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Surface homeomorphisms with zero-dimensional singular set

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Abstract

We prove that if f is an orientation-preserving homeomorphism of a closed orientable surface M^2 whose singular set $\Sigma(f)$ is totally disconnected, then f is topologically conjugate to a conformal transformation.

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1. Introduction

In a series of papers [23–28], Kérékjártó gave necessary and sufficient conditions for an orientation-preserving surface homeomorphism to be conjugate to a conformal isomorphism, answering a question of Brouwer [3]. For that purpose, he introduced the notion of regularity.

Let $f : X \to X$ be a homeomorphism of a compact metric space. A point $x \in X$ is *regular* if the family $\{f^n\}$ of all (positive and negative) iterates of f is equicontinuous at x, that is for all $\varepsilon > 0$ there exists $\delta > 0$ so that if $d(x, y) < \delta$ then $d(f^n(x), f^n(y)) < \varepsilon$ for all n. This property is obviously independent of the choice of the metric and invariant under topological conjugacy.

The purpose of this paper is to give a complete exposition of Kérékjártó's theory in a more general setting, with shorter and simpler proofs. Some of Kérékjártó's arguments have already been clarified [2,12,17,20,30,37]. However, up to the authors'

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knowledge, there is nowhere in the literature a complete, modern and elementary paper on Kérékjártó's theory. This is what we intend to do in the following.

Let f be a homeomorphism of a closed surface M^2. We define the *singular* set of f, noted $\Sigma(f)$ to be the closure of the set of nonregular points. One may ask what the connection is between $\Sigma(f)$ and the Julia set $J(f)$ of a holomorphic map of the Riemann sphere. For us, the only difference is that a point is regular if the family of all iterates of f, positive and negative, is equicontinuous whereas in the holomorphic case only positive iterates are required to form an equicontinuous family. The main result of this paper is the following:

Theorem 1.1. *Let f be an orientation-preserving homeomorphism of a closed orientable surface M^2 whose singular set, $\Sigma(f)$, is totally disconnected. Then*:

(1) *If $M^2 = \boldsymbol{S}^2$, f is conjugate to a linear fractional transformation.*
(2) *If $M^2 = \boldsymbol{T}^2$, f is periodic or f is conjugate to the map $(s,t) \mapsto (s+\alpha, t+\beta)$.*
(3) *If $\chi(M^2) < 0$, f is periodic.*

Using the fact that any periodic, orientation-preserving transformation of a closed surface is conjugate to a conformal transformation [6,10] Theorem 1.1 can be restated by saying that an orientation-preserving homeomorphism of a closed orientable surface is conjugate to a conformal transformation if and only if $\Sigma(f)$ is totally disconnected.

In order to be complete, we have also treated the case of orientation-reversing homeomorphisms, of nonorientable closed surfaces and of surfaces with boundary, orientable or not. We will show in particular that a homeomorphism of a surface with negative Euler characteristic is always periodic if its singular set is totally disconnected. We will give then the complete classification up to topological conjugacy of nonperiodic homeomorphisms with totally disconnected singular set on surfaces with nonnegative Euler characteristic: the sphere $\boldsymbol{S}^2$, the projective plane $\mathbb{RP}^2$, the disc $\boldsymbol{D}^2$, the torus $\boldsymbol{T}^2$, the Klein bottle $\boldsymbol{K}$, the annulus and the Mœbius band. The essential argument, once we know the result for closed orientable surface, is to reduce to that case by considering the two-fold orientation covering or by taking the double of the surface to remove boundary.

In general $\Sigma(f)$ is very large (and so the hypothesis that $\Sigma(f)$ is totally disconnected is very strong). There are very simple homeomorphisms of the sphere $\boldsymbol{S}^2$ with $\Sigma(f) = \boldsymbol{S}^2$. Let us present such an example.

Let h be the homeomorphism of the plane $\mathbb{R}^2$ which leaves each circle c_r, $r \in [0,\infty[$ centered at $(0,0)$ globally invariant and whose restriction to c_r is the rotation by angle $\phi(r)$ where $\phi\colon [0,\infty[\to \mathbb{R}$ is any continuous function which is constant on no interval. Let f be the homeomorphism of $\boldsymbol{S}^2$ obtained by extending f to infinity by a fixed point. Then, f is a homeomorphism of $\boldsymbol{S}^2$ such that $\Sigma(f) = \boldsymbol{S}^2$.

Another interesting example is the case of a C^1-structurally-stable diffeomorphism f of a surface S (that is, f satisfies Axiom A and the strong transversality condition). In that case, a point is regular if and only if it belongs to the intersection of the stable manifold of a periodic attractor with the unstable manifold of a periodic repellor. Indeed, a point which does not lie in such an intersection must be a limit point of stable or unstable manifolds of saddle points (according to the Shadowing Lemma [15]) but these manifolds belong to the singular set (according to the λ-Lemma [38]). There are examples of C^1-structurally-stable diffeomorphisms f on any surface M^2, which do not have any periodic attractor; each attractor of f is a nontrivial hyperbolic attractor. In that case $\Sigma(f) = M^2$.

However, it is remarkable that such a simple condition (f is a homeomorphism with $\Sigma(f)$ totally disconnected) implies such a strong rigidity (f is conjugate to a conformal transformation). In the same spirit (but in the opposite direction), let us recall a beautiful result obtained independently by Hiraide [18] and Jorge Lewowicz [31]. A homeomorphism is expansive if there is a constant $\alpha > 0$ such that, for any pair of distinct points (x, y) there exists $n \in \mathbb{Z}$ such that the distance between $f^n(x)$ and $f^n(y)$ is greater than α. Hiraide and Lewowicz showed that any expansive homeomorphism of a compact surface is conjugate to a pseudo-Anosov homeomorphism.

In the next section, we briefly review some well-known topological facts. Section 3 contains general results on the dynamics of a homeomorphism of a compact metric space. In Section 4, we study the case of surfaces with positive Euler characteristic. The main part of this section is devoted to the sphere. Section 5 contains the complete classification of regular homeomorphisms of surfaces with nonpositive Euler characteristic and in particular of the torus.

2. Preliminaries

Let (X, d) be a compact metric space, and let 2^X be the set of all nonempty closed subsets of X. The Hausdorff distance on 2^X is defined by

$$d_H(A, B) = \max\Big\{\sup_{\alpha\in A} d(a, B), \sup_{b\in B} d(b, A)\Big\}.$$

With this distance, 2^X is a compact metric space [34].

Let (A_n) be a sequence of nonempty closed subsets of X. We define $\operatorname{Lim\,Inf}(A_n)$ to be the set of points $x \in X$ such that every neighbourhood of x meets A_n for all but a finite number of values of n. In other words, a point x belongs to $\operatorname{Lim\,Inf}(A_n)$ if and only if there exists a sequence $x_n \in A_n$ converging to x. Similarly, we define $\operatorname{Lim\,Sup}(A_n)$ to be the set of points $x \in X$ such that every neighbourhood of x meets A_n for an infinite number of values of n. In other words, x belongs to $\operatorname{Lim\,Sup}(A_n)$ if and only if there exists a sequence $x_n \in A_n$ admitting a subsequence x_{n_i} converging to x. These two sets are closed and $\operatorname{Lim\,Inf}(A_n) \subset \operatorname{Lim\,Sup}(A_n)$. The sequence (A_n) is convergent for the Hausdorff metric if and only if $\operatorname{Lim\,Inf}(A_n) = \operatorname{Lim\,Sup}(A_n)$ [34].

A *continuum* is a nonempty, compact, connected metric space. It is *nondegenerate* provided that it contains more than one point. The proof of the following lemma may be found in [34].

Lemma 2.1. *Let* (A_n) *be a sequence of continua of* X *such that* $\operatorname{Lim\,Inf}(A_n) \neq \emptyset$, *then* $\operatorname{Lim\,Sup}(A_n)$ *is a continuum. In particular, the space of continua of* X, *denoted* $C(X)$, *is closed in* 2^X.

Let f be a homeomorphism of a compact metric space (X,d). Recall that a point $x \in X$ is regular if the family $\{f^n\}$ is equicontinuous at x. For every $\varepsilon > 0$, we let $\varphi(x,\varepsilon)$ be the least upper bound of positive numbers δ such that

$$d(x,y) < \delta \Rightarrow d\big(f^n(x), f^n(y)\big) < \varepsilon, \quad \forall n.$$

By its very definition $\varphi(x,\varepsilon) \leqslant \varepsilon$. In the introduction, we have defined the singular set of f, written $\Sigma(f)$ to be the closure of the set of points which are not regular. This set is clearly invariant under f and $\Sigma(f^n) = \Sigma(f)$ for all $n \neq 0$.

If $\Sigma(f)$ is empty, the family $\{f^n,\ n \in \mathbb{Z}\}$ is uniformly equicontinuous on X. Such homeomorphisms are called *regular* by Kérékjártó and they are described by the following well-known result [14].

Theorem 2.2. *Let* f *be a homeomorphism of a compact metric space* (X,d). *The following statements are equivalent:*

(1) *The family* $\{f^n,\ n \in \mathbb{Z}\}$ *is equicontinuous on* X.

(2) *The closure* G *of the family* $\{f^n,\ n \in \mathbb{Z}\}$ *is a compact subgroup of* $\operatorname{Homeo}(X)$ *with the* C^0*-uniform topology.*

(3) *There exists a metric on* X *compatible with its topology in which* f *is an isometry.*

Remark. (1) $\Rightarrow$ (2) is essentially Ascoli's theorem, whereas in (2) $\Rightarrow$ (3) an invariant metric can be constructed using Haar's measure on G.

A homeomorphism of a compact metric space is *recurrent* provided that for all $\varepsilon > 0$, there exists $n \neq 0$ such that $d(f^n, \mathrm{Id}) < \varepsilon$. Clearly, in view of Theorem 2.2, a regular homeomorphism of a compact metric space is recurrent.

3. Dynamics on a compact metric space

In this section, we investigate the connection between regularity and the limit sets of a homeomorphism f of a compact metric space X. Our primary interest is in such homeomorphisms which have a totally disconnected singular set. Recall that a space X is totally disconnected, provided any connected subset of it is a point or empty. The subject of this section may be compared to the one of [20,30].

