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On colorings of graph powers[☆]

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ABSTRACT

In this paper, some results concerning the colorings of graph powers are presented. The notion of helical graphs is introduced. We show that such graphs are hom-universal with respect to high odd-girth graphs whose $(2t+1)$th power is bounded by a Kneser graph according to the homomorphism order. Also, we consider the problem of existence of homomorphism to odd cycles. We prove that such homomorphism to a $(2k+1)$-cycle exists if and only if the chromatic number of the $(2k+1)$th power of $G^{\frac{1}{3}}$ is less than or equal to 3, where $G^{\frac{1}{3}}$ is the 3-subdivision of G. We also consider Nešetřil's Pentagon problem. This problem is about the existence of high girth cubic graphs which are not homomorphic to the cycle of size five. Several problems which are closely related to Nešetřil's problem are introduced and their relations are presented.

1. Introduction

Throughout this paper we only consider finite graphs. A *homomorphism* $f : G \longrightarrow H$ from a graph G to a graph H is a map $f : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. The existence of a homomorphism is indicated by the symbol $G \longrightarrow H$. Two graphs G and H are homomorphically equivalent if $G \longrightarrow H$ and $H \longrightarrow G$. Also, the symbol $\mathrm{Hom}(G, H)$ is used to denote the set of all homomorphisms from G to H (for more on graph homomorphisms see [2,3,8,13]). If n and d are positive integers with $n \geq 2d$, then the *circular complete graph* $K_{\frac{n}{d}}$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ in which v_i is connected to v_j if and only if $d \leq |i-j| \leq n-d$. A graph G is said to be (n, d)-colorable if G admits a homomorphism to $K_{\frac{n}{d}}$. The *circular chromatic number* (also known as the *star chromatic number* [32]) $\chi_c(G)$ of a graph G is the minimum of those ratios $\frac{n}{d}$ for which $gcd(n, d) = 1$ and such that G admits a homomorphism to $K_{\frac{n}{d}}$. It can be shown that one may only consider onto-vertex homomorphisms [34]. We denote by $[m]$ the set $\{1, 2, \ldots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. For a given subset $A \subseteq [m]$, the complement of A in $[m]$ is denoted by $\overline{A}$. The *Kneser graph* $KG(m, n)$ is the graph with vertex set $\binom{[m]}{n}$, in which A is connected to B if and only if $A \cap B = \emptyset$. It was conjectured by Kneser [16] in 1955, and proved by Lovász [20] in 1978, that $\chi(KG(m, n)) = m - 2n + 2$. A subset S of $[m]$ is called 2-*stable* if $2 \leq |x-y| \leq m-2$ for all distinct elements x and y of S. The *Schrijver graph* $SG(m, n)$ is the subgraph of $KG(m, n)$ induced by all 2-stable n-subsets of $[m]$. It was proved by Schrijver [28] that $\chi(SG(m, n)) = \chi(KG(m, n))$ and that every proper subgraph of $SG(m, n)$ has a chromatic number smaller than that of $SG(m, n)$. The *fractional chromatic number*, $\chi_f(G)$, of a graph G is defined as

$$\chi_f(G) \stackrel{\text{def}}{=} \inf\left\{\frac{m}{n} \,\middle|\, \mathrm{Hom}(G, KG(m, n)) \neq \emptyset\right\}.$$

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For more about fractional coloring see [27]. The *local chromatic number* of a graph was defined in [6] as the minimum number of colors that must appear within distance 1 of a vertex. For any positive integer k, the k-neighborhood of a vertex v of G is the set of all vertices whose distance from v is at most k. As a generalization of local chromatic number, it may be of interest to find a proper coloring which uses few colors for the k-neighborhood of every vertex.

Definition 1. Let G be a graph and k be a positive integer. Define

$$\psi_k(G) \stackrel{\text{def}}{=} \min_c \max_{v\in V(G)} |\{c(u) : u \in V(G), d_G(u, v) \le k\}|,$$

where the minimum is taken over all proper colorings c of G and $d_G(u, v)$ denotes the distance between u and v in G. □

Note that $\psi_1(G)$ is the local chromatic number of G. Set $\psi(G) \stackrel{\text{def}}{=} \psi_1(G)$ for convenience. One can define $\psi_k(G)$ via graph homomorphism. In this regard, k-universal graphs were defined in [6] as follows.

