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ScienceDirect
Discrete Applied Mathematics 156 (2008) 2793–2803
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

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# Identities via Bell matrix and Fibonacci matrix

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Received 31 July 2006; received in revised form 3 September 2007; accepted 13 October 2007
Available online 21 February 2008

**Abstract**

In this paper, we study the relations between the Bell matrix and the Fibonacci matrix, which provide a unified approach to some lower triangular matrices, such as the Stirling matrices of both kinds, the Lah matrix, and the generalized Pascal matrix. To make the results more general, the discussion is also extended to the generalized Fibonacci numbers and the corresponding matrix. Moreover, based on the matrix representations, various identities are derived.


*Keywords:* Combinatorial identities; Fibonacci numbers; Generalized Fibonacci numbers; Bell polynomials; Iteration matrix

## 1. Introduction

Recently, the lower triangular matrices have catalyzed many investigations. The Pascal matrix and several generalized Pascal matrices first received wide concern (see, e.g., [2,3,17,18]), and some other lower triangular matrices were also studied systematically, for example, the Lah matrix [14], the Stirling matrices of the first kind and of the second kind [5,6].

In this paper, we will study the Fibonacci matrix and the Bell matrix. Let us first consider a special $n \times n$ lower triangular matrix $S_n$ which is defined by

$$(S_n)_{i,j} = \begin{cases} 1, & i = j, \\ -1, & i-2 \le j \le i-1, \\ 0, & \text{else}, \end{cases} \quad \text{for } i, j = 1, 2, \dots, n. \tag{1.1}$$

Thus, we have

$$S_n = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & -1 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & -1 & 1 \end{pmatrix}.$$

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0166-218X/$ - see front matter 
doi:10.1016/j.dam.2007.10.025

By some computation, it is not difficult to find that the inverse of $S_n$ is

$$S_n^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 \\ 2 & 1 & 1 & 0 & \cdots & 0 \\ 3 & 2 & 1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \\ & \cdots & 3 & 2 & 1 & 1 \end{pmatrix}.$$

Now, one can observe that the inverse matrix $S_n^{-1}$ is related to the famous Fibonacci numbers, which are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Actually, if we denote $S_n^{-1}$ by $\mathscr{F}_n$, then

$$(\mathscr{F}_n)_{i,j} = f_{i,j} = \begin{cases} F_{i-j+1}, & i-j+1 > 0, \\ 0, & i-j+1 \leq 0, \end{cases} \quad \text{for } i, j = 1, 2, \ldots, n. \tag{1.2}$$

For this reason, $\mathscr{F}_n$ is called the $n \times n$ Fibonacci matrix.

The Fibonacci matrix and several generalizations have already been studied in many works. In [11], the authors gave the Cholesky factorization of the Fibonacci matrix $\mathscr{F}_n$ and discussed the eigenvalues of the symmetric Fibonacci matrix $\mathscr{F}_n\mathscr{F}_n^{\mathrm{T}}$. In [8,9], the authors studied the generalized Fibonacci matrix as well as the $k$-Fibonacci matrix. Moreover, the relations between the Fibonacci matrix and some other lower triangular matrices, such as the Pascal matrix, the Stirling matrices of the first kind and of the second kind as well as the Bernoulli matrix, were investigated in [10,19], respectively.

It is well known that many combinatorial sequences, for instance, the Stirling numbers and the Lah numbers, are special cases of the Bell polynomials (see [1], [4, Chapter 11] and [7, Chapter 3]). Therefore, by means of the study of the matrix related to the Bell polynomials, we will have a unified approach to various lower triangular matrices. In [15], we have studied the factorizations of the Bell matrix, and in the present paper, we will do some further researches on the relations between the Bell matrix and the Fibonacci matrix. Additionally, from the matrix representations, we will also give some combinatorial identities.

This article is organized as follows. In Section 2, we consider the relations between the matrix related to the exponential partial Bell polynomials and the Fibonacci matrix. Section 3 is devoted to the identities concerning the Fibonacci numbers and Section 4 is devoted to the Bell polynomials with respect to $\Omega$ as well as the iteration matrix. Finally, in Section 5, we extend the discussion to the generalized Fibonacci numbers and their corresponding matrix.

## 2. Relations between Bell matrix and Fibonacci matrix

The exponential partial Bell polynomials are defined as follows [7, pp. 133 and 134]:

**Definition 2.1.** The exponential partial Bell polynomials are the polynomials

$$B_{n,k} = B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$$

in an infinite number of variables $x_1, x_2, \ldots$, defined by the series expansion

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}\frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots.$$

Their exact expression is

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1!c_2!\cdots(1!)^{c_1}(2!)^{c_2}\cdots} x_1^{c_1}x_2^{c_2}\cdots,$$

where the summation takes place over all integers $c_1, c_2, c_3, \ldots \geq 0$, such that $c_1 + 2c_2 + 3c_3 + \cdots = n$ and $c_1 + c_2 + c_3 + \cdots = k$.