Given $x \in X$, we set $\mathcal{O}(x,f) = \{f^n(x);\ n \in \mathbb{Z}\}$. We define similarly $\mathcal{O}^+(x,f)$ as the set of positive iterates of x and $\mathcal{O}^-(x,f)$ as the set of negative iterates of x. The ω-limit set of a point $x \in X$ is defined by $\omega(x,f) = \{\lim f^{n_k}(x),\ n_k \to +\infty\}$ as its

α-limit set by $\alpha(x, f) = \omega(x, f^{-1})$. The limit set of x is $\lambda(x, f) = \alpha(x, f) \cup \omega(x, f)$. These sets are invariant under f. A point x is *recurrent* if $x \in \lambda(x, f)$.

Lemma 3.1.

(1) *Let $x \in X$ be a regular point and suppose that there are sequences (x_i) and (n_i) such that $x_i \to x$ and $f^{n_i}(x_i) \to z$, then $f^{n_i}(x) \to z$.*

(2) *Let x and y be two regular points. Then $f^{n_i}(x) \to y$ iff $f^{-n_i}(y) \to x$.*

(3) *A regular point x is recurrent iff $\lambda(x, f)$ contains a regular point.*

(4) *If a regular point is recurrent, then $\omega(x, f) = \alpha(x, f) = \mathrm{cl}(\mathcal{O}(x, f))$.*

Proof. (1) Let $\varepsilon > 0$ and $\delta = \varphi(x, \varepsilon)$. For i large enough, we have $d(x_i, x) < \delta$ and $d(f^{n_i}(x_i), z) < \delta$ and hence $d(f^{n_i}(x), z) < 2\varepsilon$.

(2) Let $\varepsilon > 0$ and $\delta = \varphi(x, \varepsilon)$. For i large enough, we have $d(f^{n_i}(x), y) < \delta$ and hence $d(x, f^{-n_i}(y)) < \varepsilon$.

(3) Let $y \in \lambda(x, f)$ be a regular point. Let $\varepsilon > 0$ and $\delta = \varphi(y, \varepsilon)$. We can find $n > m > 0$ such that $d(y, f^n(x)) < \delta$ and $d(y, f^m(x)) < \delta$. Hence, $d(f^{-m}(y), f^{n-m}(x)) < \varepsilon$ and $d(f^{-m}(y), x) < \varepsilon$ and so $d(x, f^{n-m}(x)) < 2\varepsilon$.

(4) If x is recurrent then $x \in \omega(x, f)$ and $x \in \alpha(x, f)$ according to (2). Hence $\mathrm{cl}(\mathcal{O}(x, f)) = \omega(x, f) = \alpha(x, f)$. $\square$

Corollary 3.2. *The set of points of $X \setminus \Sigma(f)$ which are recurrent is open and closed in $X \setminus \Sigma(f)$.*

Proof. According to Lemma 3.1(1), the set of recurrent points of $X \setminus \Sigma(f)$ is closed in $X \setminus \Sigma(f)$. Let $x \in X \setminus \Sigma(f)$ be a recurrent point. We can find a neighbourhood $U \subset X \setminus \Sigma(f)$ of x such that $\lambda(y, f) \cap X \setminus \Sigma(f) \neq \emptyset$ for each $y \in U$. According to Lemma 3.1(3), each point of U is recurrent so the set of recurrent points of $X \setminus \Sigma(f)$ is also open. $\square$

Lemma 3.3. *Let $x \in X \setminus \Sigma(f)$. If x is recurrent, then for every neighbourhood U of x there exists an integer $N \geqslant 0$ such that*

$$\mathcal{O}(x, f) \subset \bigcup_{i=0}^{N} f^i(U).$$

Proof. Let $x \in X \setminus \Sigma(f)$ be a recurrent point. According to Lemma 3.1(4), we have:

$$\mathrm{cl}\left(\mathcal{O}(x, f)\right) = \omega(x, f) = \alpha(x, f).$$

Let U be a compact neighbourhood of x such that $U \subset X \setminus \Sigma(f)$ and let $V \subset U$ be an open neighbourhood of x. For each point $y \in \lambda(x, f) \cap U$ we can find by Lemma 3.1(2) an integer $n(y) > 0$ such that $f^{n(y)}(y) \in V$ and hence we can find an open neighbourhood V_y of y such that

$$f^{n(y)}(V_y) \subset V.$$

Let $V_1, V_2, \ldots, V_r$ be a finite subcover of the covering (V_y) of $\lambda(x,f) \cap U$. To each open set V_i there corresponds a positive integer n_i such that:

$$f^{n_i}(V_i) \subset V.$$

We let $N = \max\{n_i\}$. For any $z \in \lambda(x,f) \cap U$, there exists $n \in \{0, 1, \ldots, N\}$ such that

$$f^n(z) \in V.$$

Therefore, we can construct an increasing sequence $m_i \to +\infty$ such that:

- $m_{i+1} - m_i \leqslant N$, for all i,
- $f^{m_i}(x) \in V \subset U$, for all i

and we have thus:

$$\mathcal{O}(x,f) \subset \omega(x,f) = \mathrm{cl}\left(\mathcal{O}^+(x,f)\right) \subset \bigcup_{i=0}^{N} f^i(U). \qquad \square$$

Corollary 3.4. *Let $x \in X \setminus \Sigma(f)$. Then $\lambda(x,f) \cap \Sigma(f) \neq \emptyset$ iff $\lambda(x,f) \subset \Sigma(f)$.*

Proof. Let $x \in X \setminus \Sigma(f)$ and suppose $\lambda(x,f) \not\subset \Sigma(f)$. According to Lemma 3.1(3), x is recurrent and from Lemma 3.3, we get that $\lambda(x,f) = \mathrm{cl}(\mathcal{O}(x,f)) \subset X \setminus \Sigma(f)$. $\square$

Theorem 3.5. *Let f be a homeomorphism of a locally connected, compact metric space (X,d) such that $\Sigma(f)$ is totally disconnected. Then:*

(1) *For each $s \in \Sigma(f)$ which is not a regular point, there exists $x \in X \setminus \Sigma(f)$ such that s belongs to the limit set $\lambda(x,f)$ of x.*

(2) *For each $x_0 \in X \setminus \Sigma(f)$ and any sequence $n_k \in \mathbb{Z}$ such that $\lim f^{n_k}(x_0) = s \in \Sigma(f)$, one has in fact $f^{n_k}(x) \to s$ uniformly on a neighbourhood U of x_0 and the set*

$$E_s = \left\{x \in X \setminus \Sigma(f);\ \lim f^{n_k}(x) = s\right\}$$

is open and closed in $X \setminus \Sigma(f)$.

Proof. (1) Suppose that $s \in \Sigma(f)$ is not a regular point. There exist $\varepsilon > 0$, a sequence $z_p \to s$ and a sequence n_p of integers so that:

$$d\left(f^{n_p}(z_p), f^{n_p}(s)\right) \geqslant \varepsilon, \quad \forall p.$$

Since $\Sigma(f)$ is a compact, totally disconnected, proper subset of X (otherwise $X = \Sigma(f)$ is finite and therefore $X = \Sigma(f) = \emptyset$), we can find a neighbourhood U of $\Sigma(f)$, $U \neq X$, which is the union of nonintersecting open sets of diameter less than ε (cf. 7.10 of [34]). Let

$$K = \mathrm{cl}\left(\bigcup_{n\in\mathbb{Z}} f^n(X \setminus U)\right).$$

Suppose that $s \notin K$. As X locally connected, the components of $X \setminus K$ are open and since $X \setminus K \subset U$, all of them are of diameter less than ε. Furthermore, these components are permuted by f and so for p large enough, $f^n(z_p)$ and $f^n(s)$ are in the same component of $X \setminus K$ which implies $d(f^n(z_p), f^n(s)) < \varepsilon$ for all n. This is a contradiction.

Hence $s \in K$ and we can find sequences $x_n \in X \setminus U$ and $i_n \in \mathbb{Z}$ such that $f^{i_n}(x_n)$ converges to s. Extracting a subsequence if necessary we can assume that x_n converges to a point $x \in X \setminus U$ as $n \to \infty$. From Lemma 3.1(1), one obtains that $f^{i_n}(x) \to s$ as $n \to \infty$, that is $s \in \lambda(x, f)$.

(2) Let $x_0 \in X \setminus \Sigma(f)$, and suppose that $f^{n_k}(x_0) \to s \in \Sigma(f)$ as $k \to \infty$. According to Corollary 3.4, $x_0 \notin \omega(x_0, f)$. Choose a compact and connected neighbourhood U of x_0 such that $x \notin \omega(x, f)$ for all $x \in U$ (cf. Corollary 3.2 and Lemma 3.1(4)). According to Lemma 2.1

$$\operatorname{Lim\,Sup}\left(f^{n_k}(U)\right)$$

is a continuum since $s \in \operatorname{Lim\,Inf}(f^{n_k}(U))$. If $\operatorname{Lim\,Sup}(f^{n_k}(U))$ contains some point $y \in X \setminus \Sigma(f)$, there exists a subsequence m_k of n_k, and a sequence $x_k \in U$ such that $f^{m_k}(x_k)$ converges to y. From compactness of U, if necessary extracting a subsequence, we may assume that x_k converges to a point $x \in U$, and from Lemma 3.1(1), we have $\lim f^{n_k}(x) = y$ which leads to a contradiction since $\omega(x, f) \subset \Sigma(f)$ for all $x \in U$. Hence

$$\operatorname{Lim\,Sup}\left(f^{n_k}(U)\right) \subset \Sigma(f)$$

and is therefore reduced to $\{s\}$ since $\Sigma(f)$ is totally disconnected.

