On Faddeev-Leverrier's Method for the Computation of the Characteristic Polynomial of a Matrix and of Eigenvectors

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ABSTRACT

Faddeev's method of computing the eigenvalues and eigenvectors of a matrix is presented and completed so as also to cover the case of multiple zeros of the characteristic equation.

Let A be an n-by-n matrix with coefficients in a field K of zero characteristic. The characteristic polynomial of A may be obtained by a method proposed by D. K. Faddeev [3, §47; 4]. This is a modification of a method of U. J. J. Leverrier (1840) which, according to A. S. Householder [8, p. 172], was "rediscovered and improved" in the late forties also by J. M. Souriau [9] and J. S. Frame [5]. Although this method is not optimal from the point of view of numerical computation, it is rather elegant and easy to realize in a computer program. It consists of the following steps (I denotes the

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identity matrix):

$$A_1 := A, \tag{1}$$

$$a_k := -\frac{\text{trace } A_k}{k} \qquad (1 \leqslant k \leqslant n), \tag{2}$$

$$A_{k+1} := A(A_k + a_k I) \quad (1 \leqslant k \leqslant n-1). \tag{3}$$

Putting $a_0 = 1$, one then has

$$\Phi(\lambda) := \det(\lambda I - A) = \sum_{k=0}^{n} a_k \lambda^{n-k}. \tag{4}$$

An additional definition at the same time (and under a condition to be mentioned below) also furnishes eigenvectors for given eigenvalues: Let

$$\overline{A}_0 := I, \tag{5}$$

$$\overline{A}_k := A_k + a_k I \qquad (1 \leqslant k \leqslant n-1), \tag{6}$$

and define

$$C(\lambda) := \sum_{k=0}^{n-1} \lambda^{n-1-k} \overline{A}_k. \tag{7}$$

Then

$$A \cdot C(\lambda) = \lambda C(\lambda) - \Phi(\lambda) I,$$

and therefore, if λ_0 is an eigenvalue of A, the nonzero columns of $C(\lambda_0)$ are corresponding eigenvectors. Faddeev and Faddeeva [3] mention that $C(\lambda_0) \neq 0$ if all eigenvalues of A are distinct.

Faddeev's proof, as well as that of Leverrier, Householder, and Gantmacher [7], uses the connection between the coefficients of a polynomial and the power sums of its roots. The proofs of Souriau and Frame (see also [2] and [6]) exploit the concept of the adjoint of the matrix $\lambda I - A$, and we shall repeat it very briefly for completeness. (A third proof using a companion matrix has been given in [1]). The purpose of this paper is to state the necessary and sufficient conditions for $C(\lambda_0) \neq 0$ and to show how the derivatives of C at λ_0 still may serve to find eigenvectors and generalized eigenvectors of A if $C(\lambda_0)$ vanishes.

THEOREM 1. *Let*

$$\Phi(\lambda) := \det(\lambda I - A) = \sum_{k=0}^{n} a_k \lambda^{n-k} \tag{4}$$

be the characteristic polynomial of A, *and define the matrices* $\bar{A}_k$ $(0 \leqslant k \leqslant n-1)$, A_k $(1 \leqslant k \leqslant n)$, *and* $C(\lambda)$ *by*

$$\bar{A}_0 := I, \tag{5}$$

$$\bar{A}_k := A_k + a_k I \qquad (1 \leqslant k \leqslant n-1), \tag{6}$$

$$A_1 := A, \tag{1}$$

$$A_{k+1} := A(A_k + a_k I) \qquad (1 \leqslant k \leqslant n-1), \tag{3}$$

and

$$C(\lambda) := \sum_{k=0}^{n-1} \lambda^{n-1-k} \bar{A}_k. \tag{7}$$

Then

(a) $C(\lambda) = \operatorname{adj}(\lambda I - A)$ *for* $\lambda \in K$;
(b) $a_k = (\operatorname{trace} \bar{A}_k)/(n-k)$ *for* $0 \leqslant k \leqslant n-1$;
(c) $a_k = -(\operatorname{trace} A_k)/k$ *for* $1 \leqslant k \leqslant n$.