By the definition, we can readily obtain some special values of the Bell polynomials. Particularly, we have $B_{0,0} = 1$, $B_{n,0} = 0$, $B_{n,1} = x_n$ for $n \geq 1$ and $B_{n,k} = 0$ for $n < k$. In addition to these, the following lemma holds [7, p. 135].

**Lemma 2.2.** *For positive integers $n$ and $k$, we have*

$$B_{n,k}(1,1,1,\ldots) = S(n,k), \quad (\textit{Stirling number of the second kind}),$$
$$B_{n,k}(1!,2!,3!,\ldots) = \binom{n-1}{k-1}\frac{n!}{k!} = L(n,k), \quad (\textit{Lah number}),$$
$$B_{n,k}(0!,1!,2!,\ldots) = s(n,k), \quad (\textit{unsigned Stirling number of the first kind}),$$
$$B_{n,k}(1,2,3,\ldots) = \binom{n}{k}k^{n-k}, \quad (\textit{idempotent number}).$$

The readers are referred to [1] and [4, Chapter 11] for some other sequences which can be obtained from the Bell polynomials.

Now, define the $n \times n$ Bell matrix $B_n$ by $(B_n)_{i,j} = B_{i,j}$ and denote $(S_n)_{i,j} = (\mathscr{F}_n^{-1})_{i,j} = f'_{i,j}$, where $i, j = 1, 2, \ldots, n$. In the next lemma, we will consider the matrix multiplications $S_nB_n$ and $B_nS_n$.

**Lemma 2.3.** *We have $S_nB_n = \mathscr{N}_n$ and $B_nS_n = \mathscr{M}_n$, where the $n \times n$ matrices $\mathscr{N}_n$ and $\mathscr{M}_n$ are defined by*

$$(\mathscr{N}_n)_{i,j} = q_{i,j} = B_{i,j} - B_{i-1,j} - B_{i-2,j},$$
$$(\mathscr{M}_n)_{i,j} = p_{i,j} = B_{i,j} - B_{i,j+1} - B_{i,j+2},$$

*respectively, for $i, j = 1, 2, \ldots, n$.*

**Proof.** From Definition 2.1 and the remark after it, we can determine the elements of the matrix $\mathscr{N}_n$. Especially, $q_{1,1} = B_{1,1}$, $q_{1,j} = 0$ for $j \geq 2$, $q_{2,1} = B_{2,1} - B_{1,1}$, $q_{2,2} = B_{2,2}$, and $q_{2,j} = 0$ for $j \geq 3$. The elements of $\mathscr{M}_n$ can also be determined in a similar way.

Let us first verify the equation $S_nB_n = \mathscr{N}_n$.

Since $f'_{1,j} = B_{1,j} = 0$ for $j \geq 2$, then $\sum_{k=1}^n f'_{1,k}B_{k,1} = f'_{1,1}B_{1,1} = B_{1,1} = q_{1,1}$, $\sum_{k=1}^n f'_{1,k}B_{k,j} = f'_{1,1}B_{1,j} = 0 = q_{1,j}$ for $j \geq 2$. Since $f'_{2,1} = -1$, $f'_{2,2} = 1$ and $f'_{2,j} = 0$ for $j \geq 3$, then $\sum_{k=1}^n f'_{2,k}B_{k,1} = f'_{2,1}B_{1,1} + f'_{2,2}B_{2,1} = B_{2,1} - B_{1,1} = q_{2,1}$, $\sum_{k=1}^n f'_{2,k}B_{k,2} = f'_{2,1}B_{1,2} + f'_{2,2}B_{2,2} = B_{2,2} = q_{2,2}$, and $\sum_{k=1}^n f'_{2,k}B_{k,j} = f'_{2,1}B_{1,j} + f'_{2,2}B_{2,j} = 0 = q_{2,j}$ for $j \geq 3$.

Next, let $i \geq 3$. In view of (1.1), $\sum_{k=1}^n f'_{i,k}B_{k,j} = f'_{i,i}B_{i,j} + f'_{i,i-1}B_{i-1,j} + f'_{i,i-2}B_{i-2,j} = B_{i,j} - B_{i-1,j} - B_{i-2,j} = q_{i,j}$.

Therefore, we have $S_nB_n = \mathscr{N}_n$.

Similar to the preceding process, we can also verify the equation $B_nS_n = \mathscr{M}_n$. □

Since the Fibonacci matrix $\mathscr{F}_n$ is the inverse of $S_n$, the following theorem holds.

