}$ such that

$$E(y(m+i)y(m)^*) = A_i(m) \quad m \in \mathbb{Z}, \quad i = 1, \ldots, N \tag{1.1}$$

and such that for every $i \in \mathbb{Z}$ the spectrum of the almost periodic sequence $E(y(m+i)y(m)^*)$ belongs to M. We shall always assume that there exists an $\varepsilon_0 > 0$ such that $A_0(m) > \varepsilon_0$ for every $m \in \mathbb{Z}$.

In what follows we shall often use the following.

Remark 1.3. Since the matrix $E(y(m+i)y(m)^*) = A_i(m)$ $m \in \mathbb{Z},\ i \in \mathbb{Z}$, is positive definite it is readily seen that $\|A_i(m)\| \leqslant \sup_m \|A_0(m)\|$ $m \in \mathbb{Z},\ i \in \mathbb{Z}$.

We also obtain a description of all extended matrices under the non-degeneracy assumption specified in Problem 1.2. To do this we reformulate this problem as a

Carathéodory problem in the time-varying setting. As an application we shall also obtain the results of [1] as a special case.

2. Almost periodic sequences

In this section we present some information about almost periodic sequences which will be needed in the sequel. A complex function $A(k)$ defined for all integers will be called a sequence. An integer τ is called an ε-almost period of a sequence $A(k)$ for some $\varepsilon > 0$, and is denoted by $\tau(\varepsilon)$, if the inequality

$$|A(k+\tau) - A(k)| < \varepsilon$$

holds for every $k \in \mathbb{Z}$.

Definition 2.1. A sequence of complex numbers (elements of some Banach space) $A(k)$, $k \in \mathbb{Z}$, is called an almost periodic sequence if for every $\varepsilon > 0$ the set of ε-almost periods of the sequence is relatively dense, i.e., for every $\varepsilon > 0$ there exists a natural number m_ε such that for every $j \in \mathbb{Z}$ there is at least one ε-almost period among numbers $j, j+1, \dots, j+m_\varepsilon$.

Almost periodic sequences have been investigated by many authors[12,16,18] and they are from a special case of von Neumann general theory of almost periodic functions on a group. It is perhaps worth mentioning the following theorem which makes the connection between almost periodic sequences and almost periodic functions on the real line $\mathbb{R}$ and therefore establishes a relationship between both theories.

Theorem 2.2 [5, Theorem 1.27]. *A sequence $A(k)$ $(k \in \mathbb{Z})$ is an almost periodic sequence if and only if it is the restriction of an almost periodic function on the real line $\mathbb{R}$ to the integers.*

It follows from the latter theorem and can be derived independently that every almost periodic sequence is bounded and possesses a mean value:

$$M(A(k)) = \lim_{(\delta-\gamma)\to+\infty} \frac{1}{(\delta-\gamma)} \sum_{k=\gamma}^{\delta} A(k).$$

It is easily seen that the sum and the product of two almost periodic sequences as well as the limit of a uniformly convergent sequence of almost periodic sequences are again almost periodic. As in case of almost periodic functions on the real line $\mathbb{R}$, to an arbitrary almost periodic sequence $A(k)$ corresponds a Fourier series

$$A(m) = \sum_{k=0}^{\infty} a_k \, \mathrm{e}^{2\mathrm{i}\pi\mu_k m},$$

where

$$a_k = M(A(m)\,\mathrm{e}^{-2\mathrm{i}\pi\mu_k m}). \tag{2.1}$$

The set of $\mu_k,\ k \in \mathbb{Z}$, is called the spectrum of $A(k)$; here, of course, the exponents μ_k are only determined mod 2π, since the unit circle $\mathbb{T}$ is the dual group to the group of integers. Later on we shall be especially concerned with the relationship between almost periods and spectrum of almost periodic sequences. The cornerstone for this relation is given by the following Kronecker's or Weil–Kronecker's theorem.

Theorem 2.3 [5, Theorem 6.12]. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\theta_1, \theta_2, \ldots, \theta_n$ be arbitrary real numbers. For the system of inequalities*

$$|\lambda_k t - \theta_k| < \delta \ (\mathrm{mod}\ 2\pi) \quad (k = 1, \ldots, n) \tag{2.2}$$

to have consistent real solutions for any arbitrary small δ it is necessary and sufficient that every time the relation $\sum_1^n l_i \lambda_i = 0$ holds for $l_i \in \mathbb{Z}$ we have $\sum_1^n l_i \theta_i = 0\ (\mathrm{mod}\ 2\pi)$.

The following two theorems establish the interplay between spectrum and almost periods of almost periodic sequences. They are consequences of the previous one [5,18]. We first formulate a necessary condition for a natural number m to be an almost period of sequence $\{A_n\},\ n \in \mathbb{Z}$.

Theorem 2.4. *For every $\delta > 0$ and for all natural numbers N there exists an $\varepsilon = \varepsilon(\delta, N) > 0$ such that every ε-almost period of an almost periodic sequence $\{A_n\},\ n \in \mathbb{Z}$, satisfies the system of inequalities*

$$|\mathrm{e}^{\mathrm{i}\lambda_j m} - 1| < \delta \quad (j = 1, \ldots, N), \tag{2.3}$$

where $\lambda_j\ (j = 1, \ldots, N)$ belong to the spectrum of the almost periodic sequence $\{A_n\},\ n \in \mathbb{Z}$.

Next, we present a sufficient condition for some integer number τ to be an ε-almost period.

Theorem 2.5. *For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ and a natural number $N = N(\varepsilon)$ such that every real number τ satisfying the system of inequalities* (2.3) *is an ε-almost period of the almost periodic sequence $\{A_n\},\ n \in \mathbb{Z}$.*

3. Time-varying interpolation

In this section we shall recall the basic setting of interpolation problem of the Nevanlinna–Pick type in a general setting of upper triangular operators. Problems of this kind were considered by a number of authors, see for example [4,6,8,10,19,20].