Proof. (a): Let the matrices B_k $(0 \leqslant k \leqslant n-1)$ be defined by

$$\operatorname{adj}(\lambda I - A) = \sum_{k=0}^{n-1} \lambda^{n-1-k} B_k.$$

Then we have

$$\Phi(\lambda) I = (\lambda I - A) \operatorname{adj}(\lambda I - A),$$

$$\sum_{k=0}^{n} a_k \lambda^{n-k} I = (\lambda I - A) \sum_{k=0}^{n-1} \lambda^{n-1-k} B_k$$

$$= B_0 \lambda^n + \sum_{k=1}^{n-1} \lambda^{n-k} (B_k - A B_{k-1}) - A B_{n-1},$$

and therefore

$$\begin{aligned} a_0 I &= B_0, \\ a_k I &= B_k - A B_{k-1} \qquad (1 \leqslant k \leqslant n-1), \\ a_n I &= -A B_{n-1}. \end{aligned} \tag{8}$$

We conclude

$$B_0 = I = \bar{A}_0,$$

and, by induction on k (with inductive hypothesis $B_{k-1} = \bar{A}_{k-1}$),

$$B_k = A B_{k-1} + a_k I = A \bar{A}_{k-1} + a_k I = A_k + a_k I = \bar{A}_k \qquad (1 \leqslant k \leqslant n-1).$$

This shows

$$\operatorname{adj}(\lambda I - A) = \sum_{k=0}^{n-1} \lambda^{n-1-k} \bar{A}_k = C(\lambda).$$

(b): Let A_{ii} be the $(n-1)$-by-$(n-1)$ matrix obtained by deleting the ith row and the ith column in the matrix A. The coefficient of λ in $\Phi(\lambda) = \det(\lambda I - A)$ is

$$a_{n-1} = \sum_{i=1}^{n} \det(-A_{ii}) = \operatorname{trace} \operatorname{adj}(-A).$$

By the same token, we see that in the polynomial

$$\Phi(\lambda + \mu) = \det[(\lambda + \mu)I - A] = \sum_{k=0}^{n} a_k(\lambda + \mu)^{n-k}$$

the coefficient of μ is

$$\begin{aligned}\sum_{k=0}^{n-1} a_k(n-k)\lambda^{n-1-k} &= \operatorname{trace}\operatorname{adj}(\lambda I - A)\\ &= \operatorname{trace} C(\lambda) \qquad [\text{by (a)}]\\ &= \sum_{k=0}^{n-1} \lambda^{n-1-k} \operatorname{trace} \bar{A}_k.\end{aligned}$$

(c): Since $A_k = \bar{A}_k - a_k I$, we get

$$\begin{aligned}\operatorname{trace} A_k = \operatorname{trace} \bar{A}_k - na_k &= (n-k)a_k - na_k\\ &= -ka_k \qquad (1 \leqslant k \leqslant n-1).\end{aligned}$$

Finally, from (8), we get

$$na_n = -\operatorname{trace} A_n. \qquad \blacksquare$$

REMARKS.

(1) The equation deduced in (b) of the proof can be put in the following form:

$$\frac{d}{d\lambda}\det(\lambda I - A) = \operatorname{trace}\operatorname{adj}(\lambda I - A).$$

(2) The characterization of $C(\lambda)$ according to Theorem 1(a) allows us to describe the condition $C(\lambda_0) \neq 0$ more completely. Namely, by standard

matrix theory, we obtain the following equivalences:

$C(\lambda_0) \neq 0 \quad \Leftrightarrow \quad \operatorname{rank}(\lambda_0 I - A) = n - 1$

$\Leftrightarrow \quad \dim \ker(\lambda_0 I - A) = 1$

$\Leftrightarrow$ the Jordan canonical form of A (over the algebraic closure $\bar{K}$ of K) contains one sole Jordan cell with the eigenvalue λ_0.

