The method of lower and upper solutions for fourth order singular \( m \)-point boundary value problems ✪

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Abstract

This paper investigates the existence of positive solutions for fourth order singular \( m \)-point boundary value problems. Firstly, we establish a comparison theorem, then we define a partial ordering in \( C^2[0, 1] \cap C^4(0, 1) \) and construct lower and upper solutions to give a necessary and sufficient condition for the existence of \( C^2[0, 1] \) as well as \( C^3[0, 1] \) positive solutions. Our nonlinearity \( f(t, x, y) \) may be singular at \( x, y, t = 0 \) and/or \( t = 1 \).

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1. Introduction

The singular ordinary differential equations arise in the fields of gas dynamics, Newtonian fluid mechanics, the theory of boundary layer and so on. The theory of singular boundary value problems has become an important area of investigation in recent years (see [1–5] and references therein).

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The existence of solutions of multi-point boundary value problems has been studied by many authors using nonlinear alternative of Leray–Schauder, coincidence degree theory and fixed point theorem in cones (see [6–10] and references therein).

Very recently, the existence of positive solutions and multiple positive solutions of singular second order multi-point boundary value problem have been studied by papers [11–13] using the fixed point index and approximate process; Zhang and Wang in [14] gave some sufficient conditions for the existence results of a class of singular nonlinear second order three-point boundary value problems by the upper and lower solution method and the monotone iterative technique. But they only considered the case: nonlinear function \( f(t, x) \) cannot be singular at \( x = 0 \).

On the boundary value problems of fourth order ordinary differential equation

\[
\begin{align*}
\begin{cases}
x^{(4)}(t) = f(t, x(t), -x''(t)), & t \in (0, 1), \\
x(0) = x(1) = 0, & x''(0) = x''(1) = 0.
\end{cases}
\end{align*}
\]

When the function \( f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) in (*) , i.e., \( f \) is continuous, problem (*) is nonsingular, the existence and uniqueness of positive solutions of (*) have been studied by papers [15–18].

A sufficient condition for the existence of solutions of the singular problem (*) was given by D. O’Regan in [19] with a topological transversal theorem.

In this paper, we shall consider the existence of positive solutions for fourth order singular \( m \)-point boundary value problems of the following differential equation:

\[
\begin{align*}
\begin{cases}
x^{(4)}(t) = f(t, x(t), -x''(t)), & t \in (0, 1), \\
x(0) = x(1) = 0, & x''(0) = x''(1) = 0,
\end{cases}
\end{align*}
\]

where \( 0 < \alpha_i < 1, 0 < \beta_i < 1, i = 1, 2, \ldots, m - 2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1 \), are constants, \( \sum_{i=1}^{m-2} \alpha_i < 1, \sum_{i=1}^{m-2} \beta_i < 1, m \geq 3 \) and \( f \) satisfy the following hypothesis:

(H) \( f \in C((0, 1) \times (0, \infty) \times (0, \infty), [0, \infty)) \), and there exist constants \( \lambda_i, \mu_i \) \( (-\infty < \lambda_i \leq 0 \leq \mu_i, \ i = 1, 2, \mu_1 + \mu_2 < 1, \lambda_2 < \mu_2) \) such that for \( t \in (0, 1), x, y \in (0, \infty) \),

\[
\begin{align*}
c^{\mu_1} f(t, x, y) &\leq f(t, cx, y) \leq c^{\lambda_1} f(t, x, y), \quad \text{if} \ 0 < c \leq 1, \\
c^{\mu_2} f(t, x, y) &\leq f(t, x, cy) \leq c^{\lambda_2} f(t, x, y), \quad \text{if} \ 0 < c \leq 1.
\end{align*}
\]

Remark.

(i) (1.3) implies

\[
c^{\lambda_1} f(t, x, y) \leq f(t, cx, y) \leq c^{\mu_1} f(t, x, y), \quad \text{if} \ 0 \geq c \geq 1.
\]

Conversely, (1.5) implies (1.3).

(ii) (1.4) implies

\[
c^{\lambda_2} f(t, x, y) \leq f(t, x, cy) \leq c^{\mu_2} f(t, x, y), \quad \text{if} \ 0 \geq c \geq 1.
\]

Conversely, (1.6) implies (1.4).
Typical functions that satisfy the above sub-linear hypothesis are those taking the form
\[ f(t, x, y) = \sum_{s=1}^{m} \sum_{k=1}^{n} p_{k,s}(t)x^{\ell_k} y^{r_{k,s}}; \] here \( p_{k,s}(t) \in C(0, 1), \) \( p_{k,s}(t) > 0 \) on \((0, 1),\) \( \ell_k \in R,\) \( r_s < 1, \) \( \max[\ell_k, 0] + \max[r_s, 0] < 1, \) \( k = 1, 2, \ldots, n, \) \( s = 1, 2, \ldots, m.\)

By singularity we mean that the functions \( f \) in (1.1) are allowed to be unbounded at the points \( x = 0, y = 0, t = 0 \) and \( t = 1. \) A function \( x(t) \in C^2[0, 1] \cap C^4(0, 1) \) is called a \( C^2[0, 1] \) (positive) solution of (1.1) and (1.2) if it satisfies (1.1) and (1.2) \((x(t) > 0, -x''(t) > 0 \) for \( t \in (0, 1)).\)

A \( C^2[0, 1] \) (positive) solution of (1.1) and (1.2) is called a \( C^3[0, 1] \) (positive) solution if \( x^{(3)}(0^+) \) and \( x^{(3)}(1^-) \) both exist \((x(t) > 0, -x''(t) > 0 \) for \( t \in (0, 1)).\)

In this paper, we shall study the existence of positive solutions for fourth order singular \( m \)-point boundary value problems (1.1) and (1.2). Firstly, we establish a comparison theorem, then we define a partial ordering in \( C^2[0, 1] \cap C^4(0, 1) \) and construct lower and upper solutions to give a necessary and sufficient condition for the existence of \( C^2[0, 1] \) as well as \( C^3[0, 1] \) positive solutions. Our nonlinearity \( f(t, x, y) \) may be singular at \( x, y, t = 0 \) and/or \( t = 1.\)

2. Several lemmas

To prove the main result, we need the following lemmas.
Suppose that \( 0 < a_n < \eta_1 < \eta_{m-2} < b_n, \) and
\[ F_n = \left\{ x \in C^2[a_n, b_n] \cap C^4(a_n, b_n), \ x(a_n) - \sum_{i=1}^{m-2} \alpha_i x(\eta_i) \geq 0, \ x(b_n) \geq 0, \ x''(a_n) - \sum_{i=1}^{m-2} \beta_i x''(\eta_i) \geq 0, \ x''(b_n) \geq 0 \right\}. \]

**Lemma 2.1 (Comparison theorem).** If \( x \in F_n \) such that \( x^{(4)}(t) \geq 0, \) \( t \in (a_n, b_n), \) then \( x(t) \geq 0, \) \( -x''(t) \geq 0, \) \( t \in [a_n, b_n]. \)