This shows in particular that the set E_s is open in $X \setminus \Sigma(f)$. That it is also closed results from Lemma 3.1(1). □

Corollary 3.6. *Let f be a homeomorphism of a closed surface M^2 with a totally disconnected singular set $\Sigma(f)$. Then $\Sigma(f)$ is empty unless $M^2 = S^2$ and in that case $\Sigma(f)$ contains no more than two points.*

Proof. Suppose that $\Sigma(f)$ is not empty and let $s \in \Sigma(f)$ be a nonregular point. According to Theorem 3.5, there exists $x_0 \in M^2 \setminus \Sigma(f)$ such that $s \in \lambda(x_0, f)$. To fix ideas, let us suppose that $s \in \omega(x_0, f)$ and we choose a sequence $n_k \to +\infty$ so that $\lim f^{n_k}(x_0) = s$. According to Theorem 3.5, the set

$$E_s = \left\{x \in M^2 \setminus \Sigma(f);\ \lim f^{n_k}(x) = s\right\}$$

is open and closed in $M^2 \setminus \Sigma(f)$ and the convergence is uniform on every compact subset of E_s. Since the connectedness of a surface is preserved by the removing of a totally disconnected subset, $E_s = M^2 \setminus \Sigma(f)$. Therefore, $\{s\} \subset \omega(x, f) \subset \Sigma(f)$ for all $x \in M^2 \setminus \Sigma(f)$. Let α be an arc joining x and $f(x)$ in $M^2 \setminus \Sigma(f)$, then:

$$\operatorname{Lim\,Sup}\left(f^n(\alpha)\right) = \bigcap_{n=0}^{+\infty} \overline{\bigcup_{k \geqslant n} f^k(\alpha)}$$

is a continuum which lies entirely in $\Sigma(f)$. Otherwise, there would be a point $y \in \alpha$ such that $\omega(y, f) \not\subset \Sigma(f)$ which is impossible by Corollary 3.4. Hence, $\operatorname{Lim\,Sup}(f^n(\alpha)) = \{s\}$ and $\omega(x, f) = \{s\}$ for every $x \in M^2 \setminus \Sigma(f)$.

If $\Sigma(f)$ is not reduced to s, we can find a nonregular point $s' \neq s$ and we prove similarly that $\alpha(x, f) = \{s'\}$ for all $x \in M^2 \setminus \Sigma(f)$. Therefore $\Sigma(f)$ cannot contain more than two points.

The argument above shows that $f^n(x) \to s$ as $n \to +\infty$ uniformly on all compact subsets of $M^2 \setminus \Sigma(f)$. If $M^2 \neq S^2$ there exists a closed curve J nonhomotopic to zero and we can choose such a curve in $M^2 \setminus \Sigma(f)$. Let U be a simply connected neighbourhood of s. For n large enough, $f^n(J) \subset U$ and hence $f^n(J)$ is null homotopic which gives a contradiction. $\square$

Remarks. Corollary 3.6 can be generalized easily to any compact manifold M of dimensional $\dim(M) \geqslant 2$:

(1) A compact and totally disconnected subset of a closed manifold M of dimension $\dim(M) \geqslant 2$ does not separate M. Hence, as in the proof of Corollary 3.6, the singular set, $\Sigma(f)$, of any homeomorphism of M contains less than two points or is not totally disconnected.

(2) Let f be a homeomorphism of a closed manifold M of dimension $\dim(M) \geqslant 2$. If there exists $i \in \{1, \ldots, n-1\}$ such that the homotopy group $\pi_i(M)$ is not $\{0\}$, and if $\Sigma(f)$ is totally disconnected, then $\Sigma(f) = \emptyset$. The proof is identical to the proof of Corollary 3.6.

4. The case of surfaces with $\chi(M^2) > 0$

Recall that every conformal automorphism of the Riemann sphere $\widehat{C}$ can be expressed as a fractional linear transformation

$$f(z) = (az + b)/(cz + d),$$

where the coefficients are complex numbers with determinant $ad - bc \neq 0$. Every nonidentity automorphism of this type has two distinct fixed points or one double fixed point in $\widehat{C}$. The nonidentity automorphisms of $\widehat{C}$ fall into three classes, as follows:

- An automorphism f is said to be *elliptic* if it has two distinct fixed points at which the modulus of the derivative is 1.
- The automorphism f is said to be *hyperbolic* if it has two distinct fixed points at which the modulus of the derivative is not 1.
- The automorphism f is *parabolic* if it has just a double fixed point.

It is an exercise to show that up to topological conjugacy there is only one model of parabolic automorphism, namely the translation $T(z) = z + 1$. There is only one model of hyperbolic automorphism, namely $H(z) = 2z$. But there is a one parameter family of elliptic transformations which are not topologically equivalent, namely the family $R_\alpha(z) = \mathrm{e}^{\mathrm{i}\alpha} z$.

In the first three paragraphs of this section we study the case of orientation-preserving homeomorphisms of the sphere and we prove the following theorem.

Theorem 4.1. *An orientation-preserving homeomorphism of the sphere S^2 is topologically conjugate to a linear fractionnal transformation iff its singular set contains no nondegenerate continuum or equivalently if its singular set contains at most two points. More precisely, the transformation is conjugate to an elliptic, parabolic or hyperbolic transformation according to whether the number of singular points is zero,* 1 *or* 2.

According to Corollary 3.6, the singular set $\Sigma(f)$ of a homeomorphism of the sphere which is totally disconnected contains no more than two points. The proof of Theorem 4.1 will be divided into three parts according as the number of singular points is zero, 1 or 2.

4.1. The parabolic case

Theorem 4.2. *An orientation-preserving homeomorphism of the sphere with exactly one singular point is conjugate to the map $z \mapsto z+1$ of the Riemann sphere.*

Proof. According to Theorem 3.5, we have:

$$\alpha(x,f) = \omega(x,f) = \{N\}, \quad \forall x \in S^2$$

and $f^n(x) \to N$ as $n \to \pm\infty$, uniformly on every compact subset of $S^2 \setminus \{N\}$. Hence the group $\langle f \rangle$ generated by f acts freely and properly on the plane $\Gamma = S^2 \setminus \{N\}$. The quotient space Γ_f is therefore a topological surface and the projection $\pi_f : \Gamma \to \Gamma_f$ is a regular covering. Since Γ is simply connected, the fundamental group of Γ_f is isomorphic to $\mathbb{Z}$. Hence, Γ_f is homeomorphic to the cylinder $\mathbb{R} \times S^1$. By uniqueness of the universal cover up to isomorphism, there exist a pair of homeomorphisms (h, H) and a commutative diagram:

$$\begin{tikzcd} \Gamma \arrow[r, "H"] \arrow[d] & \mathbb{R}^2 \arrow[d] \\ \Gamma_f \arrow[r, "h"] & S^1 \times \mathbb{R} \end{tikzcd}$$

In particular, H is a topological conjugacy between $f|_\Gamma$ and the translation $(x,y) \mapsto (x+1, y)$ of the plane $\mathbb{R}^2$. □

4.2. The hyperbolic case

Theorem 4.3. *An orientation-preserving homeomorphism of the sphere with exactly two singular points is conjugate to the map $z \mapsto 2z$ of the Riemann sphere.*

Proof. Let N and S be the two singular points. According to the proof of Corollary 3.6, $f^n(x) \to N$ for $n \to +\infty$ (respectively $f^n(x) \to S$ for $n \to -\infty$) uniformly on every compact subset of $S^2 \setminus \{S\}$ (respectively $S^2 \setminus \{N\}$). This shows that f acts freely and properly on the cylinder $\Gamma = \mathbf{S}^2 \setminus \{N, S\}$. Hence the quotient space $\Gamma_f = \Gamma/f$ is a topological surface and the projection π_f is a regular covering. Since f is orientation-preserving, Γ_f is orientable. Let $\gamma \subset \Gamma$ be a simple closed curve separating S and N. Since $f^n(\gamma)$ converges uniformly to S for positive integers, there exists $n \in \mathbb{N}$ such that $f^n(\gamma) \cap \gamma = \emptyset$. Since γ and $f^n(\gamma)$ are two nonintersecting essential simple closed curves, they bound a closed annulus $\mathcal{A}$ on the cylinder Γ. Hence Γ_{f^n} is a torus and since Γ_{f^n} is a cyclic regular covering of Γ_f, we also obtain that Γ_f is a torus $\mathbf{T}^2$. Since, up to isomorphism, there is only one regular covering $\pi : S^1 \times \mathbb{R} \to \mathbf{T}^2$ whose automorphism group is isomorphic to $\mathbb{Z}$, there exist a pair of homeomorphisms (h, H) and a commutative diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{H} & S^1 \times \mathbb{R} \\ \downarrow & & \downarrow \\ \Gamma_f & \xrightarrow{h} & \mathbf{T}^2 \end{array}$$

In particular, H is a topological conjugacy between $f|_\Gamma$ and the translation $(x, y) \mapsto (x, y+1)$ of the cylinder $S^1 \times \mathbb{R}$. □

4.3. *The elliptic case*

Theorem 4.4. *A regular, orientation-preserving homeomorphism of the sphere is conjugate to a rotation $R_\alpha(z) = \mathrm{e}^{\mathrm{i}\alpha} z$.*

The first step to this end is to show that around a fixed point of a regular homeomorphism of the sphere there are arbitrary small invariant simple closed curves. Compare the results of this paragraph to those of [2,12,17,37,39]. The following theorem is a very classical result in elementary conformal representation theory (see Theorem 2.6 of [36]), although it can be proved also by methods of plane topology [41]. Recall that a point $a \in K$ is a *cut point* of K if $K \setminus \{a\}$ is not connected.

Theorem 4.5. *Let K be a nondegenerate, locally connected continuum of $\mathbf{S}^2$ with no cut points. Then the boundary of each component of $\mathbf{S}^2 \setminus K$ is a simple closed curve.*

Lemma 4.6. *Let f be a regular homeomorphism of the sphere and let $D \subset \mathbf{S}^2$ be a closed disc. Then the compact set $K = \mathrm{cl}(\bigcup_{n\in\mathbb{Z}} f^n(D))$ is locally connected.*

Proof. Let us first recall that a compact metric space is locally connected if and only if for every $\varepsilon > 0$ it is the union of a finite number of compact connected sets of diameter less than ε.

Let $\varepsilon > 0$ be given. Choose a triangulation of D into a finite number of closed 2-cells, $e_1, \ldots, e_r$, each of which of diameter less than $\varphi(\varepsilon)$, where $\varphi(\varepsilon)$ was defined in Section 2. For each $n \in \mathbb{Z}$ we have

$$f^n(D) = \bigcup_{i=1}^{r} e_i^n,$$

where $\operatorname{diam}(e_i^n) < \varepsilon$ for $i = 1, \ldots, r$ (e_i^n stands for $f^n(e_i)$).