**Theorem 2.4.** *The Bell matrix $B_n$ can be factorized as*

$$B_n = \mathscr{F}_n\mathscr{N}_n = \mathscr{M}_n\mathscr{F}_n. \tag{2.1}$$

*From the factorizations, we have for $1 \leq k \leq n$ that*

$$B_{n,k} = \sum_{l=k}^{n} F_{n-l+1}(B_{l,k} - B_{l-1,k} - B_{l-2,k}) \tag{2.2}$$
$$= \sum_{l=k}^{n}(B_{n,l} - B_{n,l+1} - B_{n,l+2})F_{l-k+1}. \tag{2.3}$$

Let $E_n = (1, 1, \ldots, 1)^{\mathrm{T}}$. Since $B_nE_n = \mathscr{M}_n\mathscr{F}_nE_n$ and [10, Corollary 2.2]

$$F_1 + F_2 + \cdots + F_{n-2} = F_n - 1, \tag{2.4}$$

then (2.3) implies that

$$\sum_{k=1}^{n} B_{n,k} = \sum_{k=1}^{n}(B_{n,k} - B_{n,k+1} - B_{n,k+2})(F_{k+2} - 1). \tag{2.5}$$

Making use of the general identities (2.2), (2.3) and (2.5), we can obtain the corresponding ones for some special combinatorial sequences, which are given by the corollaries below. It should be noticed that, from the generating functions (see [7, pp. 156, 206 and 213]), $S(i, j)$, $s(i, j)$ and $L(i, j)$ will vanish for $0 \le i < j$. Additionally, we also follow the convention that $S(-1, 1) = s(-1, 1) = L(-1, 1) = 0$.

**Corollary 2.5.** *For* $1 \le k \le n$,

$$\begin{aligned} S(n,k) &= \sum_{l=k}^{n} F_{n-l+1}(S(l,k) - S(l-1,k) - S(l-2,k)) \\ &= \sum_{l=k}^{n}(S(n,l) - S(n,l+1) - S(n,l+2))F_{l-k+1}, \end{aligned} \tag{2.6}$$

$$\omega(n) = \sum_{k=1}^{n}(S(n,k) - S(n,k+1) - S(n,k+2))(F_{k+2} - 1),$$

*where* $\omega(n) := \sum_{k=1}^{n} S(n,k)$ *is the Bell number* [7, *p.* 210].

**Corollary 2.6.** *For* $1 \le k \le n$,

$$\begin{aligned} s(n,k) &= \sum_{l=k}^{n} F_{n-l+1}(s(l,k) - s(l-1,k) - s(l-2,k)) \\ &= \sum_{l=k}^{n}(s(n,l) - s(n,l+1) - s(n,l+2))F_{l-k+1}, \end{aligned}$$

$$n! = \sum_{k=1}^{n}(s(n,k) - s(n,k+1) - s(n,k+2))(F_{k+2} - 1). \tag{2.7}$$

It should be noticed that (2.6) and (2.7) have been proved in [10].

**Corollary 2.7.** *For* $1 \le k \le n$,

$$\begin{aligned} L(n,k) &= \sum_{l=k}^{n} F_{n-l+1}(L(l,k) - L(l-1,k) - L(l-2,k)) \\ &= \sum_{l=k}^{n}(L(n,l) - L(n,l+1) - L(n,l+2))F_{l-k+1}. \end{aligned}$$

For the idempotent numbers, the next corollary can be derived.

**Corollary 2.8.** *For* $1 \le k \le n$,

$$\binom{n}{k} k^{n-k} = F_{n-k+1} + F_{n-k}(k^2 + k - 1) + \sum_{l=k+2}^{n} k^{l-2-k} F_{n-l+1}\left(\binom{l}{k}k^2 - \binom{l-1}{k}k - \binom{l-2}{k}\right),$$

$$\binom{n}{k} k^{n-k} = \sum_{l=k}^{n}\left(\binom{n}{l}l^{n-l} - \binom{n}{l+1}(l+1)^{n-l-1} - \binom{n}{l+2}(l+2)^{n-l-2}\right)F_{l-k+1}.$$

## 3. Identities related to the Fibonacci numbers

Some combinatorial identities have been derived from the matrix representations in the last section. In this section, we will show that, from a general identity, more identities concerning the Fibonacci numbers can be obtained. In fact, since $B_{n,1} = x_n$, then by setting $k = 1$ in (2.2), we have the theorem below.

**Theorem 3.1.** *For sequence* $\{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\}$, *the following identity holds:*

$$x_n = F_n x_1 + F_{n-1}(x_2 - x_1) + \sum_{l=3}^{n} F_{n-l+1}(x_l - x_{l-1} - x_{l-2}). \tag{3.1}$$

Therefore, by choosing different sequences, we can obtain plenty of interesting identities. For example, if $x_n = 1$, then the summation $F_1 + F_2 + \cdots + F_{n-2} = F_n - 1$ will be obtained again. Also, for $n \geq 3$, we have

$$H_n = \frac{1}{2}F_{n+2} + \sum_{l=3}^{n} F_{n-l+1}\left(\frac{1}{l} - H_{l-2}\right),$$

$$n! = F_{n+1} + \sum_{l=3}^{n} F_{n-l+1} l(l-2)(l-2)!,$$

$$\omega(n) = F_{n+1} + \sum_{l=3}^{n} F_{n-l+1}(\omega(l) - \omega(l-1) - \omega(l-2)),$$

where $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the harmonic number and $\omega(n)$ is the Bell number.