Definition 2. Let n, r, and k be positive integers with $n \ge r$. Set $U^k(n, r)$ to be the *k-universal graph* whose vertex set contains all $(k+1)$-tuples $(A_0, A_1, \ldots, A_k)$ such that for any $0 \le i \le k, A_i \subseteq [n]$, $|A_0| = 1$, $A_0 \cap A_1 = \emptyset$, $|A_0 \cup \cdots \cup A_k| \le r$ and for any $t \le k-2, A_t \subseteq A_{t+2}$. Also, two vertices $(A_0, A_1, \ldots, A_k)$ and $(B_0, B_1, \ldots, B_k)$ of $U^k(n, r)$ are adjacent if for any $0 \le j \le k-1$, $A_j \subseteq B_{j+1}$ and $B_j \subseteq A_{j+1}$. □

Roughly speaking, the vertices of $U^k(n, r)$ encode the set of colors that can be found within distance k in a coloring. The following lemma reveals the connection between k-universal graphs and $\psi_k(G)$.

Lemma A (*[6]*). *Let G be a graph and k and r be positive integers. Then $\psi_k(G) \le r$ if and only if there exists a positive integer n such that G admits a homomorphism to $U^k(n, r)$.*

It is easy to verify that for any graph G, $\psi(G) \le \psi_k(G) \le \chi(G)$. Also, it was shown in [17] that $\chi_f(G) \le \psi(G)$ holds for any graph G.

For a graph G, let G^k be the kth power of G, which is obtained on the vertex set $V(G)$, by connecting any two vertices u and v for which there exists a walk of length k between u and v in G. Note that the kth power of a simple graph is not necessarily a simple graph itself. For instance, the kth power may have loops on its vertices provided that k is an even integer. It should be noted that the square of a graph G, denoted by G^2, is commonly considered the graph on the same vertex set of G and having edges between pair of vertices at distance at most 2. While in this paper G^2 stands for the second power of G. For a given graph G, the notation $og(G)$ stands for the odd-girth of graph G. Let G be a graph with $og(G) \ge 7$, the chromatic number of G^5 provides an upper bound for the local chromatic number of G. In [29], it was proved that $\psi(G) \le \lfloor \frac{m}{2} \rfloor + 2$ whenever $\chi(G^5) \le m$. The chromatic number of graph powers has been studied in the literature (see [1,4,7,24,29,31]).

The following simple and useful lemma was proved and used independently in [4,26,31].

Lemma B. *Let G and H be two simple graphs such that* $\mathrm{Hom}(G, H) \ne \emptyset$. *Then, for any positive integer k,* $\mathrm{Hom}(G^k, H^k) \ne \emptyset$.

Note that Lemma B trivially holds whenever H^k contains a loop, e.g., when $k = 2$. As immediate consequences of Lemma B, we obtain $\chi_c(P) = \chi(P)$ and $\mathrm{Hom}(C, C_7) = \emptyset$, where P and C are the Petersen and the Coxeter graphs, respectively, see [4].

In what follows we are concerned with some results concerning the colorings of graph powers. First, the notion of helical graphs is introduced. We show that such graphs are hom-universal with respect to high odd-girth graphs whose $(2t+1)$th power is bounded by a Kneser graph according to the homomorphism order. Then we consider the problem of existence of homomorphism to odd cycles. We prove that such homomorphism to a $(2k+1)$-cycle exists if and only if the chromatic number of the $(2k+1)$th power of $G^{\frac{1}{3}}$ is less than or equal to 3, where $G^{\frac{1}{3}}$ is the 3-subdivision of G. We also consider Nešetřil's Pentagon problem. This problem is about the existence of high girth cubic graphs which are not homomorphic to the cycle of size five. Several problems which are closely related to Nešetřil's problem are introduced and their relations are presented.

2. Helical graphs

For a given class $\mathcal{C}$ of graphs, a graph $U \in \mathcal{C}$ is called hom-universal with respect to $\mathcal{C}$ if for any $G \in \mathcal{C}$, $\mathrm{Hom}(G, U) \ne \emptyset$, in which case the class $\mathcal{C}$ is said to be bounded by the graph U. The problem of the existence of a bound with some special properties, for a given class of graphs, has been a subject of study in the theory of graph homomorphisms. In the following definition, we introduce a new family of hom-universal graphs, namely the family $H(m, n, k)$ of the helical graphs.