Since the columns of $C(\lambda_0)$ are eigenvectors, this is also equivalent with $\operatorname{rank} C(\lambda_0) = 1$. All these assertions just express the fact that λ_0 is an eigenvalue of A of geometric multiplicity 1.

Some additional information concerning the relation between the matrix $C(\lambda)$ and the generalized eigenspaces of A may be found in the literature: Gantmacher [7, p. 92] shows that, if λ_0 is an eigenvalue of A and if the elements of $C(\lambda)$ have a largest common factor $(\lambda - \lambda_0)^m$ $(m \geqslant 0)$, then the nonzero columns of the mth derivative $C^{(m)}(\lambda_0)$ are eigenvectors. Householder [8, p. 168] proves that, if λ_0 is an eigenvalue, "$C(\lambda_0)$ cannot vanish unless λ_0 is at least a double root." He states without proof that if λ_0 has multiplicity 2, then if $C(\lambda_0) \neq 0$, the nonzero columns of $C(\lambda_0)$ are eigenvectors and the nonzero columns of $C'(\lambda_0)$ are generalized eigenvectors. He continues: "If, however, $C(\lambda_0) = 0$, then $C'(\lambda_0)$ can be shown to have rank 2, and any nonvanishing column is a proper vector. For roots of higher multiplicity the situation is analogous." In an exercise the reader is invited to prove these assertions.

In the rest of this paper, we try to give a more complete account of the relation between the spans of $C(\lambda_0)$ and of its derivatives $C^{(m)}(\lambda_0)$ at an eigenvalue λ_0, on the one hand, and the generalized eigenspace of A for this eigenvalue, on the other hand. We assume in the sequel, without loss of generality, that K is algebraically closed and that A is given in Jordan canonical form. By $\operatorname{im} B$ we denote the vector space spanned by the columns of a matrix B or, what is the same, the image space of the linear mapping defined by B. If λ_0 has multiplicity l and index s (i.e., the largest Jordan cell corresponding to λ_0 is $s \times s$), then [see (b), (h) of the Corollary below] $\operatorname{im} C^{(l-1)}(\lambda_0) = \ker(\lambda_0 I - A)^s$, the space of all generalized eigenvectors to λ_0. For $1 \leqslant t \leqslant s - 1$, however, we have

$$\ker(\lambda_0 I - A)^t \supseteq \operatorname{im} C^{(l-1-s+t)}(\lambda_0)$$

with equality in place of inclusion only in special cases, to be precisely specified in Theorem 3 below. Still, the dimensions of the spaces $\ker(\lambda_0 I - A)^t$ and the Jordan structure of A, related to λ_0, can be found from the ranks of the matrices $C^{(l-1-s+t)}(\lambda_0)$.

It should be appreciated that the derivatives $C^{(k)}(\lambda_0)$ can easily be found by executing Faddeev's algorithm with the matrices $\tilde{A} = A - \lambda_0 I$ in place of A, which yields by (7) and Theorem 1(a)

$$C^{(k)}(\lambda_0) = k!\bar{\tilde{A}}_{n-1-k}, \qquad 0 \leqslant k \leqslant n-1.$$

THEOREM 2. *Let $\lambda_0 \in K$ be an eigenvalue of A of algebraic multiplicity l. Denote by $G = \ker(\lambda_0 I - A)^l$ the subspace of K^n consisting of the generalized eigenvectors with eigenvalue λ_0 (i.e. the vectors annihilated by some power of the matrix $\lambda_0 I - A$), and let $H = \operatorname{im}(\lambda_0 I - A)^l$. Let $M^{(k)} = \operatorname{im} C^{(k)}(\lambda_0)$ for $k \geqslant 0$. Then*

$$M^{(k)} = (\lambda_0 I - A)^{l-1-k} G \qquad \textit{for} \quad 0 \leqslant k \leqslant l-1,$$

$$H = \ker C^{(l-1)}(\lambda_0) \subseteq M^{(l)},$$

$$K^n = G \oplus H.$$

Proof. Let $V = K^n = \bigoplus_{h=1}^q V_h$, where $V_h \cong K^{s_h}$ is an A-invariant subspace of V in which A is represented by an $s_h \times s_h$ Jordan cell

$$A_h = \left.\begin{pmatrix} \lambda_h & 0 & 0 & \cdots & 0 \\ 1 & \lambda_h & 0 & \cdots & 0 \\ 0 & 1 & \lambda_h & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_h \end{pmatrix}\right\} s_h .$$