It is apparent that we can do more than these. Let us now define $x_n := F_n^r$, where $F_n$ is the $n$th Fibonacci number and $r$ is a positive integer, then the general identity (3.1) will lead us to the following theorem.

**Theorem 3.2.** *For each integer* $r \geq 1$,

$$F_k^r = F_k + \sum_{l=3}^{k}\sum_{j=1}^{r-1}\binom{r}{j} F_{l-1}^j F_{l-2}^{r-j} F_{k-l+1}, \quad k \geq 3, \tag{3.2}$$

$$\sum_{k=1}^{n} F_k^r = F_{n+2}^r - 1 - \sum_{l=3}^{n+2}\sum_{j=1}^{r-1}\binom{r}{j} F_{l-1}^j F_{l-2}^{r-j}. \tag{3.3}$$

**Proof.** (3.2) follows directly from (3.1) by considering the binomial expansion. Thus,

$$\begin{aligned}
\sum_{k=1}^{n} F_k^r &= F_1^r + F_2^r + \sum_{k=3}^{n} F_k + \sum_{k=3}^{n}\sum_{l=3}^{k}\left(\sum_{j=1}^{r-1}\binom{r}{j} F_{l-1}^j F_{l-2}^{r-j}\right) F_{k-l+1} \\
&= F_{n+2} - 1 + \sum_{l=3}^{n}\left(\sum_{j=1}^{r-1}\binom{r}{j} F_{l-1}^j F_{l-2}^{r-j}\right)(F_{n-l+3} - 1) \\
&= F_{n+2} - 1 + \sum_{l=3}^{n}\left(\sum_{j=1}^{r-1}\binom{r}{j} F_{l-1}^j F_{l-2}^{r-j}\right) F_{n-l+3} - \sum_{l=3}^{n}\sum_{j=1}^{r-1}\binom{r}{j} F_{l-1}^j F_{l-2}^{r-j} \\
&= F_{n+2}^r - 1 - \sum_{l=3}^{n+2}\sum_{j=1}^{r-1}\binom{r}{j} F_{l-1}^j F_{l-2}^{r-j},
\end{aligned}$$

where the last step can be obtained by the substitution of identity (3.2). □

In addition to these, by setting $x_n := n^r$ in (3.1), where $r$ is a non-negative integer, we can obtain the next result.

**Theorem 3.3.** *For each integer* $r \geq 0$,

$$k^r = F_{k-2} + 2^r F_{k-1} + \sum_{l=3}^{k} F_{k-l+1}(l^r - (l-1)^r - (l-2)^r), \quad k \geq 3, \tag{3.4}$$

$$\sum_{k=1}^{n} k^r = \frac{1}{r+1}\{B_{r+1}(n+1) - B_{r+1}\} \tag{3.5}$$

$$= F_n + 2^r(F_{n+1} - 1) + \sum_{l=3}^{n}(F_{n-l+3} - 1)(l^r - (l-1)^r - (l-2)^r), \tag{3.6}$$

*where $B_r(x)$ is the Bernoulli polynomial and $B_r = B_r(0)$ is the Bernoulli number.*

**Proof.** (3.4) is a direct consequence of (3.1). (3.5) is a well-known fact (see, e.g., [7, p. 155]). We now prove (3.6). Actually, by (2.4) and (3.4),

$$\begin{aligned}\sum_{k=1}^{n} k^r &= 1^r + 2^r + \sum_{k=3}^{n} k^r \\ &= 1 + 2^r + \sum_{k=3}^{n} F_{k-2} + 2^r \sum_{k=3}^{n} F_{k-1} + \sum_{l=3}^{n}\left(\sum_{k=l}^{n} F_{k-l+1}\right)(l^r - (l-1)^r - (l-2)^r) \\ &= F_n + 2^r(F_{n+1} - 1) + \sum_{l=3}^{n}(F_{n-l+3} - 1)(l^r - (l-1)^r - (l-2)^r).\end{aligned}$$

The proof is complete. □

Theorem 3.3 gives the following special cases.

**Corollary 3.4.** *We have*

$$k = F_{k+1} + \sum_{l=4}^{k}(3-l)F_{k-l+1}, \quad k \geq 4, \tag{3.7}$$

$$\sum_{l=1}^{n}(n-l+1)F_l = F_{n+4} - n - 3. \tag{3.8}$$

**Proof.** (3.7) follows from (3.4) by letting $r = 1$. For (3.8), identity (3.6) yields

$$\begin{aligned}\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} = F_n + 2(F_{n+1} - 1) + \sum_{l=3}^{n}(F_{n-l+3} - 1)(3-l) \\ &= F_{n+3} - 2 + 3\sum_{l=3}^{n} F_{n-l+3} - 3(n-2) - \sum_{l=3}^{n} lF_{n-l+3} + \sum_{l=3}^{n} l,\end{aligned}$$

which leads us to

$$\sum_{l=3}^{n} lF_{n-l+3} = F_{n+3} + 3F_{n+2} - 3n - 8.$$

Thus, adding $F_{n+2} + 2F_{n+1}$ to both sides of the above equation, and making use of the recurrence $F_n = F_{n-1} + F_{n-2}$, we can finally obtain $\sum_{l=1}^{n} lF_{n-l+3} = F_{n+6} - 3n - 8$, an equivalent form of formula (3.8). Note that (3.8) can also be found in [12, p. 157]. □