**Proof.** Set
\[ -x''(t) = y(t), \quad t \in [a_n, b_n], \] \[ x^{(4)}(t) = \sigma(t), \quad t \in (a_n, b_n), \] \[ x(a_n) - \sum_{i=1}^{m-2} \alpha_i x(\eta_i) = r_1, \quad x(b_n) = r_2, \] \[ -\left[ x''(a_n) - \sum_{i=1}^{m-2} \beta_i x''(\eta_i) \right] = r_3, \quad -x''(b_n) = r_4, \]
then \( r_i \geq 0, \) \( i = 1, 2, 3, 4, \) \( \sigma(t) \geq 0, \) \( t \in (a_n, b_n) \) and
\[ -y''(t) = \sigma(t), \quad t \in (a_n, b_n), \] \[ y(a_n) - \sum_{i=1}^{m-2} \beta_i y(\eta_i) = r_3, \quad y(b_n) = r_4. \]

It is not difficult to see that solution of (2.5) and (2.6) can be stated as
Assume that

\[ y(t) = \frac{y_{2n}(t)}{y_{2n}(a_n) - \sum_{i=1}^{m-2} \beta_i y_{2n}(\eta_i)} r_3 + \frac{y_{1n}(t)}{y_{1n}(b_n)} r_4 + \int_{a_n}^{b_n} G_n(t, s) \sigma(s) \, ds \]

where

\[ G_n(t, s) = \frac{1}{b_n - a_n} \begin{cases} (b_n - t)(s - a_n), & s < t, \\ (b_n - s)(t - a_n), & t \leq s, \end{cases} \]

and

\[ y_{1n}(t) = (t - a_n) + \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i} \sum_{i=1}^{m-2} \beta_i (\eta_i - a_n), \quad y_{2n}(t) = b_n - t. \]

By means of (2.6)–(2.9), we can obtain \( y(t) \geq 0, \ t \in [a_n, b_n], \) i.e., \( -x''(t) \geq 0, \ t \in [a_n, b_n]. \) Similarly, the solution of (2.1) and (2.3) can be stated as

\[ x(t) = \frac{y_{2n}(t)}{y_{2n}(a_n) - \sum_{i=1}^{m-2} \alpha_i y_{2n}(\eta_i)} r_1 + \frac{x_{1n}(t)}{x_{1n}(b_n)} r_2 + \int_{a_n}^{b_n} G_n(t, s) y(s) \, ds \]

and

\[ x_{1n}(t) = (t - a_n) + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \sum_{i=1}^{m-2} \alpha_i (\eta_i - a_n). \]

From (2.10) and (2.11), we can get \( x(t) \geq 0, \ t \in [a_n, b_n]. \) The proof is complete. □

**Lemma 2.2.** Suppose that (H) holds. Let \( x(t) \) be a \( C^3[0, 1] \) positive solution of (1.1) and (1.2). Then there are constants \( I_1 \) and \( I_2, \) \( 0 < I_1 < I_2, \) such that

\[ I_1 (1 - t) \leq -x''(t) \leq I_2 (1 - t), \quad t \in [0, 1]. \]

**Proof.** Assume that \( x(t) \) is a \( C^3[0, 1] \) positive solution of (1.1) and (1.2). Then both \( x^{(3)}(0) \) and \( x^{(3)}(1) \) exist, \( -x''(t) > 0 \) for \( t \in (0, 1). \) By integration of (1.1), we have

\[ \int_0^1 f(t, x(t), -x''(t)) \, dt = x^{(3)}(1) - x^{(3)}(0) < \infty, \]

and \( -x''(t) \) can be stated as

\[ -x''(t) = \int_0^1 G(t, s) f(s, x(s), -x''(s)) \, ds + \frac{1 - t}{1 - \sum_{i=1}^{m-2} \beta_i (1 - \eta_i)}. \]
where
\[ G(t, s) = \begin{cases} \frac{1}{1-t} s, & s < t, \\ \frac{1}{1-s} t, & t \leq s. \end{cases} \]

It is easy to see that
\[ -x''(0) > 0, \quad -x''(t) \geq (1-t)\|x''\|, \quad t \in [0, 1]. \]

Here \( \|x''\| = \max_{t \in [0,1]} |x''(t)| \). From (2.14) and (1.2), we have that
\[ \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i (1 - \eta_i)} \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\eta_i, s) f(s, x(s), -x''(s)) \, ds = \sum_{i=1}^{m-2} \beta_i [-x''(\eta_i)]. \]

For \( 0 \leq t \leq 1 \), we have from (2.16) and (2.17) that
\[ -x''(t) \geq (1-t) \sum_{i=1}^{m-2} \beta_i [-x''(\eta_i)] \geq (1-t) \sum_{i=1}^{m-2} \beta_i (1 - \eta_i) \|x''\|, \quad t \in [0, 1]. \]

From (2.13) and (2.14), we have
\[ -x''(t) \leq (1-t) \left[ \int_0^1 s f(s, x(s), -x''(s)) \, ds + \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i (1 - \eta_i)} \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\eta_i, s) f(s, x(s), -x''(s)) \, ds \right]. \]

Setting \( I_1 = \sum_{i=1}^{m-2} \beta_i (1 - \eta_i) \|x''\| \), \( I_2 = \int_0^1 s f(s, x(s), -x''(s)) \, ds \)
\[ + \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i (1 - \eta_i)} \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\eta_i, s) f(s, x(s), -x''(s)) \, ds, \]
then from (2.18) and (2.19) we have (2.12). This completes the proof. \( \square \)

**Definition 2.1.** Suppose \( \alpha \in C^2[0, 1] \cap C^4(0, 1) \). If \( \alpha \) satisfies
\[ \alpha^{(4)}(t) \leq f(t, \alpha(t), -\alpha''(t)), \quad t \in (0, 1), \]
\[ \alpha(0) - \sum_{i=1}^{m-2} \alpha_i \alpha(\eta_i) \leq 0, \quad \alpha(1) \leq 0, \]
\[ -\alpha''(0) - \sum_{i=1}^{m-2} \beta_i \alpha''(\eta_i) \leq 0, \quad -\alpha''(1) \leq 0, \]
then \( \alpha \) is called a lower solution of the singular problem (1.1) and (1.2).
Definition 2.2. Suppose $\beta \in C^2[0, 1] \cap C^4(0, 1)$. If $\beta$ satisfies
\[
\beta^{(4)}(t) \geq f(t, \beta(t), \beta''(t)), \quad t \in (0, 1),
\]
\[
\begin{cases}
\beta(0) - \sum_{i=1}^{m-2} \alpha_i \beta(\eta_i) \geq 0, \\
-\beta''(0) - \sum_{i=1}^{m-2} \beta_i \beta''(\eta_i) \geq 0.
\end{cases}
\]
then $\beta$ is called a upper solution of the singular problem (1.1) and (1.2).