Let $\rho > 0$ so that each 2-cell e_i contains a disc $B(x_i, \rho)$ in its interior. Then

$$B\big(f^n(x_i), \varphi(\rho)\big) \subset f^n(e_i), \quad \forall i, \ \forall n.$$

Therefore the family $\{e_i^n\}$ contains only finitely many pairwise nonintersecting 2-cells. Let $\{e_{i_1}^{n_1}, \ldots, e_{i_p}^{n_p}\}$ be a finite collection of pairwise nonintersecting 2-cells of maximal cardinality. Then for every $n \in \mathbb{Z}$ and every $i \in \{1, \ldots, r\}$, there exists $j \in \{1, \ldots, p\}$ so that $e_{i_j}^{n_j} \cap e_i^n \neq \emptyset$. For each $k \in \{1, \ldots, p\}$, let M_k be the closure of the union of all 2-cells e_i^n which meet $e_{i_k}^{n_k}$. Then M_k is a compact connected set of diameter less than 3ε and

$$K = \bigcup_{k=1}^{p} M_k. \qquad \square$$

Corollary 4.7. *Let f be a regular homeomorphism of the sphere and let x be a fixed point for f. There exist arbitrarily small closed discs, invariant under f, which are neighbourhoods of x.*

Proof. Let $\varepsilon > 0$, $\delta = \varphi(\varepsilon)$ and $\eta = \varphi(\delta)$. Let D° be the open disc of center x and radius η and set

$$U = \bigcup_{n \in \mathbb{Z}} f^n(D^\circ).$$

Then U is an open connected subset of $B(x, \delta)$ and $f(U) = U$. By Lemma 4.6, $\overline{U}$ is locally connected and has no cut point since the closure of any connected open set of the sphere has no cut point. According to Theorem 4.5, the boundary of each component of $\boldsymbol{S}^2 \setminus \overline{U}$ is a simple closed curve. Since

$$f\big(\boldsymbol{S}^2 \setminus B(x, \varepsilon)\big) \subset \boldsymbol{S}^2 \setminus B(x, \delta),$$

the component of $\boldsymbol{S}^2 \setminus \overline{U}$ which contains $\boldsymbol{S}^2 \setminus B(x, \delta)$ is invariant under f. Its boundary γ is an invariant simple closed curve and the open disc Δ° bounded by γ and containing x is an invariant disc contained in $B(x, \varepsilon)$. $\square$

An orientation-preserving homeomorphism f of $\boldsymbol{S}^2$ has at least one fixed point. If furthermore, f is regular, there exists an invariant disc around this point. According to Brouwer's fixed point theorem, there is another fixed point and up to conjugacy, we can suppose that f fixes the two poles N and S of the sphere $\boldsymbol{S}^2$.

Let γ be an invariant simple closed curve under f. We will denote the rotation number of the restriction of f to γ by $\rho(\gamma, f)$. Readers not familiar with rotation numbers may refer to [8] for an excellent exposition of this notion.

Lemma 4.8. *Let f be a regular homeomorphism of the sphere and suppose that there exists an invariant simple closed curve γ such that $\rho(\gamma, f) = 0$. Then f is the identity map of $\boldsymbol{S}^2$.*

Proof. Recall that if $\rho(\gamma, f) = 0$, f has a fixed point on γ. The argument below in one dimension lower shows that a regular orientation-preserving homeomorphism of the circle with a fixed point is the identity map. Hence, $\gamma \subset \mathrm{Fix}(f)$. We will now prove that the connected component Γ of γ in $\mathrm{Fix}(f)$ is open and closed in $\boldsymbol{S}^2$ which will complete the proof.

Let $x \in \Gamma$, $\varepsilon > 0$, $(2\varepsilon < \mathrm{diam}(\gamma))$ and $\delta = \varphi(\varepsilon)$. For each ball $B(x, r)$ with $r < \delta$, we construct as in the proof of Corollary 4.7 an invariant simple closed curve γ_r which bounds a disc $\Delta_r \subset B(x, \varepsilon)$ containing x. Since γ_r meets Γ, $f|_{\gamma_r} = \mathrm{Id}$ and hence $\gamma_r \subset \Gamma$. But

$$\gamma_r \subset \mathrm{cl}\left(\bigcup_{n\in\mathbb{Z}} f^n(c_r)\right),$$

where c_r is the boundary of $B(x, r)$. Hence, for every point $y \in \gamma_r$, there exists a point $x \in c_r$ and a sequence (n_k) so that $y = \lim f^{n_k}(x)$ and we have $x = \lim f^{-n_k}(y) = y$ (cf. Lemma 3.1). Therefore $\gamma_r = c_r$ and $c_r \subset \Gamma$. Since r is arbitrary, this shows that $B(x, \delta) \subset \Gamma$. □

We now fix an invariant simple closed curve γ and let G be the closure of the group generated by f in $\mathrm{Homeo}(\boldsymbol{S}^2)$. As we have seen in Section 3.3, G is a compact commutative subgroup of regular homeomorphisms. Each element of G leaves the curve γ invariant and moreover we have:

Lemma 4.9. *The map $\rho_\gamma : G \to S^1$ given by $g \mapsto \rho(\gamma, g)$ induces a bicontinuous isomorphism between G and a compact subgroup of $S^1 = \mathbb{R}/\mathbb{Z}$.*

Proof. It is a classical fact that the rotation number of a circle map depends continuously on the map. Therefore $\rho_\gamma : G \to S^1$ is continuous and from the relation

$$\rho_\gamma(f^n) = n\rho_\gamma(f), \quad \forall n \in \mathbb{Z}$$

we deduce that $\rho_\gamma : G \to S^1$ is a group morphism. The injectivity of this map results from Lemma 4.8. Since G is compact, ρ_γ is also a homeomorphism of G onto a closed subgroup of S^1. □

According to Lemma 4.9, G is either isomorphic to a finite cyclic group or to S^1. In the first case, f is periodic and it is well known that f is actually conjugate to a rotation by an angle $2k\pi/n$ about the North–South axis [7,9,21].

In all of what follows we will suppose that $G \cong S^1$. In other words, we are given a continuous and faithful action $\Psi : S^1 \times \boldsymbol{S}^2 \to \boldsymbol{S}^2$. We will establish first:

Lemma 4.10. $\mathrm{Fix}(G)$ *is reduced to two points N and S, G acts freely on $\boldsymbol{S}^2 \setminus \{N, S\}$ and the orbit of every point of $\boldsymbol{S}^2 \setminus \{N, S\}$ is an essential simple closed curve in $\boldsymbol{S}^2 \setminus \{N, S\}$.*

Proof. Let γ be the invariant simple closed curve used to define the isomorphism ρ_γ. This curve separates the two fixed points N and S and no element of G other than Id has a fixed point on γ.

If G has another fixed point say x_0, then the G-orbit of a continuous path joining x_0 to a point of γ in $S^2 \setminus \{N, S\}$ gives a homotopy between γ and x_0 in $S^2 \setminus \{N, S\}$, contradicting the fact that γ is essential. Hence every G-orbit other than N and S is a simple closed curve and this curve is necessarily essential in $S^2 \setminus \{N, S\}$.

Suppose that an element $g_0 \in G$ has a fixed point x_0 in $S^2 \setminus \{N, S\}$ and let $\gamma(x_0)$ be the orbit of x_0 under G. Then $g_0 = \mathrm{Id}$ according to Lemma 4.8. That is, G acts freely on $S^2 \setminus \{N, S\}$. $\square$

Lemma 4.11. *For all $\varepsilon > 0$, there exists $\delta > 0$ so that, if x and y are two distinct points on a G-orbit γ and $d(x, y) < \delta$, then at least one of the two arcs delimited by x and y on γ has a diameter less than ε.*

Proof. Let $\varepsilon > 0$ and γ_N (respectively γ_S) be a G-orbit in an ε-neighbourhood of N (respectively S). Let A be the G-invariant annulus bounded by γ_N and γ_S. We have only to prove the lemma for G-orbits which lie in A. We can find $\mu > 0$ such that:

$$d\big(x, g(x)\big) < \varepsilon/2$$

for all $x \in A$ and all $g = \Psi(\theta, \cdot)$ with $\theta \in I_\mu = [-\mu, +\mu]$. Since G acts freely on A, there exists $\delta > 0$ such that:

$$d\big(x, g(x)\big) \geqslant \delta$$

for all $x \in A$ and all $g \in S^1 \setminus I_\mu$. Now, if x and y are two distinct points on a G-orbit γ such that $d(x, y) < \delta$, then $y = \Psi(\theta, x)$ with $\theta \in I_\mu$ and the arc $\Psi([0, \mu], x)$ has diameter less than ε. $\square$

In order to complete the proof of Theorem 4.4, we are going to show the existence of a "transversal" arc to the G-orbit, which will permit us to construct a conjugacy between G and the group of Euclidean rotations about the South–North axis (see also the work of Whitney [40] for the existence of transversal arc to a family of curves). More precisely:

Lemma 4.12. *Given two points x and y which lie on distinct G-orbits, there exists a simple arc a joining x and y which meets each G-orbit in at most one point.*

Let γ and γ' be two simple closed curves which separate N and S. We write $\gamma \leqslant \gamma'$ (respectively $\gamma < \gamma'$) iff γ is contained in the closed (respectively open) disc bounded by γ' and containing S. This relation induces a total order on the set of G-orbits (with the convention that $S \leqslant \gamma$ and $\gamma \leqslant N$ for all G-orbit γ).

For any point $x \in S^2$, we let $\gamma(x)$ be the G-orbit of x. With this notation, we have the following definition. A finite collection of indexed points $\{x_0, x_1, \ldots, x_n\}$ such that $d(x_k, x_{k+1}) < \mu$ and $\gamma(x_k) < \gamma(x_{k+1})$ is called a *monotone μ-chain* from x_0 to x_n. We will establish first:

Lemma 4.13. *For all $\varepsilon > 0$, there exists $\delta > 0$ so that two points x and y with $d(x, y) < \delta$ and $\gamma(x) < \gamma(y)$ can be joined by a monotone μ-chain of diameter less than ε for all $\mu > 0$.*

Proof. Let $\varepsilon > 0$ and $\delta > 0$ as in Lemma 4.11. Suppose that x, y are two points lying on distinct G-orbits and such that $d(x,y) < \delta/2$. Given $\mu > 0 (\mu < \delta)$, we can find a finite sequence of G-orbits

$$\gamma_0 = \gamma(x) < \gamma_1 < \cdots < \gamma_n = \gamma(y)$$

such that $d_H(\gamma_k, \gamma_{k+1}) < \mu/3$.