**Corollary 3.5.** *We have*

$$k^2 = F_k - \sum_{l=2}^{k} F_{k-l+1}(l-1)(l-5),$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = F_{n+2} - 1 - \sum_{l=2}^{n}(F_{n-l+3} - 1)(l-1)(l-5).$$

## 4. Iteration matrix and Bell polynomials with respect to $\Omega$

Let $f(t)$ be a formal series of the form:

$$f(t) = \sum_{n\geq 1} \Omega_n f_n t^n,$$

where $\Omega_1, \Omega_2, \ldots$ is a reference sequence with $\Omega_n \neq 0$. In this way we treat at the same time the case of ordinary coefficients of $f(t)$ ($\Leftrightarrow \Omega_n = 1$), and the case of Taylor coefficients ($\Leftrightarrow \Omega_n = 1/n!$).

With every series $f(t)$ we associate the infinite lower iteration matrix with respect to $\Omega$:

$$B = B(f) := \begin{pmatrix} B_{1,1} & 0 & 0 & \cdots \\ B_{2,1} & B_{2,2} & 0 & \cdots \\ B_{3,1} & B_{3,2} & B_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $B_{n,k} = B^{\Omega}_{n,k}(f_1, f_2, \ldots)$ is the Bell polynomial with respect to $\Omega$ [7, pp. 137 and 145], defined as follows:

$$\Omega_k (f(t))^k = \sum_{n\geq k} B_{n,k} \Omega_n t^n.$$

Thus, the Pascal matrix is the iteration matrix for $f(t) = t(1-t)^{-1}$, $\Omega_n = 1$, and the Stirling matrix of the second kind is the iteration matrix for $f(t) = \mathrm{e}^t - 1$, $\Omega_n = 1/n!$.

It is easy to find that Theorem 2.4 and Eq. (2.5) also hold when Bell matrix and exponential partial Bell polynomials are replaced by iteration matrix and Bell polynomials with respect to $\Omega$, respectively.

For instance, the $n \times n$ Pascal matrix $P_n$, which is defined by $(P_n)_{i,j} = \binom{i-1}{j-1}$ for $i, j = 1, 2, \ldots, n$, can be factorized as

$$P_n = \mathscr{F}_n \tilde{\mathscr{N}_n} = \tilde{\mathscr{M}_n} \mathscr{F}_n,$$

where the $n \times n$ matrix $\tilde{\mathscr{M}_n}$ is defined by

$$(\tilde{\mathscr{M}_n})_{i,j} = \binom{i-1}{j-1} - \binom{i-1}{j} - \binom{i-1}{j+1}, \quad \text{for } i, j = 1, 2, \ldots, n;$$

because $\binom{-1}{k}$ and $\binom{-2}{k}$ do not vanish when $k$ is a non-negative integer, we should define $\tilde{\mathscr{N}_n}$ more carefully, as follows:

$$(\tilde{\mathscr{N}_n})_{i,j} = \begin{cases} 0, & j > i, \\ 1, & j = i, \\ i-2, & j = i-1, \\ \binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}, & \text{else,} \end{cases} \qquad \text{for } i, j = 1, 2, \ldots, n.$$

Moreover, the following theorem holds.

**Theorem 4.1.** *For $1 \leq k \leq n$, we have*

$$\binom{n-1}{k-1} = \sum_{l=k}^{n} \frac{(n-1)!}{(l+1)!(n-l)!} [l^2 + (n+1)l - n^2] F_{l-k+1}, \tag{4.1}$$

$$n + 1 = \sum_{l=1}^{n} \frac{(n-1)!}{(l+1)!(n-l)!} [l^2 + (n+1)l - n^2] F_{l+2}. \tag{4.2}$$

**Proof.** (4.1) follows from the factorization $P_n = \tilde{\mathscr{M}_n} \mathscr{F}_n$, and (4.2) follows from the equation $P_n E_n = \tilde{\mathscr{M}_n} \mathscr{F}_n E_n$, where $E_n = (1, 1, \ldots, 1)^{\mathrm{T}}$. $\square$

From another factorization $P_n = \mathscr{F}_n \tilde{\mathscr{N}_n}$, one can also derive some identities, for which, the readers are referred to [10, Section 2].