Definition 3. Let m, n, and k be positive integers with $m \ge 2n$. Set $H(m, n, k)$ to be the *helical graph* whose vertex set contains all k-tuples $(A_1, \ldots, A_k)$ such that for any $1 \le r \le k, A_r \subseteq [m]$, $|A_1| = n$, $|A_r| \ge n$ and for any $s \le k-1$ and $t \le k-2, A_s \cap A_{s+1} = \emptyset, A_t \subseteq A_{t+2}$. Also, two vertices $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ of $H(m, n, k)$ are adjacent if for any $1 \le i, j+1 \le k, A_i \cap B_i = \emptyset, A_j \subseteq B_{j+1}$, and $B_j \subseteq A_{j+1}$. □

Roughly speaking, analogous to k-universal graphs, the vertices of $H(m, n, k)$ encode the set of colors that can be found in certain walks in an n-tuple coloring. Also, note that $H(m, 1, 1)$ is the complete graph K_m and $H(m, n, 1)$ is the Kneser graph $KG(m, n)$. It is easy to verify that if $m > 2n$, then the odd-girth of $H(m, n, k)$ is greater than or equal to $2k+1$. The statement

is true for $k = 1$; hence, assume that $k \geq 2$. Indirectly, assume that C_{2l-1} is an odd cycle of $H(m, n, k)$ where $2 \leq l \leq k$. Suppose that $u = (A_1, \ldots, A_k) \in V(C_{2l-1})$. Consider two adjacent vertices $v = (B_1, \ldots, B_k)$ and $w = (B'_1, \ldots, B'_k)$ of C_{2l-1} at distance exactly $l-1$ from u. In view of the definition of helical graphs, we should have $A_1 \subseteq B_l$ and $A_1 \subseteq B'_l$. On the other hand, v and w are adjacent; consequently, $B_l \cap B'_l = \emptyset$ which is a contradiction.

For a given graph G and $v \in V(G)$, set

$$N_i(v) \stackrel{\text{def}}{=} \{u \mid \text{there is a walk of length } i \text{ joining } u \text{ and } v\}.$$

Also, for a coloring $c : V(G) \longrightarrow \binom{[m]}{n}$, define

$$c(N_i(v)) \stackrel{\text{def}}{=} \bigcup_{u \in N_i(v)} c(u).$$

It is worth noting that it was shown in [7] that the graph $H(m, 1, 2)$ is m-colorable and that there exists an m-coloring of $H(m, 1, 2)$ such that the neighborhoods of each color class is an independent set. Such colorings can be viewed as ordinary colorings of the third power of the graph $H(m, 1, 2)$. In the theorem below, we show that, instead of colorings, we look at the generalized coloring expressed by the existence of homomorphisms into helical graphs. In fact, we show that helical graphs are hom-universal graphs with respect to the family of high odd-girth graphs whose $(2k-1)$th power is bounded by a Kneser graph.

Theorem 1. *Let G be a non-empty graph with odd-girth at least $2k+1$. Then we have* $\mathrm{Hom}(G^{2k-1}, KG(m, n)) \neq \emptyset$ *if and only if* $\mathrm{Hom}(G, H(m, n, k)) \neq \emptyset$.

Proof. First, let $c \in \mathrm{Hom}(G^{2k-1}, KG(m, n))$. If v is an isolated vertex of G, then consider an arbitrary vertex of $H(m, n, k)$ and let it be $f(v)$. For any non-isolated vertex $v \in V(G)$, define

$$f(v) \stackrel{\text{def}}{=} (c(v), c(N_1(v)), c(N_2(v)), \ldots, c(N_{k-1}(v))).$$

If $i \leq j$ and $i \equiv j \bmod 2$, we have $N_i(v) \subseteq N_j(v)$, implying that $c(N_i(v)) \subseteq c(N_j(v))$. Also, since c is a homomorphism from G^{2k-1} to $KG(m, n)$, for any $i \leq j \leq k-1$ and $i \not\equiv j \bmod 2$, we obtain $c(N_i(v)) \cap c(N_j(v)) = \emptyset$. Hence, for any vertex $v \in V(G), f(v) \in V(H(m, n, k))$. Moreover, for any $0 \leq i, j+1 \leq k-1$, we have $N_i(v) \cap N_i(u) = \emptyset$, $N_j(v) \subseteq N_{j+1}(u)$, and $N_j(u) \subseteq N_{j+1}(v)$ provided that u is adjacent to v. Hence, f is a graph homomorphism from G to $H(m, n, k)$.