First we introduce some notation. The basic setting, which is described in more detail in [2,6], is the space $\mathscr{X}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ of bounded linear operators from the Hilbert space

$$\ell^2_{\mathscr{N}} = \bigoplus_{i=-\infty}^{\infty} \mathscr{N}_i \text{ into the Hilbert space } \ell^2_{\mathscr{M}} = \bigoplus_{i=-\infty}^{\infty} \mathscr{M}_i,$$

where the coordinate spaces $\mathscr{N}_i$ and $\mathscr{M}_i$, $i = 0, \pm 1, \ldots$, are themselves each separable Hilbert spaces. We shall assume that $\mathscr{N}_i = \mathscr{N}_0$ and $\mathscr{M}_i = \mathscr{M}_0$ for $i = \pm 1$, $\pm 2, \ldots$, in order to simplify the exposition. The more general setting can be considered analogously [8,6,10,19,20]. Every $A \in \mathscr{X}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ has a block matrix representation

$$\begin{bmatrix} & & \vdots & & \\ & & A_{-1,0} & & \\ \cdots & A_{0,-1} & \boxed{A_{00}} & A_{01} & \cdots \\ & & A_{10} & & \\ & & \vdots & & \end{bmatrix},$$

where $A_{ij} : \mathscr{N}_j \to \mathscr{M}_i$. Correspondingly, if we write $g \in \ell^2_{\mathscr{N}}$ and $f \in \ell^2_{\mathscr{M}}$ as infinite column vectors

$$g = \begin{bmatrix} \vdots \\ g_{-1} \\ \boxed{g_0} \\ g_1 \\ \vdots \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} \vdots \\ f_{-1} \\ \boxed{f_0} \\ f_1 \\ \vdots \end{bmatrix},$$

respectively, then the components of $f = Ag$ follow the usual rules of matrix multiplication

$$f_j = \sum_{k=-\infty}^{\infty} A_{jk} g_k.$$

Let $\mathscr{U}$, $\mathscr{L}$ and $\mathscr{D}$ denote the subspaces of $\mathscr{X}$ consisting of upper triangular, lower triangular and diagonal operators, respectively, in the indicated matrix representation, and let $Z \in \mathscr{U}(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{B}})$ denote the shift operator which is specified by the rule

$$(Zf)_j = f_{j+1} \quad \text{for } f \in \ell^2_{\mathscr{B}} = \bigoplus_{i=-\infty}^{\infty} \mathscr{B}_i, \tag{3.1}$$

with $\mathscr{B}_i = \mathscr{B}_0$ a fixed separable Hilbert space for $i = \pm 1, \pm 2, \ldots$ In this setting, the embedding operators

$$\pi_i : u \in \mathscr{B}_i \longrightarrow f \in \ell^2_{\mathscr{B}}, \quad \text{where } \begin{cases} f_i = u & \text{for } j = i, \\ f_j = 0 & \text{for } j \neq i, \end{cases} \tag{3.2}$$

and their adjoints

$$\pi_i^* : f \in \ell^2_{\mathscr{B}} \longrightarrow f_i \in \mathscr{B}_i,$$

can all be expressed in terms of π_0 and the shift operator Z:

$$\pi_i = Z^{*i}\pi_0.$$

We shall use the letter Z for the forward shift in other spaces too; the dependence of Z on the space in question will not be indicated explicitly. Thus for example, we shall let

$$F^{(j)} = (Z^*)^j F Z^j, \quad j = 0, \pm 1, \ldots \tag{3.3}$$

for $F \in \mathscr{X}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ even though Z^* acts from $\ell^2_{\mathscr{M}} \to \ell^2_{\mathscr{M}}$ and Z acts from $\ell^2_{\mathscr{N}} \to \ell^2_{\mathscr{N}}$ in (3.3). It is useful to note that $F^{(j)}$ slides the entries in each diagonal of F by j units in the South-East direction: $(F^{(j)})_{st} = F_{s-j,t-j}$. We note that this operator will be very useful in the sequel for restating our original problem as an interpolation one. Thus each of the spaces $\mathscr{U}$, $\mathscr{L}$ and $\mathscr{D}$ is invariant under the mapping $F \to F^{(j)}$. Now let $\mathscr{X}_2(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ denote the set of operators in $\mathscr{X}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ which are Hilbert–Schmidt. All these spaces are Hilbert spaces with respect to the inner product

$$\langle F, G\rangle_{\mathrm{HS}} = \mathrm{Tr}\, G^* F.$$

It is readily checked that

$$\langle Z^j D, Z^k E\rangle_{\mathrm{HS}} = \begin{cases} \mathrm{Tr}\ E^* D & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

for every choice of D and E in $\mathscr{D}_2(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$, and hence that the spaces $\mathscr{L}_2' = Z^*\mathscr{L}_2$, $\mathscr{D}_2$ and $\mathscr{U}_2' = Z\mathscr{U}_2$ are orthogonal with respect to this inner product

$$\mathscr{X}_2 = \mathscr{U}_2' \oplus \mathscr{D}_2 \oplus \mathscr{L}_2' = \mathscr{U}_2 \oplus \mathscr{L}_2' = \mathscr{U}_2' \oplus \mathscr{L}_2.$$

Let $\underline{\underline{p}}$, $\underline{\underline{q}}$ and $\underline{\underline{q}}'$ be the orthogonal projection of $\mathscr{X}_2$ onto $\mathscr{U}_2$, $\mathscr{L}_2$ and $\mathscr{L}_2'$, respectively, and let $\mathscr{S}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ denote the set of $S \in \mathscr{U}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ with operator norm $\|S\| \leqslant 1$. We shall say that $S \in \mathscr{S}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ is a solution of the BIP (basic interpolation problem) (see [6]) based on a given set of bounded linear operators

$$f_i = \begin{bmatrix} g_i \\ h_i \end{bmatrix}$$

from $\ell^2_{\mathscr{B}}$ to $\ell^2_{\mathscr{M}} \oplus \ell^2_{\mathscr{N}}$ with components

$$g_i \in \mathscr{U}\left(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{M}}\right) \quad \text{and} \quad h_i \in \mathscr{U}\left(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{N}}\right) \quad \text{for } i = 1, \ldots, n$$

if, for every choice of $E \in \mathscr{D}_2(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{B}})$,

$$\underline{\underline{p}} S^* g_i E = h_i E \quad \text{for } i = 1, \ldots, n. \tag{3.4}$$