Let $\overline{xy}$ be the geodesic arc connecting x and y whose length is less than $\delta/2$. This arc meets each intermediate curve γ_k. Hence, we can choose, for each k, a point x_k of $\overline{xy}$ on γ_k. If $d(x_k, x_{k+1}) < \mu$ for all k, we are done. If not, we are going to show that for each pair $\{x_r, x_{r+1}\}$ for which $d(x_r, x_{r+1}) \geqslant \mu$, we can join x_r and x_{r+1} by a monotone μ-chain which lies in a 2ε-neighbourhood of $\overline{xy}$. By joining together such chains, we obtain a monotone μ-chain from x to y of diameter less than $4\varepsilon + \delta/2$.

Choose a point x'_{r+1} on γ_{r+1} such that $d(x_r, x'_{r+1}) < \mu/3$. Hence, $d(x_{r+1}, x'_{r+1}) < \delta$ and one of the arcs delimited by these two points on $\gamma_{r+1}, x_{r+1}x'_{r+1}$, has a diameter less than ε.

Divide this arc into s subarcs of diameter less than $\mu/3$ and call the intermediate points

$$x'_{r+1} = z^0_{r+1}, z^1_{r+1}, \ldots, z^s_{r+1} = x_{r+1}.$$

We choose some curves

$$\gamma_r = \gamma^0 < \gamma^1 < \cdots < \gamma^s = \gamma_{r+1}.$$

For each $k \in \{1, \ldots, s-1\}$, $d_H(\gamma^k, \gamma_{r+1}) < \mu/3$ and we can choose on each intermediate curve γ^k a point x^k_{r+1} such that $d(x^k_{r+1}, z^k_{r+1}) < \mu/3$. Hence

$$x^0_{r+1} = x_r, x^1_{r+1}, \ldots, z^s_{r+1} = x_{r+1}$$

is a monotone μ-chain. □

Proof of Lemma 4.12. From Lemma 4.13, we can choose a sequence of numbers $\delta_n > 0$ so that any two points with distance $d(x,y) < \delta_n$ can be joined, for all $\mu > 0$ by a monotone μ-chain of diameter less than $1/2^n$.

We start by choosing a monotone δ_0-chain X_0 from x to y. Inductively, once X_n has been defined, we join each consecutive pair $\{x^n_k, x^n_{k+1}\}$ of X_n by a monotone δ_{n+1}-chain of diameter less than $1/2^{n+1}$ to obtain X_{n+1} and we set

$$X = \bigcup_{n \in \mathbb{N}} X_n.$$

It is then a standard fact (Theorem 2.27 of [19]) that the closure $\overline{X}$ of X in $\boldsymbol{S}^2$ is a simple arc joining x and y with the required property. □

To complete the proof of Theorem 4.4, we choose an arc α given by Corollary 4.12 from N to S, and we let $x(r)$, $r \in [0, +\infty]$ be a parametrization of this arc. The map

$$h : r\mathrm{e}^{\mathrm{i}\theta} \mapsto \Psi(\theta, x(r))$$

is the required conjugacy between G and the group of Euclidean rotations about the z-axis.

4.4. *Orientation-reversing homeomorphisms of the sphere*

Every orientation-reversing conformal automorphism of the Riemann sphere $\widehat{C}$ is of the form:

$$f(z) = (a\overline{z} + b)/(c\overline{z} + d),$$

where the coefficients satisfy $ad - bc \neq 0$. We call them fractional reflections following Maskit [32]. The fixed point set of a fractional reflection is either empty, one point, two points or a circle in $\widehat{C}$. Fractional reflections are classified by the number of fixed points:

- A transformation f with a circle of fixed points is a *reflection*. It is topologically conjugate to the map $z \mapsto \overline{z}$.
- A transformation with exactly two fixed points is *semi-hyperbolic*. It is topologically conjugate to the map $z \mapsto 2\overline{z}$.
- A transformation with exactly one fixed point is *semi-parabolic*. It is topologically conjugate to the map $z \mapsto \overline{z} + 1$.
- A transformation with no fixed points is *semi-elliptic*. It is topologically conjugate to the map $z \mapsto \mathrm{e}^{\mathrm{i}\theta}/\overline{z}$, with $\theta \neq 0 \bmod 2\pi$.

Theorem 4.14. *An orientation-reversing* homeomorphism *of the sphere S^2 with a totally disconnected singular set is topologically conjugate to a fractional reflection.*

Proof. (1) Suppose first that the singular $\Sigma(f)$ is empty. Then $\Sigma(f^2) = \Sigma(f) = \emptyset$ and according to Theorem 4.1, we may suppose after a topological conjugacy if necessary that f^2 is a Euclidean rotation around the vertical axis.

If f has a fixed point, there are arbitrarily small closed discs invariant under f (cf. Corollary 4.7). On the boundary of each of these discs, f reverses the order and has therefore two fixed points. In particular, f and hence f^2 have an infinite number of fixed points and therefore $f^2 = \mathrm{id}$. In that case f is conjugate to a reflection [7].

If f has no fixed points, f permutes the two fixed points of f^2. If f^2 is a periodic rotation, we refer also to [7] for a proof that f is conjugate to the map $z \mapsto \mathrm{e}^{2\mathrm{i}\pi p/q}/\overline{z}$. If f^2 is an irrational rotation by angle 2α, the family of simple closed curves invariant under f^2 is unique. Hence, f permutes these curves and induces a continuous, reversing-order involution on $S^2/G \cong [0,1]$ where G is the closure of the group generated by f^2. Therefore one and exactly one of these curves is invariant under f. This curves divides the sphere into exactly two discs which are permuted by f and each of them contains a fixed point of f^2. In one of these discs, we choose a "transverse" arc which joins the fixed point of f^2 to the boundary of this disc. We map this arc onto an arc with similar properties relatively to the map $z \mapsto \mathrm{e}^{2\mathrm{i}\pi\alpha}/\overline{z}$ and extend this map by iteration under f. This map is clearly well-defined and extends into a topological conjugacy between f and the map $z \mapsto \mathrm{e}^{2\mathrm{i}\pi\alpha}/\overline{z}$.

(2) If $\Sigma(f)$ contains exactly one point N then $f(N) = N$ and the group generated by f acts freely and properly on $S^2 \setminus \{N\}$. The quotient space Γ_f is homeomorphic to the

open Mœbius strip and we can show as in Section 4.1, that f is conjugate to the map $z \mapsto \overline{z} + 1$.

(3) If $\Sigma(f)$ contains exactly two points N and S, then $f(N) = N$ and $f(S) = S$. The group generated by f acts freely and properly on the cylinder $\Gamma = S^2 \setminus \{N, S\}$. The quotient space Γ_f is homeomorphic to the Klein bottle and we can show as in Section 4.2, that f is conjugate to the map $z \mapsto 2\overline{z}$. □

We conclude this paragraph by stating the analogous result for the closed disc $\boldsymbol{D}^2$. Let f be a homeomorphism of $\boldsymbol{D}^2$ whose singular set $\Sigma(f)$ is totally disconnected. The same property holds for the homeomorphism induced on the sphere $\boldsymbol{S}^2$ viewed as the double of the disc. According to Theorem 4.1, f has at most two singular points which lie necessarily on the boundary of $\boldsymbol{D}^2$. The proofs which have been given for the sphere may be adapted in the case of the disc without introducing any new subtlety to establish the following:

Theorem 4.15. *A homeomorphism of the closed disc with a totally disconnected singular set is topologically conjugate to a fractional linear transformation or a fractional reflection of the disc according to whether it is orientation-preserving or orientation-reversing.*

4.5. The projective plane

Let f be a homeomorphism of the projective plane $\mathbb{RP}^2$. If $\Sigma(f)$ is totally discontinuous, $\Sigma(f)$ must be empty according to Corollary 3.6. Let $\widetilde{f}$ be the unique orientation-preserving lift of f to the universal cover $\boldsymbol{S}^2$ of $\mathbb{RP}^2$. Then $\widetilde{f}$ is a regular homeomorphism which commutes with the covering involution $s : z \mapsto -1/\overline{z}$.

If $\widetilde{f}$ is nonperiodic, the closure of the group generated by $\widetilde{f}$, G, is isomorphic to $\boldsymbol{S}^1$ and there is a unique family of simple closed curves invariant under $\widetilde{f}$. Since s commutes with $\widetilde{f}$, s permutes these curves and induces a continuous, orientation-reversing involution of $\boldsymbol{S}^2/G \cong [0,1]$. One and exactly one of these curves is invariant under s. We leave it to the reader to show that in that case, the conjugacy between $\widetilde{f}$ and the standard rotation $z \mapsto e^{2i\pi\theta} z$ may be chosen to commute with s. In other words, f and the standard map induced on $\mathbb{RP}^2$ by $z \mapsto e^{2i\pi\theta} z$ are conjugate on $\mathbb{RP}^2$.

If $\widetilde{f}$ is periodic, then $\widetilde{f}$ is conjugate to a periodic rotation. The quotient space $\boldsymbol{S}^2/\widetilde{f}$ is a sphere and s induces an orientation-reversing involution of this sphere. Therefore, we can find a simple closed curve on $\boldsymbol{S}^2$, which separates the two fixed points of $\widetilde{f}$ and which is invariant both by $\widetilde{f}$ and s. These considerations may be used to show that in that case also an equivariant conjugacy between $\widetilde{f}$ and $z \mapsto e^{2i\pi p/q} z$ can be constructed. We have finally proven the following:

Theorem 4.16. *Let f be a regular homeomorphism of the projective plane. Then f is topologically conjugate to a standard rotation of the projective plane.*

5. The case of surfaces with $\chi(M^2) \leqslant 0$

Let f be a homeomorphism of a compact surface M^2 with $\chi(M^2) \leqslant 0$. If $\Sigma(f)$ contains no nondegenerate continuum, f is regular everywhere according to Corollary 3.6.