Next, let $\mathrm{Hom}(G, H(m, n, k)) \neq \emptyset$ and $f : G \longrightarrow H(m, n, k)$. Assume $v \in V(G)$ and $f(v) = (A_1, A_2, \ldots, A_k)$. Define, $c(v) \stackrel{\text{def}}{=} A_1$. Assume further that $u, v \in V(G)$ such that there is a walk of length $2t+1$ ($t \leq k-1$) between u and v in G, i.e., $uv \in E(G^{2k-1})$. Consider adjacent vertices u' and v' such that $u' \in N_t(u)$ and $v' \in N_t(v)$. Also, let $f(v) = (A_1, A_2, \ldots, A_k)$, $f(u) = (B_1, B_2, \ldots, B_k)$, $f(v') = (A'_1, A'_2, \ldots, A'_k)$, and $f(u') = (B'_1, B'_2, \ldots, B'_k)$. In view of the definition of the helical graph $H(m, n, k)$, we obtain $A_1 \subseteq A'_{t+1}$ and $B_1 \subseteq B'_{t+1}$. On the other hand, $A'_{t+1} \cap B'_{t+1} = \emptyset$, which yields $c(v) \cap c(u) = \emptyset$. Thus, $\mathrm{Hom}(G^{2k-1}, KG(m, n)) \neq \emptyset$, as desired. ■

It was conjectured in [21] that a class $\mathcal{C}$ of graphs is bounded by a graph H whose odd-girth is at least $2k+1$ provided that the set $\{\chi(G^{2k-1}) \mid G \in \mathcal{C}\}$ of numbers is bounded and that all graphs in $\mathcal{C}$ have odd-girth at least $2k+1$. It is worth noting that Theorem 1 shows the above conjecture is true. This conjecture, however, was proved by C. Tardif recently (personal communication, see [21]).

It was proved by Schrijver [28] that $SG(m, n)$ is a vertex-critical subgraph of $KG(m, n)$. Motivated by the construction of Schrijver graphs, we introduce a family of subgraphs of helical graphs.

Definition 4. Let m, n, and k be positive integers with $m \geq 2n$. Define $SG(m, n, k)$ to be the induced subgraph of $H(m, n, k)$ whose vertex set contains all k-tuples $(A_1, \ldots, A_k) \in V(H(m, n, k))$ such that for any $1 \leq r \leq k, A_r = \cup_s B_s$, where every B_s is a 2-stable n-subsets of $[m]$. □

One can deduce the following theorem whose proof is almost identical to that of Theorem 1 and the proof is omitted for the sake of brevity.

Theorem 2. *Let G be a non-empty graph with odd-girth at least $2k+1$. Then* $\mathrm{Hom}(G^{2k-1}, SG(m, n)) \neq \emptyset$ *if and only if* $\mathrm{Hom}(G, SG(m, n, k)) \neq \emptyset$.

Now, we introduce a theorem which is a generalization of Theorem 1. For two subsets A and B of the vertex set of a graph G, we write $A \bowtie B$ if every vertex of A is joined to every vertex of B. Also, for any non-negative integer s, define the graph $G^{-\frac{1}{2s+1}}$ as follows

$$V(G^{-\frac{1}{2s+1}}) \stackrel{\text{def}}{=} \{(A_1, \ldots, A_{s+1}) \mid A_i \subseteq V(G), |A_1| = 1, \emptyset \neq A_i \subseteq N_{i-1}(A_1), i \leq s+1\}.$$

Two vertices $(A_1, \ldots, A_{s+1})$ and $(B_1, \ldots, B_{s+1})$ are adjacent in $G^{-\frac{1}{2s+1}}$ if for any $1 \le i \le s$ and $1 \le j \le s+1$, $A_i \subseteq B_{i+1}$, $B_i \subseteq A_{i+1}$, and $A_j \bowtie B_j$. It is easy to verify that if s is a non-negative integer, then the odd-girth of $G^{-\frac{1}{2s+1}}$ is greater than or equal to $2s+3$.

The next theorem is a generalization of Theorem 1 and Lemma 3(ii) of [31] and its proof is almost identical to that of Theorem 1 and the proof is omitted for the sake of brevity.

Theorem 3. *Let G and H be two graphs and $2r+1 < og(G)$. We have $G^{2r+1} \longrightarrow H$ if and only if $G \longrightarrow H^{-\frac{1}{2r+1}}$.*

The aforementioned theorem also obtained by C. Tardif (personal communication). Also, it should be noted that for given positive integers k, m, and n where $m > 2n$, the helical graph $H(m, n, k)$ and the graph $KG(m, n)^{-\frac{1}{2k-1}}$ are homomorphically equivalent. Although, if $k \ge 2$ and $m > 2n \ge 4$, then the number of vertices of $H(m, n, k)$ is less than that of $KG(m, n)^{-\frac{1}{2k-1}}$.

In [7], it was proved that $\chi(H(m, 1, 2)) = m$. Later in [1,29], it was shown $\chi(H(m, 1, k)) = m$. We would like to remark that the graph $H(m, 1, k)$ is defined in a completely different way in [1,29]. Simonyi and Tardos [29] showed that $\chi(H(m, 1, k)) = m$ by proving the existence of homomorphism from $SG(a, b)$ to $H(m, 1, k)$, where $a - 2b + 2 = m$ and a is sufficiently large. Similarly, one can show that $\chi(H(m, n, k)) = \chi(SG(m, n, k)) = m - 2n + 2$, where $m \ge 2n$.