This formulation is modeled on the treatment of the classical case in [8,9]. In the present setting, upper triangular operators play the role of functions which are analytic inside the unit disc in the classical case and diagonal operators play the role of scalars. The classical case emerges upon choosing all the operators to be constant along diagonals, i.e., to be block Toeplitz. The special choice

$$g_i = \xi_i \left(I - ZV_i^*\right)^{-1} \quad \text{and} \quad h_i = \eta_i \left(I - ZV_i^*\right)^{-1} \quad \text{for } i = 1, \ldots, n,$$

with $\xi_i \in \mathscr{D}(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{M}})$, $\eta_i \in \mathscr{D}(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{N}})$, $r_{sp}(ZV_i^*) < 1$ for $i = 1, \ldots, n$, corresponds to the two-sided Nevanlinna–Pick problem in the present setting. Here r_{sp} designates the spectral radius of the indicated operator. For this choice of g_i and h_i, the interpolation conditions embodied in (3.4) can be reformulated in terms of a diagonal transform $(\xi_i^* S)^\wedge(V_i)$ as follows:

$$\left(\xi_i^* S\right)^\wedge (V_i) = \eta_i^* \quad \text{for } i = 1, \ldots, n. \tag{3.5}$$

Here, for $V \in \mathscr{D}(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{B}})$ with $r_{sp}(ZV^*) < 1$, $F \in \mathscr{U}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{B}})$ and $G \in \mathscr{U}(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{M}})$, the diagonal transform $F^\wedge(V)$ may be characterized as the unique element in $\mathscr{D}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{B}})$ such that $(Z - V)^{-1}\{F - F^\wedge(V)\} \in \mathscr{U}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{B}})$; see Theorem 3.3 of [2]. Formula (3.5) serves to display this NP problem in a form which resembles the NP problem in the classical setting. Indeed, the latter emerges from the former by suitably restricting the data. In formula (3.4) the symbol S^* should be understood as the operator M_{S^*} of multiplication by S^* on the left acting from $\mathscr{X}_2(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{M}})$ to $\mathscr{X}_2(\ell^2_{\mathscr{B}}; \ell^2_{\mathscr{N}})$.

In this paper we will deal with another particular case of the tangential interpolation problem which leads to the Carathéodory problem.

Problem 3.1. We restrict the operators $f_1, \ldots, f_n$ to the following form:

$$F = \begin{bmatrix} G \\ H \end{bmatrix} = [f_1 \ldots f_n] = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - Z_1 M)^{-1}, \tag{3.6}$$

where

$$Z_1 = \operatorname{diag}(Z, \ldots, Z) \quad n \text{ times},$$

$$C_1 = [\xi_1 \ldots \xi_n] \quad \left(\xi_i \in \mathscr{D}\left(\ell^2_{\mathbb{C}}; \ell^2_{\mathscr{M}}\right)\right),$$

$$C_2 = [\eta_1 \ldots \eta_n] \quad \left(\eta_i \in \mathscr{D}\left(\ell^2_{\mathbb{C}}; \ell^2_{\mathscr{N}}\right)\right),$$

$$M = \begin{bmatrix} (M)_{11} & \cdots & (M)_{1n} \\ \vdots & & \vdots \\ (M)_{n1} & \cdots & (M)_{nn} \end{bmatrix} \quad \left((M)_{st} \in \mathscr{D}\left(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}}\right)\right),$$

where these data are further specified as follows:

$$\ell^2_{\mathscr{M}} = \ell^2_{\mathscr{N}} = \ell^2_{\mathbb{C}}.$$

The operator ξ_1 is invertible and

$$M_{(i+1)i} = I_{\ell^2_{\mathbb{C}}} \quad M_{km} = 0, \quad \text{for } k - m \neq 1.$$

This choice of the data insures that

$$G = C_1(I - Z_1M)^{-1} \quad \text{and} \quad H = C_2(I - Z_1M)^{-1}$$

are bounded upper triangular operators. Then Theorem 2.2 of [6] (see also Theorem 2.2 of [10]) implies that the interpolation problem based on the data specified just above admits a solution if and only if the following operator (operator-matrix) P:

$$P = \sum_{j=0}^{\infty} \left(M^* Z_1^*\right)^j \left(C_1^* C_1 - C_2^* C_2\right) (Z_1 M)^j \tag{3.7}$$

is positive semidefinite. We remark that due to the specific form of the data the sum in the latter formula has in fact only $n+1$ terms.

It is worth mentioning that the operator P specified just above is a solution of the Stein equation associated with the interpolation conditions (3.4) where the data are given after formula (3.6)

$$P - M^* Z_1^* P Z_1 M = C_1^* C_1 - C_2^* C_2. \tag{3.8}$$

Theorems 7.1 and 8.1 of [11] in the case when $\ell^2_{\mathscr{M}} = \ell^2_{\mathscr{N}} = \ell^2_{\mathbb{C}}$ (which is of the main interest in the paper) imply that:

Theorem 3.2. *Assume $P \geqslant \varepsilon I$ for some $\varepsilon > 0$. Then for every solution $S \in \mathscr{S}(\mathscr{E}_1; \mathscr{E}_2)$ (where $\mathscr{E}_1 = \mathscr{E}_2 = \ell^2_{\mathbb{C}}$) of Problem* 3.1 *based on the data* (3.6), *there exists a unique choice of $\Omega \in \mathscr{S}(\mathscr{E}_1; \mathscr{E}_2)$ such that*

$$S = \Sigma_{11} + \Sigma_{12}\{I - \Omega\Sigma_{22}\}^{-1}\Omega\Sigma_{21},$$

where $\Sigma : \mathscr{E}_1 \oplus \mathscr{E}_2 \longrightarrow \mathscr{E}_2 \oplus \mathscr{E}_1$ is a unitary upper triangular operator defined by the data of the interpolation problem. Moreover, S is strictly contractive if and only if Ω is strictly contractive.