5.1. General results on regular homeomorphisms of surfaces

Let M^2 be a closed orientable surface of genus $g \geqslant 1$ and $\pi : \widetilde{M}^2 \to M^2$ the universal cover of M^2. We can identify $\widetilde{M}^2$ either to the Euclidean plane $\mathbb{R}^2$ or to the Poincaré disc $\boldsymbol{D}$ in such a way that M^2 is homeomorphic to the quotient of $\widetilde{M}^2$ by a discrete subgroup Γ of Euclidean translations or hyperbolic isometries according to whether $\widetilde{M}^2$ is $\mathbb{R}^2$ or $\boldsymbol{D}$. The metric we shall use on M^2 is the quotient metric defined on $\widetilde{M}^2/\Gamma$ by:

$$d(\pi(x), \pi(y)) = \inf_{g,h \in \Gamma} \tilde{d}(g \cdot x, h \cdot y)$$

where $\tilde{d}$ is the natural metric on $\widetilde{M}^2$.

There is another metric on $\widetilde{M}^2$ that we shall use in the following, namely the spherical metric. The Alexandroff compactification of $\widetilde{M}^2$, $\widetilde{M}^2 \cup \{\infty\}$, is homeomorphic to the sphere $\boldsymbol{S}^2$. Hence, the standard metric of $\boldsymbol{S}^2$ induces a metric that we shall call $\eth$ on $\widetilde{M}^2$. These two metrics $\tilde{d}$ and $\eth$ are not uniformly equivalent on $\widetilde{M}^2$ but $\mathrm{Id} : (\widetilde{M}^2, \tilde{d}) \to (\widetilde{M}^2, \eth)$ is uniformly continuous.

Lemma 5.1. *Let f be a regular homeomorphism of a closed orientable surface M^2 of genus $g \geqslant 1$ and $\tilde{f}$ be any lift of f to the universal cover $\widetilde{M}^2$ of M^2. If $\bar{f}$ is the unique continuous extension of $\tilde{f}$ to $\boldsymbol{S}^2 = \widetilde{M}^2 \cup \{\infty\}$ then*

$$\Sigma(\bar{f}) \subset \{\infty\}.$$

Proof. Let $\tilde{f}$ be any lift of f on $\widetilde{M}^2$. Every point $\tilde{x}$ is regular under $\tilde{f}$ for the metric $\tilde{d}$. Let $\eth$ be the standard metric on $\boldsymbol{S}^2$. Then, $\mathrm{Id} : (\widetilde{M}^2, \eth) \to (\widetilde{M}^2, \tilde{d})$ is continuous and $\mathrm{Id} : (\widetilde{M}^2, \eth) \to (\widetilde{M}^2, \eth)$ is uniformly continuous. Therefore, a point $\tilde{x}$ which is regular for $(\tilde{f}, \tilde{d})$ is regular for $(\bar{f}, \eth)$ and hence $\Sigma(\bar{f}) \subset \{\infty\}$. $\square$

Corollary 5.2. *A regular homeomorphism of a closed orientable surface of genus $g \geqslant 1$ which is homotopic to the identity and which has a fixed point is the identity.*

Proof. Let $\tilde{f}$ be any lift of f on $\widetilde{M}^2$ which has a fixed point. Since f is homotopic to the identity, $\tilde{f}$ commutes with all covering translations and has therefore an infinite number of fixed points. According to Lemma 5.1 and Theorem 4.1, $\bar{f}$ and hence $\tilde{f}$ must be equal to the identity, which completes the proof. $\square$

Theorem 5.3. *A regular homeomorphism of a compact surface of negative Euler characteristic is periodic.*

Proof. If the boundary of M^2 is not empty, the natural extension of f to the double DM^2 of M^2 is still regular and since $\chi(DM^2) = 2\chi(M^2)$, we are reduced to prove

Theorem 5.3 for closed surfaces. Moreover, by passing to the orientation covering of M^2 and by considering f^2 instead of f if necessary, we may assume that M^2 is orientable and that f is orientation-preserving. So, let f be a regular orientation-preserving of a closed orientable surface. Since f is recurrent we can find a positive integer n such that f^n is homotopic to the identity and since $\chi(M^2) < 0$, Lefschetz's formula implies that f^n has a fixed point. According to Corollary 5.2, f^n is equal to the identity, which completes the proof. □

5.2. *Orientation-preserving homeomorphisms of the torus*

The *translations* of the torus are the maps of the torus induced by standard translations $\tau_{\alpha,\beta} : (s,t) \mapsto (s+\alpha, t+\beta)$ of the plane. Each of these maps is a regular transformation of the torus and it is periodic if and only if $(\alpha,\beta) \in \mathbb{Q}^2$. Moreover, two such maps $\tau_{\alpha,\beta}$ and $\tau_{\delta,\gamma}$ are topologically conjugate if and only if the two vectors (α,β) and (δ,γ) can be mapped one onto the other by a matrix $A \in GL(2,\mathbb{Z})$. The aim of this paragraph is to establish the following:

Theorem 5.4. *A regular, orientation-preserving and nonperiodic homeomorphism of the torus $\boldsymbol{T}^2$ is topologically conjugate to a nonperiodic translation of the torus.*

Remark. A complete classification of periodic transformations of the torus has been given by Brouwer [4,5,43]. We will not give here the list of them which can be find in these references.

Let f be a regular orientation-preserving homeomorphism of the torus and let $A \in SL(2,\mathbb{Z})$ be the induced matrix on $\pi_1(\boldsymbol{T}^2) \cong H_1(\boldsymbol{T}^2) \cong \mathbb{Z}^2$. Since f is recurrent, we get that $A^n = \mathrm{id}$ for some $n > 0$.

If $A \neq \mathrm{id}$ then the Lefschetz number of f, $L(f) = 2 - \mathrm{Tr}(A) \neq 0$ and f has a fixed point. Hence, $f^n = \mathrm{id}$ according to Corollary 5.2.

If $A = \mathrm{Id}$, then for any lift $\tilde{f}$ of f on $\mathbb{R}^2$ the following relation holds:

$$\tilde{f}(\tilde{x} + v) = \tilde{f}(\tilde{x}) + v, \quad \forall \tilde{x} \in \mathbb{R}^2,\ \forall v \in \mathbb{Z}^2$$

in other words $\tilde{f}$ commutes with integer translations and $\tilde{f}(\tilde{x}) - \tilde{x}$ is uniformly bounded on $\mathbb{R}^2$.

Lemma 5.5. *Let f be a regular and orientation-preserving homeomorphism of the torus which acts trivially on $\pi_1(\boldsymbol{T}^2)$ and let $\tilde{f}$ be any lift of f on $\mathbb{R}^2$. Then*

(1) $\theta(\tilde{f}, \tilde{x}) = \lim((\tilde{f}^n(\tilde{x}) - \tilde{x})/n)$ *exists and is independent of $\tilde{x}$. We shall call it the translation vector of $\tilde{f}$ and denote it by $\theta(\tilde{f})$.*

(2) $\theta(\tilde{f}) = (0,0)$ *iff $\tilde{f}$ has a fixed point.*

Proof. (1) Let K_n be the closure of the set

$$\left\{ \frac{\tilde{f}^m(\tilde{x}) - \tilde{x}}{m};\ m \geqslant n,\ \tilde{x} \in \mathbb{R}^2 \right\}.$$

Since the map $\widetilde{f}(\widetilde{x}) - \widetilde{x}$ is bounded on $\mathbb{R}^2$, (K_n) is a decreasing sequence of compact sets. We are going to show that $\operatorname{diam}(K_n) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since f is regular and thus recurrent, we can find a positive integer r and an integer vector v such that:

$$\|\widetilde{f}^r(\widetilde{x}) - (\widetilde{x} + v)\| < \varepsilon, \quad \forall \widetilde{x} \in \mathbb{R}^2$$

and therefore

$$\left\| \frac{\widetilde{f}^{kr}(\widetilde{x}) - \widetilde{x}}{kr} - \frac{v}{r} \right\| < \frac{\varepsilon}{r}, \quad \forall k > 0,\ \forall \widetilde{x} \in \mathbb{R}^2.$$

Let M be a bound of $\|\widetilde{f}(\widetilde{x}) - \widetilde{x}\|$ on $\mathbb{R}^2$. We have then:

$$\left\| \frac{\widetilde{f}^{n}(\widetilde{x}) - \widetilde{x}}{n} - \frac{v}{r} \right\| < \frac{\varepsilon}{r} + \frac{2rM}{n}, \quad \forall n \geqslant r,\ \forall \widetilde{x} \in \mathbb{R}^2$$

which shows that $\operatorname{diam}(K_n) < 2\varepsilon$ for n big enough. Therefore:

$$\frac{\widetilde{f}^{n}(\widetilde{x}) - \widetilde{x}}{n}$$

converges uniformly on $\mathbb{R}^2$ to some constant vector $\theta(\widetilde{f})$ as $n \to +\infty$.

(2) If $\widetilde{f}$ has a fixed point, then clearly $\theta(\widetilde{f}) = 0$. Conversely, if $\widetilde{f}$ has no fixed point, then according to Lemma 5.1 and Theorem 4.1, $\widetilde{f}$ has only wandering points. Hence there exists n_0 such that for every $n \geqslant n_0$, $\|\widetilde{f}^n(0,0)\| > 1$. Choose a positive number $\varepsilon < 1/2$, and let $n_1 > n_0$ and $v \in \mathbb{R}^2$ such that:

$$\|\widetilde{f}^{n_1}(\widetilde{x}) - (\widetilde{x} + v)\| < \varepsilon, \quad \forall \widetilde{x} \in \mathbb{R}^2.$$

This inequality shows first that v cannot be the vector $(0,0)$, since $\widetilde{f}^{n_1}(0,0)$ is not contained in the ball of center $(0,0)$ and of radius ε. Then, we have:

$$\left\| \frac{\widetilde{f}^{kn_1}(\widetilde{x}) - \widetilde{x}}{kn_1} - \frac{v}{n_1} \right\| < \frac{\varepsilon}{n_1}, \quad \forall k > 0,\ \forall \widetilde{x} \in \mathbb{R}^2$$

and letting $k \to +\infty$, we obtain:

$$\left\| \theta(\widetilde{f}) - \frac{v}{n_1} \right\| \leqslant \frac{\varepsilon}{n_1},$$

which shows that $\theta(\widetilde{f})$ cannot be zero. $\square$

Remark. This lemma is still true if we replace the statement f regular by f recurrent which is a weaker hypothesis. In fact, the proof of the first part of the lemma does indeed only use this hypothesis. We can also remark that (1) says precisely that the rotation set defined by Misiurewicz and Zieman in [33] is reduced to a point for a recurrent homeomorphism of the torus.