Suppose $\lambda_h = \lambda_0$ for $1 \leqslant h \leqslant p$ and $\lambda_h \neq \lambda_0$ for $p < h \leqslant q$, so $l = \sum_{h=1}^p s_h$ and $G = \bigoplus_{h=1}^p V_h$.

In V_h the matrix $C(\lambda) = \operatorname{adj}(\lambda I - A)$ is represented by an $s_h \times s_h$ matrix [also referred to as cell $Ch(\lambda)$] of the form

$$\begin{pmatrix} (\lambda-\lambda_h)^{s_h-1} & 0 & \cdot\ \cdot & \cdot & 0 \\ (\lambda-\lambda_h)^{s_h-2} & (\lambda-\lambda_h)^{s_h-1} & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot\ \cdot & & \cdot \\ \cdot & & \cdot & \cdot & 0 \\ 1 & \cdot & \cdot\ \cdot & (\lambda-\lambda_h)^{s_h-2} & (\lambda-\lambda_h)^{s_h-1} \end{pmatrix}$$

$$\times \prod_{k \neq h} (\lambda - \lambda_k)^{s_k}.$$

If $c_{ij}(\lambda)$ is a nonzero coefficient of $C_h(\lambda)$, then the multiplicity of λ_0 as a zero of $c_{ij}(\lambda)$ is $\sum_{k=1}^{p} s_k = l$ if $h > p$, and it is

$$\sum_{\substack{k=1 \\ k \neq h}}^{p} s_k + s_h - 1 - (i-j) = l - 1 - (i-j) \qquad \text{if} \quad h \leqslant p.$$

Therefore, if $k < l$, then $C^{(k)}(\lambda_0)$ has nonzero elements on the diagonal given by $k = l - 1 - (i-j)$, i.e. $i - j = l - k - 1$, where this diagonal meets a cell C_h $(1 \leqslant h \leqslant p)$, and all elements outside the cells C_h $(1 \leqslant h \leqslant p)$ and all elements in these cells but above the diagonal $i - j = l - k - 1$ are zero. [We abuse the term "cell C_h" also for the location of $C_h(\lambda)$ as an $s_h \times s_h$ submatrix of the $n \times n$ matrix $C(\lambda)$.] Consequently, $M^{(k)}$ is spanned by those basis vectors which are numbered by the row indices corresponding to the intersection of this diagonal with the cells C_h $(1 \leqslant h \leqslant p)$. On the other hand, within the cells C_h $(1 \leqslant h \leqslant p)$, the only nonzero elements in $(\lambda_0 I - A)^{l-k-1}$ are 1's on the same diagonal, and therefore

$$(\lambda_0 I - A)^{l-k-1} G = \bigoplus_{h=1}^{p} (\lambda_0 I - A_h)^{l-k-1} V_h$$

spans exactly the same subspace of K^n. Furthermore, $C_h^{(l-1)}(\lambda_0)$, $1 \leqslant h \leqslant p$, is nonsingular on G and vanishes on $\bigoplus_{h=p+1}^{q} V_h = H$. Hence $H = \ker C^{(l-1)}(\lambda_0)$. Similarly, for $p < h \leqslant q$, the matrix $C_h^{(l)}(\lambda_0)$ is nonsingular. This implies that $M^{(l)} \supseteq H$. The last assertion follows from the decomposition $K^n = \bigoplus_{h=1}^{p} V_h \oplus \bigoplus_{h=p+1}^{q} V_h$. ■

The assertions contained in the following corollary and in the remarks thereupon are all quite immediate consequences of Theorem 2 and especially of the equation

$$M^{(k)} = (\lambda_0 I - A)^{l-1-k} \ker(\lambda_0 I - A)^l, \qquad 0 \leqslant k \leqslant l-1.$$

We state these facts in great detail simply in order to give a description as complete as possible of the situation under scrutiny, and without any pretensions of originality or depth.