The set of solutions S to the interpolation Problem 3.1 *can also be expressed as a linear fractional transformation*

$$S = (\Theta_{11}\Omega + \Theta_{12})(\Theta_{21}\Omega + \Theta_{22})^{-1}$$

in terms of the block entries of an upper triangular $J_{\mathscr{E}}$-unitary operator (*with* $J_{\mathscr{E}} = \operatorname{diag}(I_{\mathscr{E}_2}, -I_{\mathscr{E}_1})$)

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} : \begin{matrix} \mathscr{E}_2 \\ \oplus \\ \mathscr{E}_1 \end{matrix} \longrightarrow \begin{matrix} \mathscr{E}_2 \\ \oplus \\ \mathscr{E}_1 \end{matrix}$$

which is the Potapov–Ginzburg transform to the operator Σ:

$$\Theta = \begin{bmatrix} \Sigma_{12} - \Sigma_{11}\Sigma_{21}^{-1}\Sigma_{22} & \Sigma_{11}\Sigma_{21}^{-1} \\ -\Sigma_{21}^{-1}\Sigma_{22} & \Sigma_{21}^{-1} \end{bmatrix}.$$

4. Positive real interpolant

In this section we present some results on the interpolation Problem 3.1 in the class of operators $\Phi \in \mathscr{U}(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{M}})$ with strictly positive real part, i.e., with $\Phi + \Phi^* \geqslant \varepsilon I$ for some $\varepsilon > 0$ (see [11] for detail and references). We denote by $\mathscr{C}^\circ(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{M}})$ this class of operators and correspondingly let $\mathscr{S}^\circ(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{M}})$ denote the set of operators in $\mathscr{S}(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{M}})$ which are strictly contractive. We shall take advantage of the fact that the Cayley transform $T \longrightarrow (I-T)(I+T)^{-1}$ defines a one-to-one correspondence between the class of operators with strictly positive real part and the class of strictly contractive operators.

Lemma 4.1. *If* $\Phi \in \mathscr{C}^\circ(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{M}})$, *then* $S = (I-\Phi)(I+\Phi)^{-1}$ *belongs to* $\mathscr{S}^\circ(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{M}})$. *Moreover* (*since the Cayley transform is its own inverse*), *every such* S *may be obtained as the Cayley transform of such a* Φ.

Our next objective is to show that if S is a solution of the interpolation problem which is expressed in terms of M, C_1 and C_2 as in the previous section (between (3.6) and (3.7)), then Φ is the solution of a related problem which will be expressed in terms of the same P and M but with

$$\widetilde{C}_1 = \frac{C_1 + C_2}{\sqrt{2}} \quad \text{and} \quad \widetilde{C}_2 = \frac{C_1 - C_2}{\sqrt{2}}$$

(Note that $\widetilde{\widetilde{C}}_i = C_i$.)

Theorem 4.2. *Let* $S \in \mathscr{S}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ *be a solution of the interpolation problem based on* M, C_1 *and* C_2 *as set forth in Section* 3. *Then*

$$\Phi = (I-S)(I+S)^{-1}$$

belongs to $\mathscr{C}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ *and is a solution of the following interpolation problem*:

$$\underline{\underline{p}}\Phi^* \widetilde{C}_1 (I - Z_1 M)^{-1} E_1 = \widetilde{C}_2 (I - Z_1 M))^{-1} E_1, \tag{4.1}$$

for every choice of $E_1 \in \{\mathscr{D}_2(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})\}^{n\times 1}$. *Conversely, if* $\Phi \in \mathscr{C}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ *is a solution of the interpolation problem* (4.1), *then* $S = (I-\Phi)(I+\Phi)^{-1}$ *belongs to* $\mathscr{S}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ *and is a solution of the original interpolation problem.*

This analysis shows that the new interpolation problem based on (4.1) admits a solution $\Phi \in \mathscr{C}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ for each strictly positive solution P of the Stein equation (3.8). Conversely, the inequality

$$\langle PE, E\rangle_{\mathrm{HS}} = \left\langle \begin{bmatrix} I & -S \\ -S^* & I \end{bmatrix} FE, FE \right\rangle_{\mathrm{HS}} \geqslant \gamma \langle FE, FE\rangle_{\mathrm{HS}} \qquad (ND)$$

shows that the condition that P be strictly positive is necessary for the existence of such solutions providing that

$$\langle FE, FE\rangle_{\mathrm{HS}} \geqslant \varepsilon_1 \langle E, E\rangle_{\mathrm{HS}}$$

for some choice of $\varepsilon_1 > 0$.

A number of facts are easily transferred from the setting of the original interpolation problem in the class $\mathscr{S}^\circ$ to the class of the new interpolation problem based on (4.1) in the class $\mathscr{C}^\circ$ (see [10,11] for details).

Theorem 4.3. *If $P \geqslant \varepsilon I$ for some $\varepsilon > 0$, then Φ is a solution of the new interpolation problem based on* (4.1) *in the class $\mathscr{C}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ if and only if it can be expressed in the form*

$$\Phi = \left(\widetilde{\Theta}_{11}\Omega + \widetilde{\Theta}_{12}\right)\left(\widetilde{\Theta}_{21}\Omega + \widetilde{\Theta}_{22}\right)^{-1} \tag{4.2}$$

for some choice of $\Omega \in \mathscr{S}^\circ(\mathscr{E}_1; \mathscr{E}_2)$, where

$$\widetilde{\Theta} = \frac{1}{\sqrt{2}}\begin{bmatrix} -I_{\mathscr{E}} & I_{\mathscr{E}} \\ I_{\mathscr{E}} & I_{\mathscr{E}} \end{bmatrix}\Theta,$$

$\mathscr{E}_1 = \mathscr{E}_2 = \mathscr{E} = \ell^2_{\mathbb{C}}$ and Θ is given by following formulas:

$$\Theta_{11} = C_1(I - Z_1M)^{-1}P^{-1}Z_1Y\left((P^{-1})^{(1)}_{nn}\right)^{1/2}, \tag{4.3}$$

$$\Theta_{12} = C_1(I - Z_1M)^{-1}P^{-1}C_2^*\left\{I_{\mathscr{E}_1} + C_2P^{-1}C_2^*\right\}^{-1/2}, \tag{4.4}$$

$$\Theta_{21} = C_2(I - Z_1M)^{-1}P^{-1}Z_1Y\left((P^{-1})^{(1)}_{nn}\right)^{1/2}, \tag{4.5}$$

$$\Theta_{22} = \left\{I_{\mathscr{E}_1} + C_2(I - Z_1M)^{-1}P^{-1}C_2^*\right\}\left\{I_{\mathscr{E}_1} + C_2P^{-1}C_2^*\right\}^{-1/2}, \tag{4.6}$$