Lemma C ([29]). *Let $u, v \subset [a]$ be two vertices of $SG(a, b)$. If there is a walk of length $2s$ between u and v in $SG(a, b)$, then $|u \setminus v| \le s(a - 2b + 2)$.*

Theorem 4. *Let m, n, and k be positive integers with $m \ge 2n$. The chromatic number of the helical graph $H(m, n, k)$ is equal to $m - 2n + 2$. Moreover, $\chi(SG(m, n, k)) = m - 2n + 2$.*

Proof. For a given vertex $v = (A_1, A_2, \ldots, A_k) \in V(H(m, n, k))$, define $f(v) \stackrel{\text{def}}{=} A_1$. It is easy to check that f is a graph homomorphism from $H(m, n, k)$ to $KG(m, n)$. It follows that $\chi(SG(m, n, k)) \le \chi(H(m, n, k)) \le m - 2n + 2$. Now, we prove that $m - 2n + 2$ is a lower bound for the chromatic number of $SG(m, n, k)$. To this end, it suffices to show, first, that for $a \stackrel{\text{def}}{=} 2(k-1)m(m - 2n + 2) + m$ and $b \stackrel{\text{def}}{=} (k-1)m(m - 2n + 2) + n$, we have $\mathrm{Hom}(SG(a, b)^{2k-1}, SG(m, n)) \ne \emptyset$. Then Theorem 2 applies, and hence the assertion follows. Now, let $[a]$ be partitioned into m sets, each of which contains $2(k-1)(m - 2n + 2) + 1$ consecutive elements of $[a]$. In other words, $[a]$ is partitioned into m disjoint sets $D_1, \ldots, D_m$, where each D_i contains consecutive elements and $|D_i| = 2(k-1)(m-2n+2)+1$. Note that $b = (k-1)m(m-2n+2)+n$ and $\sum_{i=1}^{m} \frac{(|D_i|-1)}{2} = (k-1)m(m - 2n + 2)$. Therefore, for every 2-stable subset u of $[a]$ of size b, there are at least n indices $i_1, \ldots, i_n$ such that u contains $(k-1)(m-2n+2)+1$ elements of D_{i_j}, $1 \le j \le n$. Note also that D_i contains a unique subset of cardinality $(k-1)(m - 2n + 2) + 1$ which does not contain any two consecutive elements. Use E_i to denote this unique subset of D_i, which is readily seen to consist of the smallest elements of D_i, the third smallest elements of D_i, and so on and so forth. For any vertex $u \in SG(a, b)$, we define a coloring c by choosing n indices i_j $(1 \le j \le n)$ such that $E_{i_j} \subseteq u$ and we set $c(u) \stackrel{\text{def}}{=} \{i_1, \ldots, i_n\}$. Since u is a 2-stable subset of $[a]$, it is easy to verify that $c(u)$ is a 2-stable subset of $[m]$ too. One needs to show that for any two vertices u and v for which there is a walk of length $2r - 1$ between them, where $1 \le r \le k$, we have $c(u) \cap c(v) = \emptyset$. To prove this, suppose that $i \in c(v)$ and $v = v_0, v_1, \ldots, v_{2r-1} = u$ be a walk between u and v, where $1 \le r \le k$. By Lemma C, $|v \setminus v_{2r-2}| \le (k-1)(m - 2n + 2)$. In particular, v_{2r-2} contains all but at most $(k-1)(m - 2n + 2)$ elements of E_i. As $|E_i| = (k-1)(m - 2n + 2) + 1$, we see that $v_{2r-2} \cap E_i \ne \emptyset$. Thus, the set u, which is disjoint from v_{2r-2}, cannot contain all elements of E_i, showing that $i \notin c(u)$. This proves that $c(u) \cap c(v) = \emptyset$. Therefore, Theorem 2 applies, finishing the proof. ■

For a given graph G, if u and v are distinct vertices of G and the neighborhood of u is a subset of that of v, then the graph G is certainly not a vertex-critical graph. Note that in the graph $SG(7, 2, 2)$, the neighborhood of the vertex $(\{1, 3\}, \{4, 5, 6, 7\})$ is a subset of that of the vertex $(\{1, 3\}, \{2, 4, 5, 6, 7\})$. Hence, the graph $SG(m, n, k)$ in general is not a vertex-critical graph. This motivates us to present the following definition.