Definition 2.3. We define a partial ordering in $C^2[a, b] \cap C^4(a, b)$ by $x \preceq y$ if and only if
\[
(-1)^i x^{(2i)}(t) \leq (-1)^i y^{(2i)}(t), \quad t \in [a, b], \quad i = 0, 1.
\]

Lemma 2.3. Suppose that (H) holds. And assume that there exist lower and upper solutions of (1.1) and (1.2), respectively $\alpha(x)$ and $\beta(x)$, such that $\alpha(t), \beta(t) \in C^2[0, 1] \cap C^4(0, 1), 0 < \alpha(t) \leq \beta(t), 0 < -\alpha''(t) \leq -\beta''(t)$ for $t \in (0, 1), \alpha(1) = \beta(1) = 0, \alpha''(1) = \beta''(1) = 0$. Then problem (1.1) and (1.2) has at least one $C^2[0, 1]$ positive solution $x(t)$ such that $\alpha \preceq x \preceq \beta$. If, in addition, there exists $F(t) \in L^1[0, 1]$ such that
\[
|f(t, x, -x'')| \leq F(t), \quad \text{for } \alpha \preceq x \preceq \beta,
\]
then the solution $x(t)$ of (1.1) and (1.2) is a $C^3[0, 1]$ positive solution.

Proof. Let $\{a_n\}, \{b_n\}$ be sequences satisfying $0 < \cdots < a_{n+1} < a_n < \cdots < a_1 < \eta_1 < \eta_{n-2} < b_1 < \cdots < b_n < b_{n+1} < \cdots < 1, a_n \to 0$ and $b_n \to 1$ as $n \to \infty$, and let $\{r_n\}, \{r_n\}$ be sequences satisfying
\[
\begin{cases}
\alpha(a_n) - \sum_{i=1}^{m-2} \alpha_i \alpha(\eta_i) \leq r_n \leq \beta(a_n) - \sum_{i=1}^{m-2} \alpha_i \beta(\eta_i), \\
\alpha(b_n) \leq r_n \leq \beta(b_n), \quad r_n \to 0, \quad r_n \to 0, \quad \text{as } n \to \infty, \\
-\alpha''(a_n) - \sum_{i=1}^{m-2} \beta_i \alpha''(\eta_i) \leq r_n \leq -\beta''(a_n) - \sum_{i=1}^{m-2} \beta_i \beta''(\eta_i), \\
-\alpha''(b_n) \leq r_n \leq -\beta''(b_n), \quad r_n \to 0, \quad r_n \to 0, \quad \text{as } n \to \infty.
\end{cases}
\]
For each $n, \forall x(t) \in C^2[a_n, b_n] \cap C^4(a_n, b_n) = D_n, t \in [a_n, b_n]$, we define an auxiliary function
\[
g_n(x)(t) = \begin{cases}
f(t, \alpha(t), -\alpha''(t)), & \text{if } \alpha \neq x, \\
f(t, x(t), -x''(t)), & \text{if } \alpha \preceq x \preceq \beta, \\
f(t, \beta(t), -\beta''(t)), & \text{if } x \neq \beta.
\end{cases}
\]
By the condition (H), we have $g_n : D_n \to [0, +\infty)$ is continuous. For this $n$, consider the nonsingular problem
\[
\begin{cases}
x^{(4)}(t) = g_n(x(t)), \quad t \in [a_n, b_n], \\
x(a_n) - \sum_{i=1}^{m-2} \alpha_i x(\eta_i) = r_{n1}, \quad x(b_n) = r_{n2}, \\
-x''(a_n) - \sum_{i=1}^{m-2} \beta_i x''(\eta_i) = r_{n3}, \quad -x''(b_n) = r_{n4}.
\end{cases}
\]
For convenience, we define linear operators as follows:
\[
B_n x(t) = \frac{y_{2n}(t)}{y_{2n}(a_n) - \sum_{i=1}^{m-2} \beta_i y_{2n}(\eta_i)} r_{n3} + \frac{y_{1n}(t)}{y_{1n}(b_n)} r_{n4} + \int_{a_n}^{b_n} G_n(t, s) x(s) ds.
\]
\[ + \frac{(b_n - t)}{(b_n - a_n) - \sum_{i=1}^{m-2} \beta_i (b_n - \eta_i)} \sum_{i=1}^{m-2} \beta_i \int_{a_n}^{b_n} G_n(\eta_i, s)x(s) \, ds, \quad (2.25) \]

\[ A_n x(t) = \frac{y_2 n(t)}{y_2 n(a_n) - \sum_{i=1}^{m-2} \alpha_i y_2 n(\eta_i)} r_{n1} + \frac{x_1 n(t)}{x_1 n(b_n)} r_{n2} + \int_{a_n}^{b_n} G_n(t, s)x(s) \, ds \]

\[ + \frac{(b_n - t)}{(b_n - a_n) - \sum_{i=1}^{m-2} \alpha_i (b_n - \eta_i)} \sum_{i=1}^{m-2} \alpha_i \int_{a_n}^{b_n} G_n(\eta_i, s)x(s) \, ds, \quad (2.26) \]

where \( G_n(t, s), y_1 n \) are given by (2.8) and (2.9), and

\[ x_1 n(t) = (t - a_n) + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i (\eta_i - a_n)}. \quad (2.27) \]

Obviously, by the proof of Lemma 2.1, the problem (2.24) is equivalent to the integral equation

\[ x(t) = (A_n B_n g_n) x(t). \quad (2.28) \]

**Remark.** The method of the above decomposition of the operator into two operators \( A_n \) and \( B_n \) was also used by the first author in [19] and [21], respectively.

By the definition (2.23) of \( g_n \), we can get that \( A_n B_n g_n : D_n \to D_n \) and \( g(D_n) \) is bounded. By the continuity of \( G_n(t, s) \), we can show that \( A_n B_n \) is a compact operator. So, \( A_n B_n g_n(D_n) \) is a relatively compact set. Moreover, \( x \in D_n \) is a solution of (2.24) if and only if \( A_n B_n g_n x = x \).

Using the Schauder’s fixed point theorem, we assert that \( A_n B_n g_n \) has at least one fixed point \( x_n \in C^4[a_n, b_n] \), by \( x_n(t) = (A_n B_n g_n) x_n(t) \), we can get \( x_n \in C^4[a_n, b_n] \).