Recall that a function $k \mapsto f(k)$ on $\mathbb{Z}$ is called weakly convex if $k \mapsto f(k) - f(k-1)$ is nondecreasing.

COROLLARY. *Under the assumptions of Theorem 2 and with the additional notation* $E = \ker(\lambda_0 I - A)$, $m_{-2} = m_{-1} = 0$, $m_k = \dim M^{(k)}$ *for* $0 \leqslant k \leqslant l-1$, *the following assertions hold*:

(a) *For* $1 \leqslant k \leqslant l-1$ *one has* $M^{(k-1)} = (\lambda_0 I - A) M^{(k)}$.

(b) $M^{(0)} \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(l-1)} = G$.

(c) $0 \leqslant m_0 \leqslant m_1 \leqslant \cdots \leqslant m_{l-1} = l$.

(d) *For* $0 \leqslant k \leqslant l-1$ *one has*

$$m_k - m_{k-1} = \dim(E \cap M^{(k)}),$$

and this equals the number of Jordan cells in A belonging to λ_0 *and of size at least* $l-k$; *the function* $k \mapsto m_k$, $-1 \leqslant k \leqslant l-1$, *is weakly convex.*

(e) *Let* p_k *denote the number of Jordan cells in A belonging to* λ_0 *and of size* k $(1 \leqslant k \leqslant l)$; *then* $p_k = m_{l-k} - 2m_{l-k-1} + m_{l-k-2}$.

(f) *For* $1 \leqslant t \leqslant l$ *one has*

$$m_{l-1} - m_{l-1-t} = \dim\ker(\lambda_0 I - A)^t;$$

in particular, $m_{l-1} - m_{l-2} = \dim E$, *and this equals the number of Jordan cells in A belonging to* λ_0.

(g) *If* $m_0 > 0$, *then* $M^{(0)} = E$ *and* $m_k = k+1$ *for* $0 \leqslant k \leqslant l-1$; *in this case there is just one Jordan cell belonging to* λ_0.

(h) *Let* k_1 $(0 \leqslant k_1 \leqslant l-1)$ *be the smallest index k for which* $m_k > 0$, *i.e.*,

$$0 = m_{-1} = \cdots = m_{k_1 - 1} < m_{k_1} < \cdots < m_{l-1} = l;$$

if s denotes the size of the largest Jordan cell in A belonging to λ_0, *then* $s = l - k_1$ *and* $G = \ker(\lambda_0 I - A)^s$, $H = \operatorname{im}(\lambda_0 I - A)^s$.

(i) *For* $0 \leqslant t \leqslant s$ *one has*

$$M^{(k_1-1+t)} = (\lambda_0 I - A)^{s-t} \ker(\lambda_0 I - A)^s$$

$$= \operatorname{im}(\lambda_0 I - A)^{s-t} \cap \ker(\lambda_0 I - A)^t;$$

in particular, $(\lambda_0 I - A)^t M^{(k_1-1+t)} = \{0\}$, $M^{(k_1)} \subseteq E$.

(j) *For* $1 \leqslant t \leqslant s$ *one has*

$$(\lambda_0 I - A)^{t-1} M^{(k_1-1+t)} = M^{(k_1)} \neq \{0\}.$$

(k) *If* $l = n$ *then* $M^{(l-1)} = K^n$ *and* $M^{(l)} = \{0\}$.