where

$$Y^* = [0, 0, \ldots, I].$$

Proof. By Theorem 8.4 of [11] and Corollary after this theorem Θ is given by following formulas:

$$\Theta_{11} = \left[C_1(I - Z_1M)^{-1}P^{-1/2}Z_1 \quad I_{\mathscr{E}_2}\right]\beta^*,$$

$$\Theta_{12} = C_1(I - Z_1M)^{-1}P^{-1}C_2^*\left\{I_{\mathscr{E}_1} + C_2P^{-1}C_2^*\right\}^{-1/2},$$

$$\Theta_{21} = C_2(I - Z_1M)^{-1}P^{-1/2}Z_1[I_{\mathscr{H}^{(-1)}} \quad 0]\beta^*,$$

$$\Theta_{22} = \{I_{\mathscr{E}_1} + C_2(I - Z_1M)^{-1}P^{-1}C_2^*\}\left\{I_{\mathscr{E}_1} + C_2P^{-1}C_2^*\right\}^{-1/2},$$

where operator β is some inclusion operator. Next applying the result of Case 4 [6, p. 197] with

$$\beta^* = \begin{bmatrix} (P^{-1/2})^{(1)} Y \left((P^{-1})^{(1)}_{nn} \right)^{1/2} \\ 0 \end{bmatrix},$$

$$Y^* = [0, 0, \ldots, I],$$

we obtain the desired set of formulas. □

Now we shall present a generalization of the maximum principle and the Schwarz lemma in the setting of upper triangular operators proved in [10]. These results will be expressed in terms of the symbol $F(r)$ which is defined for each upper triangular operator $F \in \mathscr{U}(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{N}})$ as the operator in $\mathscr{U}(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{N}})$ with blocks

$$F(r)_{ij} = r^{j-i} F_{ij} \tag{4.7}$$

for $j \geq i$.

Theorem 4.4. *Let $F \in \mathscr{U}(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{N}})$. Then*

$$\|F(r)\| \leqslant \|F\| \tag{4.8}$$

for every $r \in [0, 1)$. If $F \in \mathscr{U}'(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{N}})$, i.e., if F is strictly upper triangular, then

$$\|F(r)\| \leqslant r\|F\| \tag{4.9}$$

for every choice of $r \in (0, 1)$.

By using Theorem 4.4 together with Theorem 4.3, one can easily derive the following.

Theorem 4.5. *Let $S \in \mathscr{S}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ with $S(0) = 0$ be a solution of the interpolation problem based on M, C_1 and C_2 as set forth in Section 3. Then*

$$\Phi(r) = (I - S(r))(I + S(r))^{-1}$$

belongs to $\mathscr{C}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ and is a solution of the following interpolation problem:

$$\underline{\underline{p}}\Phi(r)^* \widetilde{C}_1 \left(I - r^{-1} Z_1 M \right)^{-1} E_1 = \widetilde{C}_2 \left(I - r^{-1} Z_1 M \right)^{-1} E_1 \tag{4.10}$$

for every choice of $r \in (0, 1)$ and $E_1 \in \{\mathscr{D}_2(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})\}^{n\times 1}$. Conversely, if for every choice of $r \in [0, 1)$, $\Phi(r) \in \mathscr{C}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ such that $\Phi(0) = I$ is a solution of the second stated problem, then $S \in \mathscr{S}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$, which is uniquely defined by

$$S(r) = (I - \Phi(r))(I + \Phi(r))^{-1} \in \mathscr{S}^\circ \left(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}} \right)$$

for every choice of $r \in [0, 1)$, is a solution of the original interpolation problem.

As a consequence of Theorem 4.5 we obtain:

Theorem 4.6. *If $P \geqslant \varepsilon I$ for some $\varepsilon > 0$, then $\Phi(r)$ $(\Phi(0) = I)$ is a solution of the new interpolation problem based on (4.10) in the class $\mathscr{C}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ if and only if it can be expressed in the form*

$$\Phi(r) = \left(\widetilde{\Theta}_{11}(r)\Omega(r) + \widetilde{\Theta}_{12}(r)\right)\left(\widetilde{\Theta}_{21}(r)\Omega(r) + \widetilde{\Theta}_{22}(r)\right)^{-1} \tag{4.11}$$

for some choice of $\Omega \in \mathscr{S}(\mathscr{E}_1; \mathscr{E}_2)$, *where*

$$\widetilde{\Theta}(r) = \frac{1}{\sqrt{2}}\begin{bmatrix} -I_{\mathscr{E}} & I_{\mathscr{E}} \\ I_{\mathscr{E}} & I_{\mathscr{E}} \end{bmatrix}\Theta(r),$$

$\mathscr{E}_1 = \mathscr{E}_2 = \mathscr{E} = \ell^2_{\mathbb{C}}$ *and* Θ *is from Theorem* 4.5.

Remark 4.7. We note that in Theorems 4.5 and 4.6 the matrix corresponding to the $\Phi = \Phi(1)$ need not belong to $\mathscr{C}^\circ(\ell^2_{\mathscr{M}}; \ell^2_{\mathscr{M}})$, i.e., it does not have to be the matrix of a bounded operator as long as the other conditions of these theorems are satisfied.

This fact will prove to be useful in the following section.

5. Solution of the problem

In this section we shall solve our initial problem. To do that we restate Problem 1.2. First we mention that thanks to the well-known realization theory of stochastic processes solution of Problem 1.2 is reduced to finding some positive matrix with a priori known first $N + 1$ diagonals and some extra assumption on the behavior of each diagonal. We would also like to note that the extended matrix, if it exists, may represent an unbounded operator on $\ell^2_{\mathbb{C}}$. To avoid dealing with unbounded operators, using Remark 1.3, we restate Problem 1.2 in the following equivalent way.