We have just mentioned that, if $\Omega_n = 1$, then the iteration matrix for $f(t) = t(1-t)^{-1}$ is the Pascal matrix. Now, let us consider the power of the Pascal matrix. According to the definition given in [7, p. 147], for each complex number $x$, we can obtain the $x$th order fractional iterate $f^{\langle x \rangle}(t)$ and the corresponding iteration matrix $P^x$. More explicitly, we have (see [15, Example 3.3])

$$f^{\langle x \rangle}(t) = \sum_{n \geq 1} f_n^{\langle x \rangle} t^n = \frac{t}{1-xt},$$

$$\sum_{n \geq k} B_{n,k} t^n = t^k (1-xt)^{-k} = \sum_{n \geq k} \binom{n-1}{k-1} x^{n-k} t^n.$$

Therefore, the matrix $P_n^x$ is defined by

$$(P_n^x)_{i,j} = \binom{i-1}{j-1} x^{i-j}, \quad \text{for } i, j = 1, 2, \ldots, n,$$

which is the generalized Pascal matrix studied in [17], and we denote it by $P_n[x]$.

In view of Theorem 2.4, $P_n[x]$ has the following factorizations:

$$P_n[x] = \mathscr{F}_n \hat{\mathscr{N}_n} = \hat{\mathscr{M}_n} \mathscr{F}_n, \tag{4.3}$$

where $\hat{\mathscr{N}_n}$, $\hat{\mathscr{M}_n}$ are $n \times n$ matrices defined by

$$(\hat{\mathscr{N}_n})_{i,j} = \begin{cases} 0, & j > i, \\ 1, & j = i, \\ (i-1)x - 1, & j = i-1, \\ \binom{i-1}{j-1} x^{i-j} - \binom{i-2}{j-1} x^{i-j-1} - \binom{i-3}{j-1} x^{i-j-2}, & \text{else;} \end{cases}$$

$$(\hat{\mathscr{M}_n})_{i,j} = \binom{i-1}{j-1} x^{i-j} - \binom{i-1}{j} x^{i-j-1} - \binom{i-1}{j+1} x^{i-j-2},$$

for $i, j = 1, 2, \ldots, n$. Thus, we can obtain the next two theorems.

**Theorem 4.2.** *For $1 \leq k \leq n$, we have*

$$\begin{aligned} \binom{n-1}{k-1} x^{n-k} &= F_{n-k+1} + F_{n-k}(kx-1) \\ &\quad + \sum_{l=k+2}^{n} F_{n-l+1} \left( \binom{l-1}{k-1} x^{l-k} - \binom{l-2}{k-1} x^{l-k-1} - \binom{l-3}{k-1} x^{l-k-2} \right), \end{aligned} \tag{4.4}$$

$$\binom{n-1}{k-1} x^{n-k} = \sum_{l=k}^{n} \left( \binom{n-1}{l-1} x^{n-l} - \binom{n-1}{l} x^{n-l-1} - \binom{n-1}{l+1} x^{n-l-2} \right) F_{l-k+1}. \tag{4.5}$$

**Theorem 4.3.** *We have*

$$x^n = F_{n+1} + F_n(x-1) + (x^2 - x - 1) \sum_{j=1}^{n-1} F_j x^{n-j-1}, \tag{4.6}$$

$$2x + n - 1 = \sum_{k=1}^{n} x^{-k} \left( \binom{n-1}{k-1} x^2 - \binom{n-1}{k} x - \binom{n-1}{k+1} \right) F_{k+2}. \tag{4.7}$$

**Proof.** (4.6) follows from (4.4) by setting $k = 1$. One can also obtain it from $P_n[x] E_n = \mathscr{F}_n \hat{\mathscr{N}_n} E_n$. (4.7) follows from another matrix representation $P_n[x] E_n = \hat{\mathscr{M}_n} \mathscr{F}_n E_n$. $\square$

In the following, we will study the series $f(t) = \frac{2t}{(1-t)^2}$. For arbitrary sequence $\Omega_n$,

$$\Omega_k (f(t))^k = \Omega_k \left(\frac{2t}{(1-t)^2}\right)^k = \Omega_k 2^k \sum_{j=0}^{\infty} \binom{-2k}{j} (-1)^j t^{k+j},$$

then the corresponding Bell polynomials are defined by

$$B_{n,k} = \frac{\Omega_k}{\Omega_n} \binom{-2k}{n-k} (-1)^{n-k} 2^k = \frac{\Omega_k}{\Omega_n} \binom{n+k-1}{n-k} 2^k.$$

Thus, Eq. (2.2) will lead us to the theorem below.

