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Applications of the theory of weakly nondegenerate conditions to zero decomposition for polynomial systems

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Abstract

This paper presents some applications of the theory of weakly nondegenerate conditions obtained by analytic methods to zero decomposition of polynomial systems and sets. Based on a known algorithm, a method is presented that can compute a strong regular series of any nonempty polynomial set. An algorithm is also devised that can decompose any polynomial system into two finite sets of strong regular sets with some good properties. In addition, we propose two alternative methods for decomposing any algebraic variety and quasi-algebraic variety into equidimensional components and removing redundant components respectively without computing Gröbner bases. Some examples are given to illustrate the performance and effectiveness of the applications.

1. Introduction

Let $\mathbf{K}$ be a field of characteristic 0 and $\mathbf{K}[x_1, \ldots, x_n]$ (or $\mathbf{K}[\mathbf{x}]$ for short) the ring of polynomials in the variables $(x_1, \ldots, x_n)$ with coefficients in $\mathbf{K}$. A *polynomial set* is a finite set of nonzero polynomials in $\mathbf{K}[\mathbf{x}]$. In what follows, the number of elements of a finite set $\mathbb{P}$ is denoted $|\mathbb{P}|$; it is also called the *length* of $\mathbb{P}$. For any polynomial $P \notin \mathbf{K}$, the largest index p such that $\deg(P, x_p) > 0$ is called the *class*, x_p the *leading variable*, and $\deg(P, x_p)$ the *leading degree* of P, denoted by $\mathrm{cls}(P)$, $\mathrm{lv}(P)$ and $\mathrm{ldeg}(P)$, respectively.

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A finite nonempty ordered set $\mathbb{T} = [f_1, \ldots, f_s]$ of polynomials in $\mathbf{K}[\mathbf{x}]\backslash\mathbf{K}$ is called a *triangular set* if $\mathrm{cls}(f_1) < \cdots < \mathrm{cls}(f_s)$. Let $\mathbb{T}$ be a triangular set which can be written in the following form:

$$\mathbb{T} = [f_1(u_1, \ldots, u_r, y_1), \ldots, f_s(u_1, \ldots, u_r, y_1, \ldots, y_s)], \tag{1}$$

where $(u_1, \ldots, u_r, y_1, \ldots, y_s)$ is a permutation of $(x_1, \ldots, x_n)$. We call $u_1, \ldots, u_r$ (abbreviated to $\mathbf{u}$) the *parameters* and $y_1, \ldots, y_s$ the *dependents* of $\mathbb{T}$. $\mathbb{C}_{f_i}$ denotes the set of all the nonzero coefficients of f_i in y_i, $I_i = \mathrm{ini}(f_i)$ the leading coefficient of f_i in y_i for each i, and $\mathrm{ini}(\mathbb{T})$ the set of all I_i. $\mathbb{T}^{\{j\}}$ stands for $[f_1, \ldots, f_j]$ for $1 \le j \le s$.

For any triangular set $\mathbb{T}$ and polynomial P in $\mathbf{K}[\mathbf{u}, y_1, \ldots, y_s]\backslash\mathbf{K}[\mathbf{u}]$, the index k with $\mathrm{lv}(P) = y_k$ is called the *class* of P with respect to $\mathbb{T}$, denoted by $\mathrm{cls}(P, \mathbb{T})$. With the notation in Wang (2001), $\mathrm{prem}(P, \mathbb{T})$ stands for the *pseudo-remainder* of P with respect to $\mathbb{T}$, and $\mathrm{res}(P, \mathbb{T})$ the *resultant* of P with respect to $\mathbb{T}$, respectively. Refer to Kalkbrener (1993), Yang and Zhang (1991) and Yang et al. (1996): a triangular set $\mathbb{T} = [f_1, \ldots, f_s]$ is called a *regular set* in $\mathbf{K}[\mathbf{x}]$ if $\mathrm{res}(I_j, \mathbb{T}) \neq 0$ for $j = 2, \ldots, s$.