Problem 5.1. Given N almost periodic sequences $A_i(m)$, $i = 0, 1, \ldots, N$, where $A_0(m) > \varepsilon_0,\ m \in \mathbb{Z}$, for some ε_0, with their spectra in a module $\mathscr{K}$, to find out whether or not there exists a positive matrix $\phi_{m,m+i}$, $i,\ m \in \mathbb{Z}$, such that for every $i \in \mathbb{Z}$ the spectrum of the almost periodic sequence $\phi_{m,m+i}$ belongs to $\mathscr{K}$ and

$$\begin{aligned} &(1)\ \phi_{m,m+i} = A_i(m), \quad m \in \mathbb{Z}, \ \ i = 0, 1, \ldots, N \quad \text{and} \\ &(2)\ \phi_{m,m+i}(r) = r^{|i|}\phi_{m,m+i} \end{aligned} \tag{5.1}$$

belongs to $\mathscr{C}^\circ(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ for every $r \in (0, 1)$ (i.e., $\ell^2_{\mathscr{M}}$ of Section 4 is equal to $\ell^2_{\mathbb{C}}$).

Remark 5.2. Since $A_0(m) > \varepsilon_0,\ m \in \mathbb{Z}$, we can assume without any loss of generality that $A_0(m) = 2,\ m \in \mathbb{Z}$.

Next we establish a connection between covariance matrix of almost periodically correlated processes and interpolation in time-varying case, then we reformulate our problem as an interpolation one.

First let us note that by associating with a two-sided sequence $A(k),\ k \in \mathbb{Z}$, the diagonal operator D_A ($(D_A)_{ii} = A(i),\ i \in \mathbb{Z}$) that belongs to $\mathscr{X}(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ the definition of ε-almost periods can be equivalently defined as follows: An integer τ is called an ε-almost period of sequence $A(k)$ for some $\varepsilon > 0$, and is denoted by $\tau(\varepsilon)$, if the inequality

$$\|Z^{*\tau} D_{A(k)} Z^{\tau} - D_{A(k)}\|_{\mathscr{X}(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})} < \varepsilon,$$

where $D_{A(k)}$ is the operator associated with the sequence $A(k)$.

In what follows we shall say that τ is an ε-almost period of an operator that belongs to $\mathscr{X}(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ if it satisfies the latter inequality or in other words if all its diagonals are almost periodic sequences with the same ε-almost periods.

Theorem 5.3. *Problem* 5.1 *has a strictly positive solution if and only if the interpolation problem* (4.10) *with*

$$\widetilde{C}_1 = [I, 0, \ldots, 0] \quad \textit{and} \quad \widetilde{C}_2 = [D_{1/2A_0(k)}, \ldots, D_{A_N(k)}],$$

has a strictly positive almost periodic solution, i.e., there exists $\Phi(r)$ $(\Phi(0) = I)$ *which belongs to* $\mathscr{C}^{\circ}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ *such that*

$$\underline{\underline{p}}\Phi(r)^* \widetilde{C}_1 (I - r^{-1} Z_1 M)^{-1} E_1 = \widetilde{C}_2 (I - r^{-1} Z_1 M)^{-1} E_1,$$

for every choice of $r \in (0, 1)$ *and* $E_1 \in \{\mathscr{D}_2(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})\}^{n \times 1}$.

In other words, we fix the first $(N + 1)$ upper diagonals of the matrix Φ.

Proof. Let us first assume that the extension Problem 5.1 has a strictly positive almost periodic solution, i.e., there exists a strictly positive almost periodic operator Ψ which has $D_{A_0(k)}, \ldots, D_{A_N(k)}$ as its first upper diagonals. Then, due to Remark 1.3, for every $0 \leqslant r < 1$, $\Psi(r)$ belongs to the class $\mathscr{C}^{\circ}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ which means that the upper part of this operator is an almost periodic solution to the interpolation problem specified in the statement of this theorem.

Conversely, if an upper triangular operator F is an almost periodic solution to the interpolation problem (4.10) with

$$\widetilde{C}_1 = [I,\ 0, \ldots, 0],$$

$$\widetilde{C}_2 = [D_{1/2A_0(k)}, \ldots, D_{A_N(k)}],$$

then $(F(r) + F(r)^*)$ is obviously a solution to the extension Problem 5.1. $\square$

Next we give a condition for the latter interpolation problem to have a solution which is a necessary and sufficient condition under the added assumption that condition (ND) of Section 4 holds true.

Theorem 5.4. *The interpolation problem specified in Theorem* 5.3 *admits an almost periodic solution with spectrum in the module* $\mathscr{K}$ *if the operator P given by formula* (3.7):

$$P = \sum_{j=0}^{\infty} \left(M^* Z_1^*\right)^j \left(C_1^* C_1 - C_2^* C_2\right) (Z_1 M)^j,$$

where C_j, $j = 1, 2$, are given by the formulas

$$C_1 = \frac{\widetilde{C}_1 + \widetilde{C}_2}{\sqrt{2}} \quad and \quad C_2 = \frac{\widetilde{C}_1 - \widetilde{C}_2}{\sqrt{2}},$$

is strictly positive. Moreover, in this case, an almost periodic operator $\Phi(r)$ ($\Phi(0) = I$) with spectrum in the module $\mathscr{K}$ is a solution of the interpolation problem described in Theorem 5.3 *if and only if it can be expressed in the form*

$$\Phi(r) = \left(\widetilde{\Theta}_{11}(r)\Omega(r) + \widetilde{\Theta}_{12}(r)\right)\left(\widetilde{\Theta}_{21}(r)\Omega(r) + \widetilde{\Theta}_{22}(r)\right)^{-1}, \tag{5.2}$$

where the matrix $\widetilde{\Theta}_{ij}$, $i, j = 1, 2$, is specified in Theorem 4.6, *for some choice of $\Omega \in \mathscr{S}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ which is almost periodic with spectrum in the same module $\mathscr{K}$.*

Let us note that as it is readily seen that the solution of the latter equation can be written as follows:

$$P = \begin{pmatrix} D_{A_0} & D_{A_1^{(1)}} & D_{A_2^{(2)}} & \cdots & D_{A_m^{(m)}} \\ D_{A_1^{*(1)}} & D_{A_0^{(1)}} & D_{A_1^{(2)}} & \cdots & D_{A_{m-1}^{(m)}} \\ D_{A_2^{*(2)}} & D_{A_1^{*(2)}} & D_{A_0^{(2)}} & \cdots & D_{A_{m-2}^{(m)}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ D_{A_m^{*(m)}} & D_{A_{m-1}^{*(m)}} & D_{A_{m-2}^{*(m)}} & \cdots & D_{A_0^{(m)}} \end{pmatrix},$$

Proof. Note that if the operator P is strictly positive, then, by Theorem 4.6, there exists a solution to the interpolation problem specified in Theorem 5.3 (without the restriction for solutions to be almost periodic) and the set of all the solutions can be described as it is stated in Theorem 4.6. Therefore, to complete the proof of the theorem, it is enough to prove the part concerning the restriction for the solutions to be almost periodic.