**Theorem 4.4.** *For $1 \le k \le n$, we have*

$$\frac{1}{\Omega_n}\binom{n+k-1}{n-k} = \frac{1}{\Omega_k} F_{n-k+1} + F_{n-k}\left(\frac{2k}{\Omega_{k+1}} - \frac{1}{\Omega_k}\right) + \sum_{l=k+2}^{n} F_{n-l+1}\left(\frac{1}{\Omega_l}\binom{l+k-1}{l-k} - \frac{1}{\Omega_{l-1}}\binom{l+k-2}{l-k-1} - \frac{1}{\Omega_{l-2}}\binom{l+k-3}{l-k-2}\right). \quad (4.8)$$

When $\Omega_n = 1$, identity (4.8) yields

$$\binom{n+k-1}{n-k} = F_{n-k+1} + \sum_{l=k+1}^{n} F_{n-l+1} \frac{(k^2+4kl-6k-l^2+2)(l+k-3)!}{(l-k)!(2k-1)!}.$$

If $k = 1$, the identity above will reduce to

$$n + F_n = \sum_{l=1}^{n} (3-l) F_{n-l+1},$$

which can also be derived by adding $F_{k-1} + 2F_k$ to both sides of Eq. (3.7).

When $\Omega_n = (-1)^n$, (4.8) gives

$$(-1)^n \binom{n+k-1}{n-k} = (-1)^k F_{n-k+1} + \sum_{l=k+1}^{n} (-1)^l F_{n-l+1} \frac{(l^2+4lk-4l-k^2-2k+2)(l+k-3)!}{(l-k)!(2k-1)!}.$$

Setting $k = 1$, we have

$$(-1)^n n - F_n = \sum_{l=1}^{n} (-1)^l (l+1) F_{n-l+1}.$$

Many identities can be obtained by choosing different $\Omega_n$. For example, we can let $\Omega_n = \binom{-\lambda}{n}$ or $\Omega_n = \frac{\langle 1+\alpha+\beta\rangle_{2n}}{2^{2n}\langle 1+\alpha\rangle_n}$, where $-\lambda$ is not a non-negative integer and $\langle x\rangle_n := x(x+1)\cdots(x+n-1)$ is the rising factorial. More such sequences can be found in [13, pp. 68 and 69].

Additionally, we can also derive some identities from Eqs. (2.3) and (2.5). These are left to the interested readers.

## 5. Generalized Fibonacci numbers

In this section, we will discuss a generalization of the Fibonacci numbers. Let $a, b$ be integers satisfying $a^2 + 4b > 0$, then one can define the generalized Fibonacci numbers by $G_0 = 0$, $G_1 = 1$ and $G_n = aG_{n-1} + bG_{n-2}$ for $n \ge 2$. It is evident that when $a = b = 1$, $G_n$ reduce to the classical Fibonacci numbers. Moreover, from [16] we know that if $a = 2, b = 1$, then $G_n$ are the Pell numbers, and if $a = 1, b = 2$, then $G_n$ are the Jacobsthal numbers.

The properties of the generalized Fibonacci numbers $G_n$ are similar to those of the classical Fibonacci numbers $F_n$. For instance, from the recurrence relation, we can obtain the generating function of $G_n$, that is,

$$G(t) = \sum_{n=0}^{\infty} G_n t^n = \frac{t}{1-at-bt^2}.$$

Let $\alpha, \beta$ be the roots of $x^2 - ax - b = 0$, then $G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. In addition to these, if we define an $n \times n$ lower triangular matrix $T_n$ by

$$T_n = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ -a & 1 & 0 & \cdots & \cdots & 0 \\ -b & -a & 1 & 0 & \cdots & 0 \\ 0 & -b & -a & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -b & -a & 1 \end{pmatrix},$$

then the inverse of $T_n$ is

$$\mathscr{G}_n = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ G_2 & 1 & 0 & \cdots & \cdots & 0 \\ G_3 & G_2 & 1 & 0 & \cdots & 0 \\ G_4 & G_3 & G_2 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \\ G_n & \cdots & G_4 & G_3 & G_2 & 1 \end{pmatrix}.$$

Therefore, we can obtain the following theorem.

**Theorem 5.1.** *The Bell matrix $B_n$ can be factorized as*

$$B_n = \mathscr{G}_n \mathscr{V}_n = \mathscr{U}_n \mathscr{G}_n,$$

*where the $n \times n$ matrices $\mathscr{V}_n$ and $\mathscr{U}_n$ are defined by*

$$(\mathscr{V}_n)_{i,j} = B_{i,j} - aB_{i-1,j} - bB_{i-2,j}, \qquad (\mathscr{U}_n)_{i,j} = B_{i,j} - aB_{i,j+1} - bB_{i,j+2},$$

*respectively, for $i, j = 1, 2, \ldots, n$. From the factorizations, we have for $1 \le k \le n$ that*

$$B_{n,k} = \sum_{l=k}^{n} G_{n-l+1}(B_{l,k} - aB_{l-1,k} - bB_{l-2,k}) \tag{5.1}$$

$$= \sum_{l=k}^{n} (B_{n,l} - aB_{n,l+1} - bB_{n,l+2})G_{l-k+1}. \tag{5.2}$$

By setting $k = 1$ in (5.1), we can establish the next theorem.

**Theorem 5.2.** *For sequence $\{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\}$, the following identity holds:*

$$x_n = G_n x_1 + G_{n-1}(x_2 - ax_1) + \sum_{l=3}^{n} G_{n-l+1}(x_l - ax_{l-1} - bx_{l-2}). \tag{5.3}$$

Just as what we have done for Theorem 3.1, by choosing different sequences, many combinatorial identities can be obtained.