We claim that

\[ \alpha \preceq x_n \preceq \beta, \]

that is

\[ (-1)^i \alpha^{(2i)}(t) \leq (-1)^i x_n^{(2i)}(t) \leq (-1)^i \beta^{(2i)}(t), \quad t \in [a_n, b_n], \quad i = 0, 1, \quad (2.29) \]

and hence \( x_n(t) \in C^4[a_n, b_n] \) and satisfies

\[ x_n^{(4)}(t) = f(t, x_n(t), -x_n''(t)), \quad t \in [a_n, b_n]. \quad (2.30) \]

Indeed, suppose by contradiction that \( x_n \not\equiv \beta \). By the definition of \( g \), then

\[ (g_n x_n)(t) = f(t, \beta(t), -\beta''(t)), \quad t \in [a_n, b_n], \]

and therefore

\[ x_n^{(4)}(t) = f(t, \beta(t), -\beta''(t)), \quad t \in [a_n, b_n]. \quad (2.31) \]

On the other hand, since \( \beta \) is an upper solution of (1.1) and (1.2), we also have

\[ \beta^{(4)}(t) \geq f(t, \beta(t), -\beta''(t)), \quad t \in (0, 1). \quad (2.32) \]

Then setting

\[ z(t) = \beta(t) - x_n(t), \quad t \in [a_n, b_n], \]
by (2.22), (2.31) and (2.32), we obtain
\[
z^{(4)}(t) \geq 0, \quad t \in (a_{n}, b_{n}), \quad z \in C^{2}[a_{n}, b_{n}] \cap C^{4}(a_{n}, b_{n}).
\]
\[
z(a_{n}) - \sum_{i=1}^{m-2} \alpha_{i} z(\eta_{i}) \geq 0, \quad z(b_{n}) \geq 0,
\]
\[
- \left[ z''(a_{n}) - \sum_{i=1}^{m-2} \beta_{i} z''(\eta_{i}) \right] \geq 0, \quad -z''(b_{n}) \geq 0.
\]

By Lemma 2.1, we can conclude that \(z(t) \geq 0, -z''(t) \geq 0, \quad t \in [a_{n}, b_{n}],\) a contradiction with the assumption \(x_{n} \neq \beta.\) Therefore \(x_{n} \neq \beta\) is impossible.

Similarly, we can show that \(\alpha \leq x_{n}.\) So, we have shown that (2.29) and (2.30) hold.

Since \([a_{1}, b_{1}] \subset [a_{n}, b_{n}], \quad n = 1, 2, \ldots,\) there is, for each \(n, t_{n} \in [a_{1}, b_{1}]\) such that
\[
|x_{n}'(t_{n})| = |(x_{n}(b_{1}) - x_{n}(a_{1}))/ (b_{1} - a_{1})| \leq (2/(b_{1} - a_{1}))(\beta(b_{1}) + \beta(a_{1})),
\]
\[
|x_{n}^{(3)}(t_{n})| = |(x_{n}''(b_{1}) - x_{n}''(a_{1}))/ (b_{1} - a_{1})| \leq (2/(b_{1} - a_{1}))(\beta''(b_{1}) + \beta''(a_{1})).
\]

This allows us to assume (substituting by subsequences if necessary) \(t_{n} \to t_{0} \in [a_{n}, b_{n}],\)
\[
x_{n}(t_{n}) \to x_{0} \in [\alpha(t_{0}), \beta(t_{0})], \quad x_{n}'(t_{n}) \to x_{0}' \in R, \quad as \quad n \to \infty.
\]
\[
-x_{n}''(t_{n}) \to -x_{0}'' \in [-\alpha''(t_{0}), -\beta''(t_{0})], \quad x_{n}^{(3)}(t_{n}) \to x_{0}^{(3)} \in R, \quad as \quad n \to \infty.
\]

From [20, Theorem 3.2, p. 14], there is a solution \(x(t)\) of the equation
\[
x^{(4)}(t) = f(t, x(t), -x''(t)),
\]
with the maximum existence interval \((\omega^{-}, \omega^{+})\) such that \(x(t_{0}) = x_{0}, \ x'(t_{0}) = x_{0}', \ x''(t_{0}) = x_{0}''\),
\[
x^{(3)}(t_{0}) = x_{0}^{(3)}\) and there are subsequences of \(x_{n}^{(i)}(t), \ i = 0, 1, 2—\)we denote it again by \(x_{n}^{(i)}(t), \ i = 0, 1, 2,\) such that \(x_{n}^{(i)}(t), \ i = 0, 1, 2,\) converge uniformly to \(x^{(i)}(t), \ i = 0, 1, 2—\)
on any compact subintervals of \((\omega^{-}, \omega^{+}).\) Because \([a_{n}, b_{n}] \subset [a_{n+1}, b_{n+1}], \cup_{n=1}^{\infty}[a_{n}, b_{n}] = (0, 1),\) and \(\alpha(t) \leq x_{n}(t) \leq \beta(t), \quad t \in [a_{n}, b_{n}],\) \(-\alpha''(t) \leq -x''_{n}(t) \leq -\beta''(t), \quad t \in [a_{n}, b_{n}],\) one can easily see that \(\alpha(t) \leq x(t) \leq \beta(t), \) \(-\alpha''(t) \leq -x''(t) \leq -\beta''(t)\) for \(t \in (\omega^{-}, \omega^{+}) \cap (0, 1).\) This leads additionally to the fact that \((\omega^{-}, \omega^{+}) \supset (0, 1),\) from the Extension Theorem. Also, \(x(t)\) satisfies \(x(1) = 0, \ x''(1) = 0,\) because \(\alpha(t)\) and \(\beta(t)\) do. From (2.22) and (2.24), we have
\[
x(0) - \sum_{i=1}^{m-2} \alpha_{i} x(\eta_{i}) = 0, \quad x''(0) - \sum_{i=1}^{m-2} \beta_{i} x''(\eta_{i}) = 0.
\]

Thus \(x(t)\) is a \(C^{2}[0, 1]\) positive solution of problem (1.1) and (1.2).

In addition, if (2.21) holds, then \(|x^{(4)}(t)| \leq F(t),\) and hence \(x^{(4)}(t)\) is absolutely integrable on \([0, 1].\) This implies \(x(t) \in C^{3}[0, 1],\) so \(x(t)\) is a \(C^{3}[0, 1]\) positive solution of the problem (1.1) and (1.2). The proof is complete. \(\Box\)

3. The main results

Theorem 3.1. Suppose (H) holds, then a necessary and sufficient condition for problem (1.1) and (1.2) to have \(C^{2}[0, 1]\) positive solutions is that the following integral conditions hold:
0 < \int_0^1 s(1-s)f(s, (1-s), 1) \, ds < \infty, \quad (3.1)

\lim_{t \to 0^+} t \int_t^1 (1-s)f(s, (1-s), 1) \, ds = 0, \quad (3.2)

\lim_{t \to 1^-} (1-t) \int_0^t sf(s, (1-s), 1) \, ds = 0. \quad (3.3)

**Theorem 3.2.** Suppose (H) holds, then a necessary and sufficient condition for problem (1.1) and (1.2) to have \( C^3[0, 1] \) positive solutions is that the following integral conditions hold:

\[ 0 < \int_0^1 f(s, (1-s), (1-s)) \, ds < \infty. \quad (3.4) \]

**Proof of Theorem 3.1.** Necessity. Let \( x(t) \in C^2[0, 1] \cap C^4(0, 1) \) be a positive solution of (1.1) and (1.2). By the proof of Lemma 2.2, there exist constants \( I_1 \) and \( I_2, 0 < I_1 < I_2 \) such that

\[ I_1(1-t) \leq x(t) \leq I_2(1-t), \quad t \in [0, 1], \quad (3.5) \]

and there is \( t_0 \in (0, 1) \) such that \( x^{(3)}(t_0) = 0 \). Let \( c_0 \) be a constant such that \( c_0 I_2 \leq 1, 1/c_0 \geq 1, -c_0 x''(t) \leq 1 \). Similar to the proof of Theorem 3.2 in [21] and from (1.3)–(1.6) and (3.5), we have

\[ f(t, x(t), -x''(t)) \geq c_0^{\mu_1-\lambda_1} I_1^{\mu_1} c_0^{\mu_2-\lambda_2} (-x''(t))^\mu_2 f(t, (1-t), 1), \quad t \in (0, 1), \]