Let us first notice that the operator P defined by (3.7) is an almost periodic diagonal operator with spectrum in the module $\mathscr{K}$. Indeed, it is readily checked that due to the structure of the data of the problem for every $k \in \mathbb{Z}$ the following equality holds:

$$\begin{aligned} P - Z^k P Z^{*k} = \sum_{j=0}^{\infty} \left(M^* Z_1^*\right)^j \Big\{ & \left(C_1^* C_1 - C_2^* C_2\right) \\ & - Z^{*k}\left(C_1^* C_1 - C_2^* C_2\right) Z^k \Big\} (Z_1 M)^j. \end{aligned} \tag{5.3}$$

Therefore, since the number of term in (5.3) is finite, our assertion follows from Theorems 2.4 and 2.5. Next, from formulas (4.3)–(4.6), it follows that the matrix $\widetilde{\Theta}_{ij}$, $i, j = 1, 2$, is also an almost periodic operator with spectrum in the module $\mathscr{K}$. Hence, the sufficiency drops easily from the general properties of almost periodic operators.

To prove the necessity part we first recall that the correspondence between the solutions and parameters given by (5.2) is one-to-one. This follows from the fact that

(5.2) is the composition of the Cayley transform and the linear fractional transformation based on the J-unitary matrix $\Theta_{ij},\ i, j = 1, 2$, specified by (4.3)–(4.6), both of which are one-to-one. Since, as it is easily seen, the Cayley transform respects the property of an operator to be almost periodic with its spectra in a given module $\mathscr{K}$, i.e., both operators related by the Cayley transform are almost periodic with spectra in a given module $\mathscr{K}$ if any one of them is, in order to prove the necessity part, it is enough to check that the linear fractional transformation based on the J-unitary matrix $\Theta_{ij},\ i, j = 1, 2$, specified by (4.3)–(4.6) also "respects" this property. From Theorems 3.2–3.5 of [7] it follows that the linear fractional transformation based on the J-unitary matrix $\Theta_{ij},\ i, j = 1, 2$, specified by (4.3)–(4.6)

$$T_\Theta = (\Theta_{11}\Omega + \Theta_{12})(\Theta_{21}\Omega + \Theta_{22})^{-1} \tag{5.4}$$

is invertible with the inverse transform given by the linear fractional transformation (5.4) based on Θ^{-1}. Since the matrix $\Theta_{ij},\ i, j = 1, 2$, specified by (4.3)–(4.6) is J-unitary, its inverse can be written in the form $\Theta^{-1} = J\Theta^* J$, which obviously implies that Θ^{-1} is an almost periodic operator with its spectrum in the module $\mathscr{K}$. Therefore ${T_\Theta}^{-1} = T_{\Theta^{-1}}$ "respects" the property of an operator to be almost periodic with its spectrum in a given module $\mathscr{K}$. This completes the proof. □

Next, we would like to mention that the latter theorem can also be applied to Problem 1.1. Indeed, since T-periodic sequences are obviously a particular case of almost periodic sequences, as a consequence of Theorem 5.4 we have:

Theorem 5.5. *The interpolation problem specified in Theorem* 5.3 (*where in this case the operators C_1 and C_2 are assumed to be T-periodic*) *admits a T-periodic solution if the T-periodic operator P given by formula* (3.7)

$$P = \sum_{j=0}^{\infty} \left(M^* Z_1^*\right)^j \left(C_1^* C_1 - C_2^* C_2\right) (Z_1 M)^j,$$

where $C_j,\ j = 1, 2$, are given by the formulas

$$C_1 = \frac{\widetilde{C}_1 + \widetilde{C}_2}{\sqrt{2}} \quad \textit{and} \quad C_2 = \frac{\widetilde{C}_1 - \widetilde{C}_2}{\sqrt{2}},$$

is strictly positive. Moreover, *in this case*, *a T-periodic operator $\Phi(r)$ $(\Phi(0) = I)$ is a solution of the interpolation problem described in Theorem* 5.3 *if and only if it can be expressed in the form*

$$\Phi(r) = \left(\widetilde{\Theta}_{11}(r)\Omega(r) + \widetilde{\Theta}_{12}(r)\right)\left(\widetilde{\Theta}_{21}(r)\Omega(r) + \widetilde{\Theta}_{22}(r)\right)^{-1}, \tag{5.5}$$

where the matrix $\widetilde{\Theta}_{ij},\ i, j = 1, 2$, is specified in Theorem 4.6, *for some choice of $\Omega \in \mathscr{S}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ which is a T-periodic operator.*

Remark 5.6. It is readily seen that under the assumption that the associated Pick operator is strictly positive from Theorem 5.5 follows the existence of solutions to Problem 1.1 in the same way as it was carried out for the almost periodic case.

Remark 5.7. It is easily checked that our condition that the associated T-periodic Pick operator is strictly positive is equivalent to the condition that appeared in [1].

6. The maximum entropy problem

In this section we shall apply the results of the preceding sections to formulate and solve a maximum entropy problem for operators with positive real part. We shall be using the results of [11] on a maximum entropy problem for operators from the Schur class in the setting of upper triangular operators. Similar treatments of the latter problem in the setting of upper triangular operators may be found in [15,17]. For another approach to entropy problems, see the paper of Gohberg et al. [14]. We continue to assume that $P \geqslant \varepsilon I$ for some $\varepsilon > 0$ and shall let $\mathscr{C}$ denote the set of all strictly positive bounded solutions $\Phi(r) \in \mathscr{C}^{\circ}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ to the interpolation problem which is discussed in Theorem 5.3 and $\mathscr{K}$ denote the set of all strictly contractive solutions $S(r) \in \mathscr{S}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{M}})$ to the corresponding interpolation problem in the Schur class. Then by a theorem of Arveson [3], the operators $(\Phi(r)^* + \Phi(r))/2$ and $I - S(r)^*S(r)$ admit a spectral factorization, i.e., they can be expressed in the form