Definition 5. Let m, n, and k be positive integers with $m \ge 2n$. Define $SH(m, n, k)$ to be the induced subgraph of $H(m, n, k)$ whose vertex set contains all k-tuples $(A_1, \ldots, A_k) \in V(H(m, n, k))$ such that for any $1 \le r \le k$, $A_r = \cup_s B_s$ and $\overline{A_r} = \cup_t C_t$, where B_s's and C_t's are all 2-stable n-subsets of $[m]$. □

One can check that $SH(m, n, k)$ has the property that for any two distinct vertices $u, v \in V(SH(m, n, k))$, $N(u) \not\subseteq N(v)$ and $N(v) \not\subseteq N(u)$. Also, it is straightforward to see that $SH(m, n, k)$ is the maximal subgraph of $SG(m, n, k)$ with the aforementioned property. To prove this, we modify the graph $SG(m, n, k)$ by performing the following **WHILE**-loop.

WHILE there exist two distinct vertices $u = (A_1, \ldots, A_k)$ and $v = (B_1, \ldots, B_k)$, where $N(u) \subseteq N(v)$, then **DO** the following: remove the vertex u.

We claim that in the **WHILE-loop** algorithm when the input is the graph $SG(m, n, k)$ with $m \ge 2n$, then the output is the graph $SH(m, n, k)$. To show this, note that in the **WHILE**-loop each time we search in the new graph for the bad vertex u. So a vertex u may be good at the beginning, and become bad later. Suppose that **WHILE-loop** is not completed yet. In the last graph obtained from the **WHILE-loop** algorithm, let i be the greatest positive integer for which there exists at least a vertex $u = (A_1, \ldots, A_k) \in V(SG(m, n, k))$ such that $\overline{A_i}$ is not a union of 2-stable n-subsets of $[m]$. Note that as $|A_1| = n$, it is easy to

verify that $\overline{A_1}$ is a union of 2-stable n-subsets of $[m]$, and hence $i \geq 2$. Also, by the assumption, for any $i < j$, $\overline{A_j}$ is a union of 2-stable n-subsets of $[m]$. Set $v \stackrel{\text{def}}{=} (A_1, \ldots, A_{i-1}, A_i \cup B, A_{i+1}, \ldots, A_k)$, where

$$B \stackrel{\text{def}}{=} \{j \mid j \in \overline{A_i} \text{ and } j \text{ does not appear in any 2-stable } n\text{-subsets of } \overline{A_i}\}.$$

Note that $A_{i-1} \subseteq \overline{A_i}$ and that A_{i-1} is a union of 2-stable n-subsets of $[m]$. Hence, for any $j \in B$, one can show that $\{j-1, j+1\} \subseteq A_{i-1} \subseteq \overline{A_i}$ (mod m) since otherwise one can extend $\{j\}$ to a 2-stable n-subsets of $A_{i-1} \cup \{j\} \subseteq \overline{A_i}$ which is a contradiction. Hence, for any $j \in B$, $A_i \cap \{j-1, j+1\} = \emptyset$ which implies that $A_i \cup B$ is a union of 2-stable n-subsets of $[m]$. Also, by considering the assumption, we should have $B \subseteq A_{i+2}$. Thus, $v \in V(SG(m, n, k))$ and also $N(u) \subseteq N(v)$. Consequently, when the **WHILE**-loop is completed, we obtain the graph $SH(m, n, k)$. Also, this shows that $SH(m, n, k)$ and $SG(m, n, k)$ are homomorphically equivalent. In view of the above observation, we suggest the following question.

Question 1. *Let m, n, and k be positive integers with $m \geq 2n$. Is it true that the graph $SH(m, n, k)$ is a vertex-critical graph?*

The problem whether the circular chromatic number and the chromatic number of Kneser graphs and Schrijver graphs are equal has received attention and has been studied in several papers [5,9,15,19,23,29]. Johnson, Holroyd, and Stahl [15] proved that $\chi_c(\mathrm{KG}(m, n)) = \chi(\mathrm{KG}(m, n))$ if $m \leq 2n + 2$ or $n = 2$. They also conjectured that the equality holds for all Kneser graphs.