According to (1.1), we have

\[ f(t, (1-t), 1) \leq a_0 (-x''(t))^{-\mu_2} x^{(4)}(t), \quad t \in (0, 1), \quad (3.6) \]

where \( a_0 = (c_0^{\mu_1-\lambda_1} I_1^{\mu_1} c_0^{\mu_2-\lambda_2})^{-1} \). For \( t \in (0, t_0) \), by integration of (3.6), we obtain

\[
\int_0^{t_0} f(s, s(1-s), 1) \, ds \leq a_0 \left[ -x'''(s) \right] (-x'')^{-\mu_2}(s) \bigg|_0^{t_0} \\
+ \int_0^{t_0} a_0 \left[ -\mu_2 (-x'')^{-\mu_2-1}(s) \right] [-x'''(s)]^2 \, ds \\
\leq a_0 \left[ -x'' \right]^{-\mu_2}(t) \left[ -x'''(t) \right], \quad t \in (0, t_0). \quad (3.7)
\]

Integrating (3.7), we have

\[
\int_0^{t_0} \int_0^{t_0} f(s, (1-s), 1) \, ds \, dt \leq a_0 \frac{(-x''(1-t_0))^{1-\mu_2}(t_0) - (-x'')^{1-\mu_2}(0)}{1-\mu_2} < \infty. \quad (3.8)
\]
Exchanging the integral order of (3.8), we get
\[ 0 < \int_{0}^{t_0} sf(s, (1-s), 1) \, ds < \infty. \]  
(3.9)

Similarly, by integration of (3.7), we obtain
\[ 0 < \int_{t_0}^{1} (1-s) f(s, (1-s), 1) \, ds < \infty. \]  
(3.10)

(3.9) and (3.10) imply that (3.1) holds.

For \( t \in (0, t_0) \), by integration of (3.7), we have
\[
\int_{t}^{t_0} \int_{0}^{s} f(\tau, (1-\tau), 1) \, d\tau \, ds \leq a_0 \frac{(-x''(1-\mu_2) - (-x'')(1-\mu_2)}{1-\mu_2},
\]
therefore,
\[
t \int_{t}^{t_0} f(\tau, (1-\tau), 1) \, d\tau \leq a_0 \frac{(-x''(1-\mu_2) - (-x'')(1-\mu_2)}{1-\mu_2}. \]  
(3.11)

Letting \( t \to 0 \) in (3.11), we have \( \lim_{t \to 0^+} t \int_{t}^{t_0} f(s, (1-s), 1) \, ds = 0 \). These imply that (3.2) holds. Similarly, we can prove (3.3).

**Sufficiency.** Suppose that (3.1)–(3.3) hold.

Firstly, we define the linear operators \( A \) and \( B \) as follows:
\[
B x(t) = \int_{0}^{1} G(t, s) x(s) \, ds + \frac{1-t}{1 - \sum_{i=1}^{m-2} \beta_i (1-\eta_i)} \times \sum_{i=1}^{m-2} \beta_i \int_{0}^{1} G(\eta_i, s) x(s) \, ds, \]  
(3.12)
\[
A x(t) = \int_{0}^{1} G(t, s) x(s) \, ds + \frac{1-t}{1 - \sum_{i=1}^{m-2} \alpha_i (1-\eta_i)} \times \sum_{i=1}^{m-2} \alpha_i \int_{0}^{1} G(\eta_i, s) x(s) \, ds, \]  
(3.13)

where \( G(t, s) \) is given by (2.15).

Choose a constant \( M \geq 2 \) such that \( M(\mu_2 - \lambda_2) > 1 \), and let
\[
q(t) = B f(t, (1-t), 1), \quad h(t) = AB f(t, (1-t), 1), \quad t \in [0, 1],
\]
(3.14)
\[
Q(t) = \left[q(t)\right]^{1/(M(\mu_2 - \lambda_2))}.
\]  
(3.15)

Then \( q(t), Q(t) \in C[0, 1] \cap C^2(0, 1) \) satisfying \( q(t) > 0, Q(t) > 0, t \in (0, 1) \), and
\[
-q''(t) = h^{(4)}(t) = f(t, (1-t), 1), \quad -h''(t) = q(t), \quad -Q''(t) \geq 0, \quad \text{for } t \in (0, 1)
\]
and from (3.1)–(3.3), we have
\[ q(1) = h(1) = Q(1) = 0, \quad h(0) - \sum_{i=1}^{m-2} \alpha_i h(\eta_i) = 0, \]
\[ q(0) - \sum_{i=1}^{m-2} \beta_i q(\eta_i) = 0, \quad Q(0) - \sum_{i=1}^{m-2} \beta_i Q(\eta_i) \geq 0, \]

and
\[
\int_0^1 G(s,s)f(s, (1-s), 1) \, ds \leq \int_0^1 s(1-s) f(s, (1-s), 1) \, ds < +\infty, \\
q(t) \leq \left( 1 + \left[ 1 - \sum_{i=1}^{m-2} \beta_i (1-\eta_i) \right]^{-1} \right) \int_0^1 G(s,s)f(s, (1-s), 1) \, ds < +\infty.
\]

\[
(1-t) \int_0^t s Q^{-(\mu_2-\lambda_2)}(s) f(s, (1-s), 1) \, ds \\
\leq (1-t) \int_0^t \left( (1-s) \int_0^s \tau f(\tau, (1-\tau), 1) \, d\tau \right)^{-1/M} f(s, (1-s), 1) \, ds \\
\leq (1-t)^{1-1/M} \int_0^t \left( \int_0^s \tau f(\tau, (1-\tau), 1) \, d\tau \right)^{-1/M} f(s, (1-s), 1) \, ds \\
= (1-t)^{1-1/M} (1-1/M)^{-1} \left( \int_0^t s f(s, (1-s), 1) \, ds \right)^{1-1/M} \\
\leq (1-1/M)^{-1} \left( \int_0^1 (1-s)s f(s, (1-s), 1) \, ds \right)^{1-1/M} < \infty. \tag{3.16}
\]

Similarly, we have
\[
t \int_t^1 (1-s) Q^{-(\mu_2-\lambda_2)}(s) f(s, (1-s), 1) \, ds \\
\leq (1-1/M)^{-1} \left( \int_0^1 (1-s)s f(s, (1-s), 1) \, ds \right)^{1-1/M} < \infty. \tag{3.17}
\]