If $x_n = 1$, identity (5.3) yields

$$\sum_{k=1}^{n-2} G_k = \frac{G_n + (1-a)G_{n-1} - 1}{a + b - 1}. \tag{5.4}$$

By substituting $G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ into the identity above, we can derive an expression of $\sum_{k=1}^{n-2} G_k$, which is only depend on $a$ and $b$.

If $x_n = G_n^r$, where $r$ is a non-negative integer, then (5.3) gives

$$G_n^r = G_n + G_{n-1}(a^r - a) + \sum_{l=3}^{n} G_{n-l+1}(G_l^r - aG_{l-1}^r - bG_{l-2}^r), \quad n \geq 3.$$

If $x_n = n^r$, where $r$ is a non-negative integer, then

$$k^r = bG_{k-2} + 2^r G_{k-1} + \sum_{l=3}^{k} G_{k-l+1}(l^r - a(l-1)^r - b(l-2)^r), \quad k \geq 3.$$

By appealing to (5.4), we have

$$\begin{aligned}\sum_{k=1}^{n} k^r &= 1 + 2^r G_{n-1} + (b + 2^r)\frac{G_n + (1-a)G_{n-1} - 1}{a+b-1}\\ &\quad + \frac{1}{a+b-1}\sum_{l=3}^{n}(G_{n-l+3} + (1-a)G_{n-l+2} - 1)(l^r - a(l-1)^r - b(l-2)^r).\end{aligned}$$

Analogous to Section 4, we can also study the iteration matrix and the Bell polynomials with respect to $\Omega_n$, and give more identities on the generalized Fibonacci numbers. However, we chose not to present them. The readers may do it themselves.


## Acknowledgments

We would like to thank the anonymous referees for their detailed comments and suggestions which have improved the presentation of the paper. Particularly, one of the referees suggested considering the generalized Fibonacci numbers, and this suggestion expands our knowledge of some special combinatorial sequences.


## References


[1] M. Abbas, S. Bouroubi, On new identities for Bell's polynomials, Discrete Math. 293 (1–3) (2005) 5–10.
[2] R. Brawer, M. Pirovino, The linear algebra of the Pascal matrix, Linear Algebra Appl. 174 (1992) 13–23.
[3] G.S. Call, D.J. Velleman, Pascal's matrices, Amer. Math. Monthly 100 (4) (1993) 372–376.
[4] C.A. Charalambides, Enumerative combinatorics, in: CRC Press Series on Discrete Mathematics and its Applications, Chapman & Hall/CRC, Boca Raton, FL, 2002.
[5] G.-S. Cheon, J.-S. Kim, Stirling matrix via Pascal matrix, Linear Algebra Appl. 329 (1–3) (2001) 49–59.
[6] G.-S. Cheon, J.-S. Kim, Factorial Stirling matrix and related combinatorial sequences, Linear Algebra Appl. 357 (2002) 247–258.
[7] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Co., Dordrecht, 1974.
[8] G.-Y. Lee, J.-S. Kim, The linear algebra of the generalized Fibonacci matrices, Fibonacci Quart. 41 (5) (2003) 451–465.
[9] G.-Y. Lee, J.-S. Kim, The linear algebra of the $k$-Fibonacci matrix, Linear Algebra Appl. 373 (2003) 75–87.
[10] G.-Y. Lee, J.-S. Kim, S.-H. Cho, Some combinatorial identities via Fibonacci numbers, Discrete Appl. Math. 130 (3) (2003) 527–534.
[11] G.-Y. Lee, J.-S. Kim, S.-G. Lee, Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices, Fibonacci Quart. 40 (3) (2002) 203–211.
[12] J. Riordan, Combinatorial Identities, Robert E. Krieger Publishing Co., Huntington, NY, 1979. Reprint of the 1968 original.
[13] S. Roman, The theory of the umbral calculus. I, J. Math. Anal. Appl. 87 (1) (1982) 58–115.
[14] M. Tan, T. Wang, Lah matrix and its algebraic properties, Ars Combin. 70 (2004) 97–108.
[15] W. Wang, T. Wang, Matrices related to the Bell polynomials, Linear Algebra Appl. 422 (1) (2007) 139–154.
[16] E.W. Weisstein, Lucas Sequence, From MathWorld — A Wolfram Web Resource. http://mathworld.wolfram.com/LucasSequence.html.
[17] Z. Zhang, The linear algebra of the generalized Pascal matrix, Linear Algebra Appl. 250 (1997) 51–60.
[18] Z. Zhang, M. Liu, An extension of the generalized Pascal matrix and its algebraic properties, Linear Algebra Appl. 271 (1998) 169–177.
[19] Z. Zhang, J. Wang, Bernoulli matrix and its algebraic properties, Discrete Appl. Math. 154 (2006) 1622–1632.