(l) *If* $l < n$ *then*

$$(\lambda_0 I - A) M^{(l)} \not\subseteq M^{(l-1)},$$

$$(\lambda_0 I - A)^t M^{(l)} = H \neq \{0\} \qquad \textit{for} \quad t \geqslant s,$$

$$K^n = M^{(l-1)} \oplus (\lambda_0 I - A)^s M^{(l)}.$$

Proof. Assertion (a) follows directly from Theorem 2. Since $(\lambda_0 I - A) G \subseteq G$, Theorem 2 implies $M^{(k-1)} \subseteq M^{(k)}$ for $1 \leqslant k \leqslant l-1$. This proves (b); (c) follows, since $\dim G = l$. The first statement of (d) follows from (a), since $E \cap M^{(k)}$ is the kernel of the mapping $\lambda_0 I - A$ restricted to $M^{(k)}$. Since $M^{(k)} = (\lambda_0 I - A)^{l-1-k} G$, $\dim(E \cap M^{(k)})$ is the maximal number of linearly independent eigenvectors $\mathbf{x}$ with eigenvalue λ_0 which may be written as $\mathbf{x} = (\lambda_0 I - A)^{l-1-k}\mathbf{y}$, $\mathbf{y} \in G$, i.e. which belong to a subspace V_h $(h \leqslant p)$ corresponding to a Jordan cell of size at least $l-k$. Furthermore, (b) implies that the sequence $\{m_k - m_{k-1}\}_{k=0}^{l-1}$ is nondecreasing. Assertion (e) follows from (d), since $(m_{l-k} - m_{l-k-1}) - (m_{l-k} - m_{l-k-2})$ equals the number of Jordan cells of size $l-k$. Assertion (f) follows from $M^{(l-1-t)} = (\lambda_0 I - A)^t M^{(l-1)}$. To prove (g) we first note that, by Remark 2 above, $m_0 > 0$ implies $M^{(0)} = E$ and $m_0 = \dim E = 1$; then (b) and (d) imply $m_k = k+1$ $(0 \leqslant k \leqslant l-1)$.

If s denotes the largest size of a Jordan cell in A belonging to the eigenvalue λ_0, then $G = \ker(\lambda_0 I - A)^s$ and $\ker(\lambda_0 I - A)^{s-1}$ is a proper subspace of G. By Theorem 2 this implies $M^{(l-s-1)} = \{0\} \neq M^{(l-s)}$, so $k_1 = l - s$ and (h) is proven. The first line of (i) is a reformulation of Theorem 2, taking account of (h). The second line follows from the first line because, generally, $P \ker(QP) = \operatorname{im} P \cap \ker Q$. (j) follows from (i) on account of the definition of k_1.

If $l = n$, then $M^{(l)} = M^{(n)} = \{0\}$, since $C(\lambda)$ is a polynomial of degree $n-1$ [compare (7)], and $M^{(l-1)} = G = K^n$. This proves (k). If $l < n$, then the subspace H is nontrivial by Theorem 2. Since $\ker(\lambda_0 I - A) \subseteq G$ and $G \cap H = \{0\}$, the mapping $\lambda_0 I - A$ acts as an automorphism on H. Again by Theorem 2

$$(\lambda_0 I - A) M^{(l)} \supseteq (\lambda_0 I - A) H = H,$$

so certainly

$$(\lambda_0 I - A) M^{(l)} \not\subseteq M^{(l-1)} = G.$$

Finally, for $t \geqslant s$,

$$(\lambda I - A)^t M^{(l)} \subseteq \operatorname{im}(\lambda_0 I - A)^s = H = (\lambda_0 I - A)^t H \subseteq (\lambda_0 I - A)^t M^{(l)}.$$

This shows $(\lambda_0 I - A)^t M^{(l)} = H$ and completes the proof. ∎

REMARKS.

(1) If $k_1 = l - 1$ $(s = 1)$, then one has

$$M^{(0)} = \cdots = M^{(l-2)} = \{0\}, \qquad M^{(l-1)} = \ker(\lambda_0 I - A)$$

with $\dim M^{(l-1)} = l$. This is the case considered in the quotation from Householder. In this case, all Jordan cells corresponding to λ_0 have size 1.