Let \( c_1 > 0 \) such that \( (1/c_1) Q(t) \leq 1, \ c_1 \geq 1 \). From (1.3)–(1.6), we have
\[
Q^{-\mu_2} f(t, (1-t), Q) \leq Q^{-\mu_2} (Q/c_1)^{\lambda_2} f(t, (1-t), c_1) \\
\leq Q^{-\mu_2} (Q/c_1)^{\lambda_2} c_1^{\mu_2} f(t, (1-t), 1) \\
= c_1^{\mu_2-\lambda_2} Q^{\lambda_2-\mu_2} f(t, (1-t), 1). \tag{3.18}
\]
Let
\[ h_1(t) = AB(1 - t)^{\mu_2} f(t, (1 - t), 1), \quad t \in [0, 1], \]
\[ h_2(t) = AB Q^{-\mu_2}(t) f(t, (1 - t), Q(t)) + AQ(t), \quad t \in [0, 1]. \]

Thus, (3.16)–(3.18) imply that
\[ 0 \leq h_1(t) < \infty, \quad 0 \leq -h_1''(t) < \infty, \quad 0 \leq h_2(t) < \infty, \quad 0 \leq -h_2''(t) < \infty, \]
for \( t \in [0, 1]. \)

One can check that \( h_i \in C^2[0, 1] \cap C^3(0, 1), h_i(1) = 0, h_i''(1) = 0, i = 1, 2, \) and
\[ h_j(0) - \sum_{i=1}^{m-2} \alpha_i h_j(\xi_i) = 0, \quad j = 1, 2, \]
\[ \left[ h_1''(0) - \sum_{i=1}^{m-2} \beta_i h_1'(\xi_i) \right] = 0, \quad \left[ h_2''(0) - \sum_{i=1}^{m-2} \beta_i h_2'(\xi_i) \right] \geq 0, \]
\[ L_1(1 - t) \leq -h_1''(t) \leq \| h_1'' \|, \quad Q(t) \leq -h_2''(t) \leq \| h_2'' \|, \quad t \in [0, 1]; \]
\[ h_1^{(4)}(t) = (t(1 - t))^{\mu_2} f(t, t(1 - t), 1), \quad t \in (0, 1), \] (3.19)
\[ h_2^{(4)}(t) \geq Q^{-\mu_2}(t) f(t, t(1 - t), Q(t)), \quad t \in (0, 1); \] (3.20)
\[ a_1(1 - t) \leq h_1(t) \leq a_2(1 - t), \quad t \in [0, 1], \] (3.21)
\[ b_1(1 - t) \leq h_2(t) \leq b_2(1 - t), \quad t \in [0, 1]. \] (3.22)

Here,
\[ L_1 = \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i (1 - \xi_i)} \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\xi_i, s)(1 - s)^{\mu_2} f(s, (1 - s), 1) \, ds, \]
\[ a_1 = \frac{L_1}{1 - \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i, s)(1 - s) \, ds, \]
\[ a_2 = \| h_1'' \| \left[ \frac{1}{2} + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i, s) \, ds \right], \]
\[ b_1 = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i, s) Q(s) \, ds, \]
\[ b_2 = \| h_2'' \| \left[ \frac{1}{2} + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i, s) \, ds \right], \]
\[ \| h_j'' \| = \max_{0 \leq t \leq 1} | h_j''(t) |, \quad j = 1, 2. \]
Let $\alpha(t) = k_1 h_1(t)$, $\beta(t) = k_2 h_2(t)$, $t \in [0, 1]$; here $k_1$, $k_2$ are constants satisfying $0 < k_1 \leq 1 \leq k_2$ and will be determined later. Suppose $c_2, c_3$ are constants such that $c_2 \|h''_1\| \leq 1$, $c_2 a_2 \leq 1$, $1/c_2 \geq 1$, $c_3 \geq 1$, $c_3 b_1 \geq 1$, $1/c_3 \leq 1$. From (1.5), (1.6), we have
\[
\begin{align*}
    f(t, \alpha(t), -\alpha''(t)) \\
    \geq (1/c_2)^{\lambda_2} f(t, \alpha(t), -c_2 \alpha''(t)) \\
    \geq (c_2)^{\mu_2 - \lambda_2} (-\alpha'')^{\mu_2} f(t, \alpha(t), 1) \\
    \geq (c_2)^{\mu_2 - \lambda_2} (k_1 L_1)^{\mu_2} (1 - t)^{\mu_2} f(t, \alpha(t), 1) \\
    \geq (c_2)^{\mu_2 - \lambda_2} (k_1 L_1)^{\mu_2} (1 - t)^{\mu_2} (c_2)^{\mu_1 - \lambda_1} (k_1 a_1)^{\mu_1} f(t, (1 - t), 1) \\
    = (c_2)^{\mu_2 + \mu_1 - \lambda_2 - \lambda_1} (k_1)^{\mu_2 + \mu_1} L_1^{\mu_2} a_1^{\mu_1} (1 - t)^{\mu_2} f(t, (1 - t), 1) \\
    \geq k_1 (1 - t)^{\mu_2} f(t, (1 - t), 1) = \alpha''^{(4)}(t), \quad t \in (0, 1),
\end{align*}
\]
(3.23)
\[
\begin{align*}
   f(t, \beta(t), -\beta''(t)) \\
   \leq (c_3)^{\mu_2 - \lambda_2} \left( \frac{\beta''(t)}{Q(t)} \right)^{\mu_2} f(t, \beta(t), Q(t)) \\
   \leq (c_3)^{\mu_2 - \lambda_2} (k_2 \|h''_2\|)^{\mu_2} Q^{-\mu_2} f(t, \beta(t), Q(t)) \\
   \leq (c_3)^{\mu_2 - \lambda_2} (c_3)^{\mu_1 - \lambda_1} (k_2 b_2)^{\mu_1} Q^{-\mu_2} f(t, (1 - t), Q(t)) \\
   \leq k_2 Q^{-\mu_2} f(t, (1 - t), Q(t)) = \beta''^{(4)}(t), \quad t \in (0, 1).
\end{align*}
\]
(3.24)

By virtue of (1.3), (1.4), we can find $k_0$ such that $f(t, (1 - t), Q(t)) \geq k_0 Q^{\mu_2} f(t, (1 - t), 1)$, and hence, from the definitions of $h_1(t)$, $h_2(t)$, we have, when $k > k_0^{-1}$, $h_1(t) \leq kh_2(t)$ for $t \in [0, 1]$. Now we choose
\[
    k_1 = \min \left\{ 1, \left( L_1^{\mu_2 - \mu_1} c_2^{\mu_2 + \mu_1 - \lambda_2 - \lambda_1} a_1^{\mu_1} \right)^{1/(1 - \mu_2 - \mu_1)} \right\}
\]
and
\[
    k_2 = \max \left\{ 1, k_0^{-1}, \left( \|h''_2\|^{\mu_2} c_3^{\mu_2 + \mu_1 - \lambda_2 - \lambda_1} b_2^{\mu_1} \right)^{1/(1 - \mu_2 - \mu_1)} \right\}.
\]

Then $\alpha(t), \beta(t) \in C^2[0, 1] \cap C^4(0, 1)$, $0 < \alpha(t) \leq \beta(t)$, $0 < -\alpha''(t) \leq -\beta''(t)$ for $t \in (0, 1)$,
\[
    \alpha(1) = \beta(1) = 0, \quad \alpha''(1) = \beta''(1) = 0,
\]
and
\[
    \alpha(0) - \sum_{i=1}^{m-2} \alpha_i \alpha(\eta_i) = 0, \quad \beta(0) - \sum_{i=1}^{m-2} \alpha_i \beta(\eta_i) = 0,
\]
\[
    \left[ \alpha''(0) - \sum_{i=1}^{m-2} \beta_i \alpha''(\eta_i) \right] = 0, \quad -\left[ \beta''(0) - \sum_{i=1}^{m-2} \beta_i \beta''(\eta_i) \right] \geq 0.
\]

From (3.23) and (3.24), we obtain that for such choice of $k_1$ and $k_2$, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of (1.1), respectively.