To prove (2), we need to know the fact that an orientation-preserving and fixed point free homeomorphism of the plane has only wandering points which is a corollary of Brouwer's Lemma on translation arcs [11,16] which we have not used here. (2) can be considered as a particular case of a result of Franks [13].

Let $\widetilde{g}$ be another lift of f. Then $\theta(\widetilde{f}) - \theta(\widetilde{g}) \in \mathbb{Z}^2$ and the class of $\theta(\widetilde{f})$ modulo $\mathbb{Z}^2$ is independent of the particular choice of the lift $\widetilde{f}$. We shall call it the *rotation vector* of f and denote it by $\rho(f)$.

As in Theorem 2.2, let G be the closure of the family $\{f^n;\ n \in \mathbb{Z}\}$. G is a commutative compact group of regular homeomorphisms of $\boldsymbol{T}^2$ and each element of G acts trivially on the fundamental group of $\boldsymbol{T}^2$. Moreover, we have:

Lemma 5.6. *Let f be a regular homeomorphism of the torus $\boldsymbol{T}^2$ acting trivially on the fundamental group, and let G be the closure of the group generated by f. Then the map $\rho: G \to \boldsymbol{T}^2$, $g \mapsto \rho(g)$ induces a bicontinuous isomorphism from G onto a compact subgroup of $\boldsymbol{T}^2$.*

Proof. We will show first that ρ is a group morphism. Let $g, h \in G$ and let $\widetilde{g}, \widetilde{h}$ be lifts of g and h, respectively. Since g and h commute, there exists $v \in \mathbb{Z}^2$ such that

$$(\widetilde{h}^{-1} \circ \widetilde{g} \circ \widetilde{h}) - \widetilde{g} = v.$$

Since $\widetilde{h}$ and $\widetilde{g}$ commute with the integer translations and since other lifts are obtained by composing $\widetilde{g}$ and $\widetilde{h}$ with integer translations, one shows easily that the vector v does not depend on the lifts $\widetilde{g}$ and $\widetilde{h}$. Furthermore, from the commutativity of $\widetilde{g}$ and $\widetilde{h}$ with the integer translations, one obtains inductively

$$\widetilde{h}^{-n} \circ \widetilde{g} \circ \widetilde{h}^n - \widetilde{g} = nv$$

and this relation also holds for any other lift of g and h^n. Since h is recurrent, there exists $n > 0$ such that h^n is arbitrarily close to identity. Hence, we can find a lift of h^n close to identity and therefore $v = 0$; that is $\widetilde{g}$ and $\widetilde{h}$ commute. Hence, we can write:

$$\frac{(\widetilde{g} \circ \widetilde{h})^n(x) - x}{n} = \frac{\widetilde{g}(\widetilde{h}^n(x)) - \widetilde{h}^n(x)}{n} + \frac{\widetilde{h}^n(x) - x}{n}$$

and since $(\widetilde{g}^n(x) - x)/n$ converges uniformly to $\theta(\widetilde{g})$ on $\mathbb{R}^2$, we get

$$\theta(\widetilde{g} \circ \widetilde{h}) = \theta(\widetilde{g}) + \theta(\widetilde{h}).$$

That is $\rho: G \to \boldsymbol{T}^2$ is a group morphism. According to Lemma 5.5 and Corollary 5.2, this morphism is necessarily injective. The continuity of ρ results from the fact that

$$\|\widetilde{g} - \mathrm{Id}\| < \varepsilon \Rightarrow \|\theta(\widetilde{g})\| \leqslant \varepsilon. \qquad \square$$

A compact subgroup of $\boldsymbol{T}^2$ is either a finite group, $\boldsymbol{T}^2$ or the product of S^1 by a finite cyclic group [1]. In the first case f is periodic. We shall leave aside the third case until the end of this section.

Lemma 5.7. *Assume that $\rho(G) = \boldsymbol{T}^2$. Then there is a topological conjugacy between G and the group of translations of $\boldsymbol{T}^2$.*

Proof. Fix a point x_0 in $\boldsymbol{T}^2$ and let $\phi:\boldsymbol{T}^2 \to \boldsymbol{T}^2$ be defined as follows. For each $t \in \boldsymbol{T}^2$ there is a unique $g_t \in G$ with $\rho(g_t) = t$ and we define a continuous map ϕ by the following:

$$\phi(t) = g_t(x_0).$$

If $\phi(t) = \phi(s)$, then $(g_s^{-1} \circ g_t)(x_0) = x_0$ and so $g_s^{-1} \circ g_t$ is the identity map (Corollary 5.2). As ρ is an isomorphism this implies that $s = t$. Hence ϕ is one-to-one and is thus a homeomorphism of $\boldsymbol{T}^2$ onto $\phi(\boldsymbol{T}^2) = \boldsymbol{T}^2$, by the invariance of domain.

It remains to be shown that ϕ is a conjugacy between the group of translations and G. Given $s \in \boldsymbol{T}^2$, we denote by τ_s the translation by s. For every $t \in \boldsymbol{T}^2$, we have

$$\phi \circ \tau_s(t) = \phi(t+s) = g_{s+t}(x_0) = g_s \circ g_t(x_0) = g_s\bigl(\phi(t)\bigr)$$

which completes the proof. □

Remark. We have proved in fact that every faithful and continuous action of $\boldsymbol{T}^2$ on $\boldsymbol{T}^2$ is isomorphic to the standard action.

From now on, we shall assume that $\rho(G)$ is (up to a linear conjugacy of $\boldsymbol{T}^2$) the subgroup

$$\left(\frac{1}{q} \cdot \mathbb{Z}\right)/\mathbb{Z} \times S^1 \subset \boldsymbol{T}^2.$$

Let g be the unique element of G whose rotation vector is $(1/q, 0)$ and let G_0 be the connected component of the identity. G is clearly the direct product of G_0 by $\langle g \rangle$, the finite group (isomorphic to $\mathbb{Z}/q\mathbb{Z}$) generated by g. Each G-orbit is a family of q distinct simple closed curves since no element of G has a fixed point. These curves divide the torus into q distinct topological annulus $A_0, A_1, \ldots, A_{q-1}$ which are permuted by g and each one of these annulus is invariant under G_0. According to the results of Section 4, the restriction of the action of G_0 on each annuli A_i is conjugate to the standard action of S^1. From these considerations, we deduce the following results which complete the proof of Theorem 5.4.

Lemma 5.8. *There exists a simple path $\sigma_0 : [0, 1/q] \to \boldsymbol{T}^2$ joining $(0,0)$ to $g(0,0)$ such that for any $s, t \in [0, 1/q]$, $s \neq t$ and $\{s,t\} \neq \{0, 1/q\}$, $\sigma_0(s)$ and $\sigma_0(t)$ are on distinct G_0-orbits.*

Corollary 5.9. (1) *Let $\sigma : [0,1] \to \boldsymbol{T}^2$ be defined by $\sigma(1/q + t) = g^i(\sigma_0(t))$, $t \in [0, 1/q]$. Then σ is a simple closed curve invariant under g, which meets each orbit of G_0 in exactly one point.*

(2) *Let $\phi : \boldsymbol{T}^2 \to \boldsymbol{T}^2$ be the map defined by $\phi(s,t) = g_{(0,t)}(\sigma(s))$ where $g_{(0,t)}$ is the unique element of G whose rotation vector is equal to $(0,t)$. Then ϕ is a conjugacy between G and the subgroup of translations by elements of $(1/q \cdot \mathbb{Z})/\mathbb{Z} \times S^1 \subset \boldsymbol{T}^2$.*

5.3. Orientation-reversing homeomorphisms of the torus

Let f be an orientation-reversing homeomorphism of the torus $\boldsymbol{T}^2$. According to Theorem 1.1(2) f^2 is periodic or is conjugate to a translation by a vector with at least one irrational coordinate. The two families of maps $(s,t) \mapsto (-s, t+\alpha)$ and $(s,t) \mapsto (-s, t+s+\alpha)$, where $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ are standard examples of regular, orientation-reversing and nonperiodic homeomorphism of the torus $\boldsymbol{T}^2$. Two such maps are not conjugate. We shall show in this paragraph that, up to topological conjugacy, there are no other such map. More precisely:

Theorem 5.10. *Let f be a regular, orientation-reversing and nonperiodic homeomorphism of the torus $\boldsymbol{T}^2$. Then, there exists $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ such that f is topologically conjugate either to $(s,t) \mapsto (-s, t+\alpha)$ or $(s,t) \mapsto (-s, t+s+\alpha)$.*

Proof. Let $A \in GL(2,\mathbb{Z})$ be the induced matrix on $\pi_1(\boldsymbol{T}^2) \cong H_1(\boldsymbol{T}^2) \cong \mathbb{Z}^2$. For any lift $\widetilde{f}$ of f to the universal cover $\mathbb{R}^2$ of $\boldsymbol{T}^2$ we have:

$$\widetilde{f}(\widetilde{x}+v) = \widetilde{f}(\widetilde{x}) + A \cdot v,$$

for every integer vector v. Furthermore, since f is not periodic, $A^2 = I$ according to the results of Section 5.2. Let $\theta(\widetilde{f^2}, \widetilde{x})$ be the function defined in Lemma 5.5. An easy computation shows that:

$$\theta(\widetilde{f} \circ \widetilde{f^2} \circ \widetilde{f}^{-1}, \widetilde{f}(\widetilde{x})) = A \cdot \theta(\widetilde{f^2}, \widetilde{x})$$

since the function $\widetilde{f}(\widetilde{x}) - A(\widetilde{x})$ is bounded on $\mathbb{R}^2$. Moreover, since $\theta(\widetilde{f^2}, \widetilde{x})$ has been shown to be independent of $\widetilde{x}$, we get:

$$A \cdot \theta(\widetilde{f^2}) = \theta(\widetilde{f^2}).$$

In other words $\theta(\widetilde{f^2})$ is an eigenvector of A.