(2) In Theorem 2 and its Corollary, the value of the multiplicity l of the eigenvalue λ_0 is presupposed to be known from the beginning, e.g. by decomposing the characteristic polynomial given in Theorem 1. Alternatively, l can be characterized in terms of the numbers m_k or the spaces $M^{(k)}$. If $m_0 = 0$, it follows from (d) of the Corollary that

$$l = \min\{k : 1 \leqslant k \text{ and } m_{k-1} = k\}.$$

Part (g) of the Corollary shows that this is not true if $m_0 > 0$. Other characterizations are based on the distinction between the spaces $M^{(l)}$ and $M^{(k)}$ for $k < l$, as expressed by parts (a), (k), and (l). It is easily verified that

$$l = \max\left\{m : 1 \leqslant m \leqslant n \text{ and } M^{(k-1)} \subseteq \ker(\lambda_0 I - A)^k \text{ for } 1 \leqslant k \leqslant m\right\}.$$

With knowledge of k_1, which eventually can be obtained from the definition, viz.

$$k_1 = \min\{k : 0 \leqslant k \text{ and } M^{(k)} \neq \{0\}\},$$

the last characterization by parts (i) and (l) can be sharpened to

$$l = \max\left\{m : 1 \leqslant m \leqslant n \text{ and } M^{(k-1)} \subseteq \ker(\lambda_0 I - A)^{k-k_1} \text{ for } k_1 \leqslant k \leqslant m\right\}.$$

Finally, observe that also (setting $M^{(-1)} = \{0\}$)

$$l = \max\left\{m : 1 \leqslant m \leqslant n \text{ and } (\lambda_0 I - A) M^{(k-1)} \subseteq M^{(k-2)} \text{ for } 1 \leqslant k \leqslant m\right\}.$$

THEOREM 3. *Let A, λ_0, l, E, G, $M^{(k)}$, m_k, k_1, s be defined as in Theorem 2 and its Corollary. Then the following assertions are equivalent:*

(a) $M^{(k_1)} = E$;
(b) $m_{k_1} = \dim E$;
(c) $m_k - m_{k-1}$ *is independent of k for* $k_1 \leqslant k \leqslant l - 1$;
(d) $m_k - m_{k-1} = \dim E$ *for* $k_1 \leqslant k \leqslant l - 1$;
(e) $M^{(k)} = \ker(\lambda_0 I - A)^{k-k_1+1}$ *for some* k, $k_1 \leqslant k < l - 1$;
(f) $M^{(k)} = \ker(\lambda_0 I - A)^{k-k_1+1}$ *for* $k_1 \leqslant k \leqslant l - 1$;
(g) *all the Jordan cells of A belonging to the eigenvalue λ_0 have the same size.*

Proof. Since $M^{(k_1)} \subseteq E$ by (i) of the Corollary, the equivalence of assertions (a) and (b) is obvious. Furthermore, $M^{(k_1)} = (\lambda_0 I - A)^{s-1} G$ equals E if and only if all the Jordan cells of A belonging to the eigenvalue λ_0 have the same size s. Therefore, assertions (a) and (g) are equivalent. Obviously, (d) implies (b) as well as (c) and, conversely, (c) implies (d) by (f) of the Corollary. Also, assertion (a), which is equivalent to (b), implies (d) by (b) and (d) of the Corollary. Since (f) $\Rightarrow$ (e) is trivial, the proof will be complete upon showing that (c) $\Rightarrow$ (f) and (e) $\Rightarrow$ (c). Let $k = k_1 - 1 + t$ $(1 \leqslant t \leqslant s)$. Then we have to deal with the relations

$$M^{(k_1 - 1 + t)} = \ker(\lambda_0 I - A)^t. \tag{9}$$

Since, by (i) of the Corollary, $M^{(k_1-1+t)} \subseteq \ker(\lambda_0 I - A)^t$, (9) is true iff $m_{k_1-1+t} = \dim\ker(\lambda_0 I - A)^t$. The latter dimension equals $m_{l-1} - m_{l-1-t}$ by (f) of the Corollary, and hence, if we define