From the first conclusion of Lemma 2.3, we conclude that the problem (1.1) has at least a $C^2[0, 1]$ positive solution $x(t)$ satisfying $\alpha \leq x \leq \beta$. This completes the proof of Theorem 3.1. □
Proof of Theorem 3.2. Necessity. Suppose that $x(t)$ is a $C^3[0, 1]$ positive solution of (1.1) and (1.2). Then both $x^{(3)}(0)$ and $x^{(3)}(1)$ exist. By Lemma 2.2 and the proof of Theorem 3.1, there are constants $I_1, I_2, I_3$ and $I_4$, $0 < I_1 < I_2, 0 < I_3 < I_4$ such that
\[
I_1(1-t) \leq -x''(t) \leq I_2(1-t), \quad t \in [0, 1],
\]
\[
I_3(1-t) \leq x(t) \leq I_4(1-t), \quad t \in [0, 1].
\]
(3.25)
(3.26)
Let $c_4$ be a constant satisfying $c_4I_2 \leq 1, c_4I_4 \leq 1, 1/c_4 \geq 1$. Then (1.3)–(1.6) and (3.25), (3.26), lead to
\[
f(t, x(t), -x''(t)) \geq (1/c_4)^{\lambda_1} f(t, c_4x(t)(1-t)/(1-t), -x''(t))
\]
\[
\geq (c_4)^{\mu_1-\lambda_1} f(t, (1-t), -x''(t))
\]
\[
\geq (c_4)^{\mu_1-\lambda_1} I_3^{\mu_1} f(t, (1-t), -x''(t))
\]
\[
\geq (c_4)^{\mu_1-\lambda_1} I_3^{\mu_1} (c_4)^{\mu_2} f(t, (1-t), (1-t)), \quad t \in (0, 1).
\]
Consequently,
\[
\int_0^1 f(t, (1-t), (1-t)) \, dt \leq (c_4)^{\lambda_2-\mu_2} I_1^{-\mu_2} (c_4)^{\lambda_1-\mu_1} I_3^{-\mu_1} \int_0^1 f(t, x(t), -x''(t)) \, dt
\]
\[
= (c_4)^{\lambda_2-\mu_2} I_1^{-\mu_2} (c_4)^{\lambda_1-\mu_1} I_3^{-\mu_1} (x^{(3)}(1) - x^{(3)}(0)) < \infty.
\]
Thus (3.4) holds.

Sufficiency. Suppose that (3.4) holds. Let
\[
h(t) = ABf(t, (1-t), (1-t)), \quad t \in [0, 1].
\]
Then $h(t) \in C^3[0, 1] \cap C^4(0, 1)$ and (3.25), (3.26) hold if $x(t)$ is replaced by $h(t)$, and
\[
I_1 = \sum_{i=1}^{m-2} \beta_i (1-\eta_i) \eta_i \|h''\|
\]
\[
I_2 = \int_0^1 s f(s, (1-s), (1-s)) \, ds + \frac{1}{1-\sum_{i=1}^{m-2} \beta_i (1-\eta_i)}
\]
\[
\times \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\eta_i, s) f(s, (1-s), (1-s)) \, ds,
\]
hold, where
\[
I_3 = \frac{I_1}{1-\sum_{i=1}^{m-2} \alpha_i (1-\eta_i)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s)(1-s) \, ds,
\]
\[
I_4 = I_2 \left[ \int_0^1 s(1-s) \, ds + \frac{1}{1-\sum_{i=1}^{m-2} \alpha_i (1-\eta_i)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s)(1-s) \, ds \right].
\]
Suppose that constant $c_5$ satisfies $c_5 I_2 \leq 1$, $c_5 I_4 \leq 1$, $1/c_5 \geq 1$, $c_6 I_1 \geq 1$, $c_6 I_3 \geq 1$, $1/c_6 \leq 1$. Let $\alpha(t) = k_1 h(t)$, $\beta(t) = k_2 h(t)$, $t \in [0, 1]$; here

\[
k_1 = \min \left\{ 1, \left( I_1^{\mu_2} I_3^{\mu_1} c_5^{\mu_2 - \lambda_2} c_5^{1 - \lambda_1} \right)^{1/(1 - \mu_1 - \mu_2)} \right\}
\]

and

\[
k_2 = \max \left\{ 1, \left( I_2^{\mu_2} I_4^{\mu_1} c_6^{\mu_2 - \lambda_2} c_6^{1 - \lambda_1} \right)^{1/(1 - \mu_1 - \mu_2)} \right\}.
\]

A similar argument to that we have checked in the sufficiency proof of Theorem 3.1 yields that

\[
f(t, \alpha(t), -\alpha''(t))
\]

\[
\geq (1/c_5)^{\lambda_2} f(t, \alpha(t), -c_5 \alpha''(t))
\]

\[
\geq (c_5)^{\mu_2 - \lambda_2} \left( -\alpha''(t)/(1 - t) \right)^{\mu_2} f(t, \alpha(t), (1 - t))
\]

\[
\geq (c_5)^{\mu_2 - \lambda_2} (k_1 I_1)^{\mu_2} f(t, \alpha(t), (1 - t))
\]

\[
\geq (c_5)^{\mu_2 - \lambda_2} (k_1 I_1)^{\mu_2} (c_5)^{\mu_1 - \lambda_1} (k_1 I_3)^{\mu_1} f(t, (1 - t), (1 - t))
\]

\[
\geq k_1 f(t, (1 - t), (1 - t)) = a(4)(t), \quad t \in (0, 1),
\]

(3.27)

\[
f(t, \beta(t), -\beta''(t))
\]

\[
\leq (1/c_6)^{\lambda_2} f(t, \beta(t), -c_6 \beta''(t))
\]

\[
\leq (c_6)^{\mu_2 - \lambda_2} \left( -\beta''(t)/(1 - t) \right)^{\mu_2} f(t, \beta(t), (1 - t))
\]

\[
\leq (c_6)^{\mu_2 - \lambda_2} (k_2 I_2)^{\mu_2} f(t, \beta(t), (1 - t))
\]

\[
\leq (c_6)^{\mu_2 - \lambda_2} (k_2 I_2)^{\mu_2} (c_6)^{\mu_1 - \lambda_1} (k_2 I_4)^{\mu_1} f(t, (1 - t), (1 - t))
\]

\[
\leq k_2 f(t, (1 - t), (1 - t)) = b(4)(t), \quad t \in (0, 1).
\]