Since f reverses orientation, $\det A = -1$ and the eigenvalues of A are $+1$ and -1. In $GL(2,\mathbb{Z})$ there are two conjugacy classes of such matrices:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{(Type 1)}$$

and

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{(Type 2).}$$

In both cases, we can find a new system of coordinates on the torus in which we have $\theta(\widetilde{f^2}) = (0, 2\alpha)$. Since f^2 is not periodic by assumption, α must be an irrational number and the closure, G, of the group generated by f^2 is isomorphic to $\boldsymbol{S}^1$. Each G-orbit is a simple closed curve whose homology class is $(0,1)$ in this system of coordinates.

According to Corollary 5.9, we can suppose, after a change of coordinates if necessary, that f^2 is the map defined by $(s,t) \mapsto (s, t+2\alpha)$. The circles $\{s\} \times S^1$ become then the closures of f^2-orbits. In particular, f maps each such circle onto another one and its analytic expression must be $(s,t) \mapsto (i(s), \varphi(s,t))$. Notice that i is an involution of

$\boldsymbol{S}^1$. Moreover, since the homology class of each circle $\{s\} \times S^1$ is an eigenvector for the eigenvalue 1 for the induced map by f on homology, i must reverse the order and is therefore conjugate to the map $s \mapsto -s$. In these new coordinates, f is expressed by $(s,t) \mapsto (-s, \phi_s(t))$ whereas f^2 is still the map $(s,t) \mapsto (s, t+2\alpha)$. Since f commutes with f^2, so does ϕ_s with the irrational rotation $t \mapsto t + 2\alpha$. Hence ϕ_s must also be a rotation and we can write $\phi_s(t) = t + \alpha(s)$. The formula $f^2(s,t) = (s, t+2\alpha)$ leads to $\alpha(s) + \alpha(-s) = 2\alpha$. In particular, we have $2\alpha(0) = 2\alpha(1/2) = 2\alpha$ and hence $\alpha(0) - \alpha(1/2) \in \{0, 1/2\}$.

(1) If $\alpha(0) = \alpha(1/2) = \alpha$, we let $\beta : \boldsymbol{S}^1 \to S^1$ be the continuous map which is 0 on $[0, 1/2]$ and $\alpha(0) - \alpha(s)$ on $[-1/2, 0]$. The continuity of this map at $1/2 = -1/2 \in \boldsymbol{S}^1$ results from the fact that $\alpha(0) = \alpha(1/2)$. An easy computation shows that for all $s \in \boldsymbol{S}^1$ we have:

$$\alpha(s) + \beta(s) - \beta(-s) = \alpha.$$

We let then B be the homeomorphism of the torus defined by $B(s,t) = (s, t + \beta(s))$ and we can check that:

$$B^{-1} \circ f \circ B(s,t) = (-s, t + \alpha).$$

(2) If $\alpha(0) - \alpha(1/2) = 1/2$, we let $\beta : \boldsymbol{S}^1 \to \boldsymbol{S}^1$ be the continuous map which is s on $[0, 1/2]$ and $\alpha(0) - \alpha(s)$ on $[-1/2, 0]$. The continuity of this map at $1/2 = -1/2 \in \boldsymbol{S}^1$ results from the fact that $\alpha(0) - \alpha(1/2) = 1/2$. An easy computation shows that for all $s \in \boldsymbol{S}^1$ we have:

$$\alpha(s) + \beta(s) - \beta(-s) = s + \alpha(0).$$

Again, we let B be the homeomorphism of the torus defined by $B(s,t) = (s, t + \beta(s))$ and we write $\alpha(0) = \alpha$. An easy computation shows that:

$$B^{-1} \circ f \circ B(s,t) = (-s, t + s + \alpha)$$

which completes the proof. □

5.4. *The case of the Klein bottle*

Let θ_0 be the involution of the torus $\boldsymbol{T}^2$ defined by $(s,t) \mapsto (-s, t + 1/2)$. Since this involution is fixed point free, the quotient space $\boldsymbol{T}^2/\theta_0$ is a closed surface and since θ_0 is orientation reversing, this surface must be the Klein bottle $\boldsymbol{K}$. The canonical map $\pi : \boldsymbol{T}^2 \to \boldsymbol{T}^2/\theta_0$ is the orientation covering of K and θ_0 is the unique automorphism of this covering.

For $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$, we note ϕ_α and ψ_α the homeomorphisms of $\boldsymbol{T}^2$ defined by $\phi_\alpha(s,t) = (s, t+\alpha)$ and $\psi_\alpha(s,t) = (s + 1/2, t + \alpha)$. Since these maps commute with θ_0, they induces two homeomorphisms of $\boldsymbol{K}$ that we shall denote by Φ_α and Ψ_α, respectively. These maps define two distinct families of regular and nonperiodic homeomorphisms of $\boldsymbol{K}$ and we note that:

(1) For every irrational number α, the closure of each Φ_α-orbit is a simple closed curve which is the projection by π of the two circles $\{s\} \times S^1$ and $\{-s\} \times S^1$ of

$\boldsymbol{T}^2$. The rotation number of the restriction of Φ_α restricted to this curve is equal to α if $s \neq -s$ (i.e., $s \notin \{0, 1/2\}$) or to 2α if $s \in \{0, 1/2\}$.

(2) For every irrational number α, the closure of each Ψ_α-orbit is the projection by π of the circles $\{s\} \times S^1$, $\{-s\} \times S^1$, $\{s + 1/2\} \times S^1$ and $\{-s + 1/2\} \times S^1$ of $\boldsymbol{T}^2$. If $s \notin \{1/4, -1/4\}$, these circles are mapped into two simple closed curves. If $s \in \{1/4, -1/4\}$, these circles are mapped into just one circle on which the rotation number is $\alpha + 1/2$. Note that $\Psi_\alpha^2 = \Phi_{2\alpha}$.

The aim of this paragraph is to establish the following theorem:

Theorem 5.11. *Let f be a regular, nonperiodic homeomorphism of the Klein bottle $\boldsymbol{K}$. Then there exist $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ such that f is topologically conjugate either to Φ_α or to Ψ_α.*

Recall first that f has exactly to lifts on $\boldsymbol{T}^2$, one of which f_+ preserves orientation while the other $f_- = \theta_0 \circ f_+$ reverses orientation. Note also that f_+ and f_- commute with θ_0.

Since f_- and θ_0 commute, so do the induced maps on homology. According to the results of Section 5.3, the homology matrix of f_- is therefore of type 1 and there are coordinates on the torus in which f_- can be written $(s, t) \mapsto (-s, t + \alpha)$. Since θ_0 commutes with f_-, its expression in these coordinates must be $(s, t) \mapsto (i_-(s), t + \alpha(s))$ where i_- is an involution of $\boldsymbol{S}^1$ which reverses the order. The expression of f_+ follows from the relation $f_+ = \theta_0 \circ f_-$ and we have $f_+(s, t) = (i_+(s), t + \alpha(s) + \alpha)$ where $i_+(s) = -i_-(s)$ is an involution of $\boldsymbol{S}^1$ which preserves the order.

It is a straightforward exercise to show that given two commuting involutions of $\boldsymbol{S}^1$, i_+ and i_- such that i_+ preserves the order and i_- reverses the order, there is a homeomorphism of $\boldsymbol{S}^1$ which conjugate the pair (i_+, i_-) to either $(s \mapsto s, s \mapsto -s)$ or $(s \mapsto s + 1/2, s \mapsto -s)$.

From this fact, we deduce that there are coordinates on the torus in which we can write either:

(1) $f_-(s,t) = (-s, t+\alpha)$, $\theta_0(s,t) = (-s, t+\alpha(s))$ and $f_+(s,t) = (s, t+\alpha(s)+\alpha)$,

or

(2) $f_-(s,t) = (-s, t+\alpha)$, $\theta_0(s,t) = (-s, t+\alpha(s))$ and $f_+(s,t) = (s+1/2, t+\alpha(s)+\alpha)$.

Since θ_0 is an involution, we have $\alpha(s) + \alpha(-s) = 0$. In particular, $2\alpha(0) = 2\alpha(1/2) = 0$ and we get $\alpha(0) = \alpha(1/2) = 1/2 \in S^1$ because θ_0 is fixed point free.

To achieve the proof of the theorem, we let $\beta : \boldsymbol{S}^1 \to S^1$ be the continuous map which is 0 on $[0, 1/2]$ and $1/2 - \alpha(s)$ on $[-1/2, 0]$. The continuity of this map at 0 and $1/2$ results from the fact that $\alpha(0) = \alpha(1/2) = 1/2$. We verify then that, for all $s \in \boldsymbol{S}^1$, we have:

$$\alpha(s) + \beta(s) - \beta(-s) = 1/2$$

and we let B be the homeomorphism of the torus defined by $B(s,t) = (s, t + \beta(s))$. In the first case, we can check that:

- $B^{-1} \circ \theta_0 \circ B(s,t) = (-s, t+1/2)$,
- $B^{-1} \circ f_+ \circ B(s,t) = (s, t+\alpha+1/2) = \Phi_{\alpha+1/2}(s,t)$.

In the second case, we have $\alpha(s+1/2) = \alpha(-s)$ since f_+ commutes with θ_0 and we obtain:

- $B^{-1} \circ \theta_0 \circ B(s,t) = (-s, t+1/2)$,
- $B^{-1} \circ f_+ \circ B(s,t) = (s+1/2, t+\alpha+1/2) = \Psi_{\alpha+1/2}(s,t)$

which completes the proof. □

5.5. *The annulus and the Mœbius strip*

To obtain the complete classification of regular homeomorphisms of the annulus $[-1,1] \times \boldsymbol{S}^1$ and the Mœbius strip $[-1,1] \times \boldsymbol{S}^1/(s,t) \mapsto (-s, t+1/2)$, it suffices to consider their doubles and then to retain, among the list of regular homeomorphisms of the torus and the Klein bottle, those which keep invariant an annulus or a Mœbius strip. We state here the result without any proof.

Theorem 5.12. *Let f be a regular and nonperiodic homeomorphism of the annulus. There exists $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ such that:*
(1) *$f(s,t) = (s, t+\alpha)$ if f is orientation-preserving,*
(2) *$f(s,t) = (-s, t+\alpha)$ if f is orientation-reversing.*

Theorem 5.13. *Let f be a regular and nonperiodic homeomorphism of the Mœbius strip. There exists $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ such that f is conjugate to the map induced by the homeomorphism $(s,t) \mapsto (s, t+\alpha)$ of the annulus.*

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