$$\frac{\Phi(r)^* + \Phi(r)}{2} = \Psi^*_{\Phi(r)}\Psi_{\Phi(r)} \quad \text{and} \quad I - S(r)^*S(r) = F^*_{S(r)}F_{S(r)},$$

where $\Psi_{\Phi},\ F_S \in \mathscr{U}(\ell^2_{\mathscr{N}}; \ell^2_{\mathscr{N}'})$ are bounded invertible operators with $\Psi_{\Phi}^{-1},\ F_S^{-1} \in \mathscr{U}(\ell^2_{\mathscr{N}'}; \ell^2_{\mathscr{N}})$ for some suitable choice of $\ell^2_{\mathscr{N}'}$. Note that if $\|S\| < 1$ and consequently Φ is bounded, then $\Psi_{\Phi(r)}$ and $F_{S(r)}$ are equal to $\Psi_{\Phi}(r)$ and $F_S(r)$, respectively. Let $\Delta(\Psi_{\Phi(r)}),\ \Delta(F_{S(r)})$ denote the diagonal of $\Psi_{\Phi(r)}$ and $F_{S(r)}$, respectively, i.e., in terms of the notation which was introduced in Section 5,

$$\Delta(\Psi_{\Phi(r)}) = \Psi_{\Phi(r)}(0)$$

or alternatively, in terms of the diagonal transform which was discussed briefly just after formula (3.5)

$$\Delta(\Psi_{\Phi(r)}) = \Psi^{\wedge}_{\Phi(r)}(0).$$

The maximum entropy problem in this setting is to (1) evaluate

$$\lim_{r \to 1} \sup\{\Delta(\Psi_{\Phi(r)})^*\Delta(\Psi_{\Phi(r)}) : \Phi(r) \in \mathscr{C}\},$$

and (2) find those Φ which achieve this supremum, if any exist.

Theorem 6.1. *If $P \geqslant \varepsilon I$, then*

$$\lim_{r\to 1} \Delta(\Psi_{\Phi(r)})^* \Delta(\Psi_{\Phi(r)}) \leqslant \lim_{r\to 1} I_{\mathscr{E}_1} - C_2\left(P_r + C_2^* C_2\right)^{-1} C_2^* = I_{\mathscr{E}_1} - C_2\left(P + C_2^* C_2\right)^{-1} C_2^*, \tag{6.1}$$

where P_r is the Pick operator associated with the interpolation problem (4.10) *and given by the formula*

$$P_r = \sum_{j=0}^{\infty} \left(M^* r^{-1} Z_1^*\right)^j \left(C_1^* C_1 - C_2^* C_2\right)(r^{-1} Z_1 M)^j$$

(*recall that the latter sum has only $n+1$ terms, therefore P_r makes sense*) *with equality if*

$$\Phi = \widetilde{\Theta}_{12}\widetilde{\Theta}_{22}^{-1} = \widetilde{\Sigma}_{12} = (\Theta_{22} - \Theta_{12})(\Theta_{22} + \Theta_{12})^{-1},$$

which is upper triangular almost periodic bounded operator (*or, which is due to the Cayley transform is equivalent to, $S = \Sigma_{11} = \Theta_{12}\Theta_{22}^{-1}$, which, by formulas* (4.13) *and* (4.15), *is equal to*

$$S(r) = C_1(I - rZ_1M)^{-1}P^{-1}C_2^*\left\{I_{\mathscr{E}_1} + C_2(I - rZ_1M)^{-1}P^{-1}C_2^*\right\}^{-1},$$

which is upper triangular almost periodic strictly contractive operator).

Proof. The fact that $\Phi(S)$ is upper triangular almost periodic bounded operator (strictly contractive operator) is a consequence of the results of Section 4, since Φ admits the representation 5.2 with $\Omega = 0$. Therefore it is enough to prove the other assertion of the theorem. Under the given assumptions, every $\Phi \in \mathscr{C}$ can be expressed in the form (4.2) with $S \in \mathscr{K}$ and then dropped from the notation

$$\frac{\Phi^*(r) + \Phi(r)}{2} = \Psi^*(r)(I_{\mathscr{E}_1} - S^*(r)S(r))\Psi(r),$$

where

$$\Psi(r) = (I_{\ell^2_{\mathbb{C}}} - S(r))$$

belongs to $\mathscr{U}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$ and, by Theorem 4.4 (recall that $S(0) = 0$), is invertible with $\Psi(r)^{-1} \in \mathscr{U}(\ell^2_{\mathbb{C}}; \ell^2_{\mathbb{C}})$. Therefore,

$$\Psi_{\Phi(r)} = F_{S(r)}\Psi(r)$$

and

$$\Delta(\Psi_{\Phi(r)}) = \Delta(F_{S(r)})\Delta(\Psi(r)),$$

up to left unitary diagonal factors which do not affect the final calculation. Moreover, since $\Delta(\Psi(r)) = I_{\mathbb{C}}$, by Theorem 9.1 of [11],

$$\Delta(\Psi_{\Phi(r)}) = \Delta(F_{S(r)}) = \Delta(F_{\Omega})\Sigma_{21}^r(0)$$

(here the upper index r means that Σ^r is the corresponding to the l interpolation problem with index r Σ-operator) and hence

$$\Delta(\Psi_{\Phi(r)})^*\Delta(\Psi_{\Phi(r)}) = \Sigma^r_{21}(0)^*\Delta(F_\Omega)^*\Delta(F_\Omega)\Sigma^r_{21}(0).$$

Therefore, it follows that

$$\Delta(\Psi_{\Phi(r)})^*\Delta(\Psi_{\Phi(r)}) \leqslant \Sigma^r_{21}(0)^*\Sigma^r_{21}(0)$$

with equality if $\Omega = 0$, which is equivalent to the statement that $S = \Sigma^r_{11}$. The proof is now completed by checking that

$$\begin{aligned}\lim_{r\to 1} \Sigma^r_{21}(0)^*\Sigma^r_{21}(0) &= \lim_{r\to 1} I_{\mathscr{E}_1} - C_2\left(P_r + C_2^*C_2\right)^{-1}C_2^* \\ &= I_{\mathscr{E}_1} - C_2\left(P + C_2^*C_2\right)^{-1}C_2^*, \qquad (6.2)\end{aligned}$$

which is a straightforward calculation. □

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