(3.28)

So, $\alpha(t), \beta(t) \in C^3[0, 1] \cap C^4(0, 1)$ are respectively lower and upper solutions of (1.1) and (1.2) satisfying $0 < \alpha(t) \leq \beta(t)$, $0 < -\alpha''(t) \leq \beta''(t)$, for $t \in (0, 1)$, and

\[
\alpha(1) = \beta(1) = 0, \quad \alpha''(1) = \beta''(1) = 0,
\]

and

\[
\alpha(0) - \sum_{i=1}^{m-2} \alpha_i \alpha(\eta_i) = 0, \quad \beta(0) - \sum_{i=1}^{m-2} \alpha_i \beta(\eta_i) = 0,
\]

\[
\left[ \alpha''(0) - \sum_{i=1}^{m-2} \beta_i \alpha''(\eta_i) \right] = 0, \quad - \left[ \beta''(0) - \sum_{i=1}^{m-2} \beta_i \beta''(\eta_i) \right] = 0,
\]

Additionally, when $\alpha \preceq x \preceq \beta$, we have

\[
0 \leq f(t, x, -x'') \leq \left( \frac{k_1}{c_6} \right)^{\lambda_1} f\left( t, \frac{c_6 x}{k_1 (1 - t)}, (1 - t), -x'' \right)
\]

\[
\leq \left( \frac{k_1}{c_6} \right)^{\lambda_1} \left( \frac{c_6 x}{k_1 (1 - t)} \right)^{\mu_1} f(t, (1 - t), -x'')
\]

\[
\leq \left( \frac{k_1}{c_6} \right)^{\lambda_1 - \mu_1} (k_2 I_4)^{\mu_1} \left( \frac{k_1}{c_6} \right)^{\lambda_2 - \mu_2} (k_2 I_2)^{\mu_2} f(t, (1 - t), (1 - t)) = F(t).
\]
From (3.4), we have \( \int_0^1 F(t) \, dt < \infty \). By Lemma 2.3, we assert that problem (1.1) admits a positive solution \( x(t) \in C^3[0, 1] \cap C^4(0, 1) \) such that \( \alpha \preceq x \preceq \beta \). The proof of Theorem 3.2 is complete.

Remark 1. Consider the singular \( m \)-point boundary value problem

\[
\begin{aligned}
& x^{(4)}(t) = f(t, x(t), -x''(t)), \quad t \in (0, 1), \\
& x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \\
& x''(0) = 0, \quad x''(1) = \sum_{i=1}^{m-2} \beta_i x''(\eta_i).
\end{aligned}
\]  

(3.29)

Firstly, we define the linear operators \( A_1 \) and \( B_1 \) as follows:

\[
\begin{aligned}
B_1 x(t) &= \int_0^1 G(t, s) x(s) \, ds + \frac{t}{1 - \sum_{i=1}^{m-2} \beta_i \eta_i} \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\eta_i, s) x(s) \, ds, \\
A_1 x(t) &= \int_0^1 G(t, s) x(s) \, ds + \frac{t}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\eta_i, s) x(s) \, ds,
\end{aligned}
\]  

(3.30) \hspace{1cm} (3.31)

where \( G(t, s) \) is given by (2.15).

By analogous methods, we have the following results.

Assume that \( x(t) \) is a \( C^2[0, 1] \) positive solution of problem (3.29). Then \( x(t) \) can be stated as

\[ x(t) = A_1 B_1 f(t, x(t), -x''(t)). \]

Theorem 3.3. Suppose \( (H) \) holds, then a necessary and sufficient condition for problem (3.29) to have \( C^2[0, 1] \) positive solutions is that the following integral conditions hold:

\[
0 < \int_0^1 s(1 - s) f(s, s, 1) \, ds < \infty, \hspace{1cm} (3.32)
\]

\[
\lim_{t \to 0^+} t \int_t^1 (1 - s) f(s, s, 1) \, ds = 0, \hspace{1cm} (3.33)
\]

\[
\lim_{t \to 1^-} (1 - t) \int_0^t s f(s, s, 1) \, ds = 0. \hspace{1cm} (3.34)
\]

Theorem 3.4. Suppose \( (H) \) holds, then a necessary and sufficient condition for problem (3.29) to have \( C^3[0, 1] \) positive solutions is that the following integral conditions hold:

\[
0 < \int_0^1 f(s, s, s) \, ds < \infty. \hspace{1cm} (3.35)
\]
4. An example

Consider the following singular three-point boundary value problem:

\[
\begin{aligned}
&x^{(4)}(t) = t^p(1-t)^q x^\lambda(t)[-x''(t)]^\mu, \quad t \in (0, 1), \\
x(0) = \frac{1}{2} x\left(\frac{1}{2}\right), \quad x(1) = 0, \\
x''(0) = \frac{1}{2} x''\left(\frac{1}{2}\right), \quad x''(1) = 0, 
\end{aligned}
\] (4.1)

where \( p, q, \lambda, \mu \in R \), and \( \max\{0, \lambda\} + \max\{0, \mu\} < 1 \), \( m = 3 \), \( \alpha_1 = \frac{1}{2} \), \( \beta_1 = \frac{1}{3} \), \( \eta_1 = \frac{1}{2} \), \( f(t, x, y) = t^p(1-t)^q x^\lambda y^\mu \), and we have (H) holds. So, from Theorems 3.1 and 3.2, we obtain that

**Conclusion 4.1.** A necessary and sufficient condition for problem (4.1) to have \( C^2[0, 1] \) positive solutions is \( p > -2, q + \lambda > -2 \).

**Conclusion 4.2.** A necessary and sufficient condition for problem (4.1) to have \( C^3[0, 1] \) positive solutions is \( p > -1, q + \lambda + \mu > -1 \).

Similarly, for singular three-point boundary value problem

\[
\begin{aligned}
&x^{(4)}(t) = t^p(1-t)^q x^\lambda(t)[-x''(t)]^\mu, \quad t \in (0, 1), \\
x(0) = 0, \quad x(1) = \frac{1}{2} x\left(\frac{1}{2}\right), \\
x''(0) = 0, \quad x''(1) = \frac{1}{2} x''\left(\frac{1}{2}\right), 
\end{aligned}
\] (4.2)

where \( \alpha, \beta, \lambda \in R \), and \( \lambda < 1 \), we have

**Conclusion 4.3.** A necessary and sufficient condition for problem (4.2) to have \( C^2[0, 1] \) positive solutions is \( q > -2, p + \lambda > -2 \).

**Conclusion 4.4.** A necessary and sufficient condition for problem (4.2) to have \( C^3[0, 1] \) positive solutions is \( q > -1, p + \lambda + \mu > -1 \).

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References