Conjecture 1 ([15]). *For all $m \geq 2n + 1$, $\chi_c(\mathrm{KG}(m, n)) = \chi(\mathrm{KG}(m, n))$.*

It was shown in [9] that if $m \geq 2n^2(n-1)$, then the circular chromatic number of $\mathrm{KG}(m, n)$ is equal to its chromatic number. Later, it was proved independently in [23,29] that $\chi(\mathrm{KG}(m, n)) = \chi_c(\mathrm{KG}(m, n)) = m - 2n + 2$ whenever m is an even natural number. Also in [1,29], it was shown that $\chi(H(m, 1, k)) = m$. Simonyi and Tardos [29] used the fact that $\mathrm{Hom}(SG(a, b), H(m, 1, k)) \neq \emptyset$, where $a - 2b + 2 = m$, and hence $m - 1$ is a lower bound for the co-index of a particular box complex of $H(m, 1, k)$. Note that there are several similar, but somewhat different box complexes defined in the literature, see [22].

Theorem A ([23,29]). *If $\mathrm{coind}(B_0(G))$ is odd for a graph G, then $\chi_c(G) \geq \mathrm{coind}(B_0(G)) + 1$.*

It was shown in [29] that the circular chromatic number and the chromatic number of $H(m, 1, k)$ are equal whenever m is an even natural number.

Theorem 5. *Let m, n, and k be positive integers, where $m \geq 2n$ and m is an even positive integer. Then $\chi_c(SG(m, n, k)) = \chi_c(H(m, n, k)) = m - 2n + 2$. Furthermore, $\chi_c(SH(m, n, k)) = m - 2n + 2$.*

Proof. As proved in Theorem 4, if $a - 2b = m - 2n$ and $a = 2(k-1)m(m-2n+2) + m$, then $\mathrm{Hom}(SG(a, b), SG(m, n, k)) \neq \emptyset$. This implies $\mathrm{coind}(B_o(SG(a, b))) \leq \mathrm{coind}(B_o(SH(m, n, k)))$. Also, it is well known that $\mathrm{coind}(B_o(SG(a, b))) = a - 2b + 1$, see [29]. Thus, by Theorem A, we have $\chi_c(SG(m, n, k)) = \chi_c(H(m, n, k)) = m - 2n + 2$. Also, two graphs $SH(m, n, k)$ and $SG(m, n, k)$ are homomorphically equivalent. Thus, $\chi_c(SH(m, n, k)) = m - 2n + 2$. ■

In [29], the authors made use of Theorem A to prove that $\chi_c((SG(a, b))) = \chi((SG(a, b)))$ provided that a is an even positive integer. In view of $\chi_c((SG(a, b))) = \chi((SG(a, b)))$, where a is an even integer number, one can present an alternate proof of Theorem 5. However, note that the equality $\mathrm{coind}(B_o(H(m, n, k))) = m - 2n + 1$ provides more information about the colorings of the helical graph $H(m, n, k)$ (see [29,30]).

It was conjectured in [19] and proved in [9], that for every fixed n, there is a threshold $t(n)$ such that $\chi_c(SG(m, n)) = \chi(SG(m, n))$ for all $m \geq t(n)$. Note that $H(3, 1, 2)$ is the nine cycle and that $\chi_c(H(3, 1, 2)) = \frac{9}{4}$. Hence, the following question arises naturally.

Question 2. *Given positive integers n and k, does there exist a natural number $t(n, k)$ such that the equality $\chi_c(SH(m, n, k)) = \chi_c(H(m, n, k)) = \chi(H(m, n, k)) = m - 2n + 2$ holds for all $m \geq t(n, k)$?*

3. Homomorphism to odd cycles

In this section, we investigate the problem of existence of homomorphisms to odd cycles. A graph H is said to be a subdivision of a graph G if H is obtained from G by subdividing some of the edges. The graph $G^{\frac{1}{s}}$ is said to be the s-subdivision of a graph G if $G^{\frac{1}{s}}$ is obtained from G by replacing each edge by a path with exactly $s - 1$ inner vertices. Note that $G^{\frac{1}{1}}$ is isomorphic to G. Hereafter, for a given graph G, we use the following notation for convenience. Set

$$G^{\frac{t}{s}} \stackrel{\text{def}}{=} (G^{\frac{1}{s}})^t.$$

In the following theorem, we prove that a homomorphism to $(2k + 1)$-cycle exists if and only if the chromatic number of $G^{\frac{2k+1}{3}}$ is less than or equal to 3.

Theorem 6. *Let G be a graph with odd-girth at least $2k+1$. Then $\chi(G^{\frac{2k+1}{3}}) \leq 3$ if and only if* $\mathrm{Hom}(G, C_{2k+1}) \neq \emptyset$.

Proof. First, if there exists a homomorphism from G to C_{2k+1}, then it is obvious to see that there is a homomorphism from $G^{\frac{1}{3}}$ to $C_{2k+1}^{\frac{1}{3}} = C_{6k+3} = H(3, 1, k+1)$. In view of Theorem 1, we have $\chi(G^{\frac{2k+1}{3}}) \leq 3$.