$$\phi(t) := m_{k_1-1+t}, \qquad 0 \leqslant t \leqslant s,$$

then $\phi(0) = 0$ and (9) proves to be equivalent with

$$\phi(t) + \phi(s-t) = \phi(0) + \phi(s). \tag{10}$$

Now (c) means that ϕ is linear, which fact clearly implies (10) and thus (9) for $0 \leqslant t \leqslant s$. Hence (c) $\Rightarrow$ (f) is proven. In order to show (e) $\Rightarrow$ (c), we infer from (d) of the Corollary that ϕ is weakly convex in the interval $0 \leqslant t \leqslant s$. Hence there are but two possibilities: either ϕ is linear or ϕ is strictly sublinear. In the latter case (10) does not hold for any t with $0 < t < s$, i.e., (e) is false. Consequently, (e) implies the linearity of ϕ and hence (c). ■

APPENDIX

Finally, we want to present a proof of Theorem 2 which does not use the Jordan decomposition.

If $\Phi(\lambda) = \det(\lambda I - A)$ as above, then we obtain from the identity $\Phi(\lambda)I = (\lambda I - A)C(\lambda)$ by differentiation that

$$\Phi^{(k)}(\lambda)I = (\lambda I - A)C^{(k)}(\lambda) + kC^{(k-1)}(\lambda) \qquad \text{for} \quad k \geqslant 1.$$

[This equation is a paraphrase of the recursion formulae (3) and (6) with the matrix $A - \lambda I$ in place of A.] If $\lambda_0 \in K^n$ is an eigenvalue of multiplicity l, then $\Phi^{(k)}(\lambda_0) = 0$ for $0 \leqslant k \leqslant l-1$ and hence

$$C^{(k-1)}(\lambda_0) = -\frac{1}{k}(\lambda_0 I - A)C^{(k)}(\lambda_0), \quad M^{(k-1)} = (\lambda_0 I - A)M^{(k)},$$
$$1 \leqslant k \leqslant l-1.$$

Next we show $M^{(l-1)} = G$. Note that

$$(\lambda_0 I - A)^l C^{(l-1)}(\lambda_0) = (-1)^{l-1}(l-1)!(\lambda_0 I - A)C(\lambda_0)$$
$$= (-1)^{l-1}(l-1)!\Phi(\lambda_0)I = 0,$$

and hence $M^{(l-1)} = \operatorname{im} C^{(l-1)}(\lambda_0) \subseteq G$. If we denote $\ker C^{(l-1)}(\lambda_0)$ by N, then

$$\dim M^{(l-1)} + \dim N = n = \dim G + \dim H,$$

and hence we can conclude that $M^{(l-1)} = G$ and $N = H$ once we have proven that $N \subseteq H$. To this end, restrict the linear mapping

$$\Phi^{(l)}(\lambda_0)I = (\lambda_0 I - A)C^{(l)}(\lambda_0) + lC^{(l-1)}(\lambda_0)$$

to N, in which case the last term vanishes. Since $\Phi^{(l)}(\lambda_0) \neq 0$, it follows that

$$N = (\lambda_0 I - A)C^{(l)}(\lambda_0)N \subseteq (\lambda_0 I - A)N \subseteq N.$$

[Notice that the derivatives of $C(\lambda)$ commute with one another. Hence $C^{(l)}(\lambda_0)N \subseteq N$. Moreover, these derivatives also commute with A, and this implies $(\lambda_0 I - A)N \subseteq N$.] Consequently, $N = (\lambda_0 I - A)^l N \subseteq H$. This finishes the proof of $M^{(l-1)} = G$ and $N = H$.

Incidentally, it has been shown that $(\lambda_0 I - A)^l H = H$, which implies $G \cap H = \ker(\lambda_0 I - A)^l \cap H = \{0\}$. Taking account of dimensions, $G \oplus H = K^n$ follows.

Finally, since $C^{(l)}(\lambda_0)(\lambda_0 I - A)N = (\lambda_0 I - A)C^{(l)}(\lambda_0)N = N$, it also follows that $M^{(l)} = \operatorname{im} C^{(l)}(\lambda_0) \supseteq N = H$.

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