Next, if $\chi(G^{\frac{2k+1}{3}}) \leq 3$, then in view of Theorem 1, we have $\mathrm{Hom}(G^{\frac{1}{3}}, C_{6k+3}) \neq \emptyset$. Consequently, $\mathrm{Hom}(G^{\frac{3}{3}}, C_{6k+3}^{3}) \neq \emptyset$. Also, it is easy to verify that G and $G^{\frac{3}{3}}$ are homomorphically equivalent and that there is a homomorphism from C_{6k+3}^{3} to C_{2k+1}. Therefore, we have $\mathrm{Hom}(G, C_{2k+1}) \neq \emptyset$. ■

Considering Theorem 6, it is worth studying the following question.

Question 3. *Let G be a non-bipartite graph. What is the value of*

$$\sup\left\{\frac{2t+1}{2s+1} \mid \chi(G^{\frac{2t+1}{2s+1}}) = \chi(G), \frac{2t+1}{2s+1} < og(G)\right\}?$$

In [25], Nešetřil proposed the Pentagon problem.

Problem 1 (*Nešetřil's Pentagon Problem [25]*). Let G be a cubic graph of sufficiently large girth, is it true that $\mathrm{Hom}(G, C_5) \neq \emptyset$?

It should be noted that if in the problem C_5 is replaced by C_3, then the answer is affirmative; and in fact it is a quick consequence of Brooks' theorem. On the other hand, the answer is known to be negative if one replaces C_5 by C_{11}, C_9 or C_7 [10,18,33].

In view of Theorem 6, it is possible to rephrase the Pentagon Problem as follows.

Question 4. *Let G be a cubic graph of sufficiently large girth, is it true that $\chi(G^{\frac{5}{3}}) \leq 3$?*

If the answer to the Pentagon problem is affirmative, then it follows from Lemma B that there exists a number g_0 with the property that the chromatic number of the third power of any cubic graph with girth larger than g_0 is less than six.

Question 5 (*[4]*). *Is it true that for any natural number g_0, there exists a cubic graph G whose girth is larger than g_0 and $\chi(G^3) \geq 6$?*

It is interesting to find $\max_{g(G)\geq g} \chi(G^3)$, where maximum is taken over all cubic graphs with girth at least g. It should be noted that by considering a greedy coloring this maximum is less than or equal to 16. In view of Theorem 1, the following question is equivalent to Question 5.

Question 6. *Is it true that for any natural number g_0, there exists a cubic graph G whose girth is larger than g_0 and* $\mathrm{Hom}(G, H(5, 1, 2)) = \emptyset$?

Note that $H(3, 1, 2)$ is the nine cycle. It was proved in [33] that the above question has an affirmative answer when $H(5, 1, 2)$ is replaced by $H(3, 1, 2)$. This motivates us to suggest the following question.

Question 7. *Is it true that for any natural number g_0, there exists a cubic graph G whose girth is larger than g_0 and* $\mathrm{Hom}(G, H(4, 1, 2)) = \emptyset$?

The fractional chromatic number of graphs with odd-girth greater than 3 has been studied in several papers [11,12]. Heckman and Thomas [12] posed the following conjecture.

Conjecture 2 (*[12]*). *Every triangle free graph with maximum degree at most 3 has the fractional chromatic number at most $\frac{14}{5}$.*

Helical graphs bound high girth graphs. Thus, it may be interesting to compute their fractional chromatic number and their local chromatic number.

Question 8. *Let m, n, and k be positive integers with $m \geq 2n$. What are the values of $\chi_f(H(m, n, k))$ and $\psi(H(m, n, k))$?*

Let $\mathcal{P}_{2k+1}$ be the class of planar graphs of odd-girth at least $2k+1$. Naserasr [24] conjectured an upper bound for the chromatic number of planar graph powers as follows.

Conjecture 3 (*[24]*). *For every $G \in \mathcal{P}_{2k+1}$ we have $\chi(G^{2k-1}) \leq 2^{2k}$.*

Again in view of Theorem 1, one can rephrase Naserasr's conjecture in terms of helical graphs. The following conjecture is Jaeger's modular orientation conjecture restricted to planar graphs.

Conjecture 4 (*Jaeger's Conjecture [14]*). *Every planar graph with girth at least $4k$ has a homomorphism to C_{2k+1}.*

Considering Theorem 6, one can reformulate Jaeger's conjecture as follows.

Conjecture 5. *Let P be a planar graph with girth at least $4k$. Then we have $\chi(P^{\frac{2k+1}{3}}) \leq 3$.*

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