The extension field $\tilde{\mathbf{K}}$ of $\mathbf{K}$ considered in this paper is an algebraically closed field. While speaking about a *polynomial system*, we refer to a pair $[\mathbb{P}, \mathbb{Q}]$ of polynomial sets. The set of all zeros of $[\mathbb{P}, \mathbb{Q}]$ is defined as

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) \triangleq \{\mathbf{z} \in \tilde{\mathbf{K}}^n : P(\mathbf{z}) = 0, Q(\mathbf{z}) \neq 0, \forall P \in \mathbb{P}, Q \in \mathbb{Q}\}.$$

For any triangular set $\mathbb{T}$ and polynomial P, $\mathbb{T}$ is said to be *strongly independent* of P if $\mathrm{Zero}(\mathbb{T} \cup \{P\}) = \emptyset$. Given a nonempty polynomial set $\mathbb{P}$, using the Wu method, one can compute a *characteristic series* (Wang, 2001; Wu, 1978) $\{\mathbb{T}_1, \ldots, \mathbb{T}_e\}$ of $\mathbb{P}$ such that

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathrm{ini}(\mathbb{T}_i)).$$

Wang (1993, 1998, 2000) proposed two methods, by which one can decompose any polynomial system $[\mathbb{P}, \mathbb{Q}]$ into e *fine triangular systems* (Wang, 1993, 1998, 2000) $[\mathbb{T}_i, \mathbb{U}_i]$ such that

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i).$$

In particular, any polynomial system can be decomposed into finitely many *regular systems* (Wang, 2000, 2001) by the algorithm RegSer[1] described in Wang (2000, 2001).

There exist three algorithms for decomposing any polynomial set $\mathbb{P}$ into a *regular series* (Wang, 2001) $\{\mathbb{T}_1, \ldots, \mathbb{T}_e\}$ such that

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathrm{ini}(\mathbb{T}_i)) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathrm{sat}(\mathbb{T}_i)), \tag{2}$$

where $\mathrm{sat}(\mathbb{T})$ is the *saturation* (Wang, 2001) of $\mathbb{T}$. One of those algorithms is presented by Lazard in Lazard (1991), and the other two algorithms are respectively applications of the

[1] One can obtain an implementation of RegSer in the Epsilon library (http://www-calfor.lip6.fr/~wang/epsilon).

algorithms RegSer and RegSer*, of which the latter is adapted from Kalkbrener's method given in Kalkbrener (1993); see Wang (2001) for details.

With the same notation as in Li et al. (2002), $\mathrm{Reg}(\mathbb{P})$ denotes a regular series of $\mathbb{P}$. Referring to Aubry and Moreno Maza (1999), one can choose one of these algorithms according to efficiency, conciseness and legibility of the output. In practical computation, the algorithm RegSer is a powerful tool for some hard examples; we hereby recommend one to compute $\mathrm{Reg}(\mathbb{P})$ by $\mathrm{RegSer}(\mathbb{P}, \emptyset)$. The following assertion proved by Aubry and others in Aubry et al. (1999) (see also Theorem 6.2.4 in Wang, 2001) plays an important role in this paper.

Theorem 1.1. *A triangular set* $\mathbb{T}$ *is a regular set if and only if* $\mathrm{sat}(\mathbb{T}) = \{P \in \mathbf{K}[\mathbf{x}] : \mathrm{prem}(P, \mathbb{T}) = 0\}$.

For any $\bar{\mathbf{z}} = (\bar{z}_1, \ldots, \bar{z}_n) = (\bar{\mathbf{u}}, \bar{y}_1, \ldots, \bar{y}_s) \in \mathrm{Zero}(\mathbb{T})$, we write $\bar{\mathbf{z}}^{\{i\}}$ for $\bar{\mathbf{u}}, \bar{y}_1, \ldots, \bar{y}_i$ or $(\bar{\mathbf{u}}, \bar{y}_1, \ldots, \bar{y}_i)$ with $\bar{\mathbf{z}} = \bar{\mathbf{z}}^{\{s\}}$ and $\bar{\mathbf{u}} = \bar{\mathbf{z}}^{\{0\}}$. $\bar{\mathbf{z}}$ is said to be *regular* if either $\bar{z}_i = x_i$ or x_i is a dependent of $\mathbb{T}$ for any $1 \le i \le n$. The set of all regular zeros of $\mathbb{T}$ is denoted by $\mathrm{RegZero}(\mathbb{T})$. Let $\mathbb{T} = [f_1, \ldots, f_s]$ be a regular set in $\mathbf{K}[\mathbf{x}]$. A zero $\mathbf{z}_0 \in \mathrm{Zero}(\mathbb{T})$ is called a *quasi-normal zero*, or in other words, it satisfies the weakly nondegenerate condition, if $\mathbf{z}_0^{\{i\}} \notin \mathrm{Zero}(\mathbb{C}_{f_i})$ for any $1 \le i \le s$. $\mathbb{T}$ is called a *strong regular set* if every zero of $\mathbb{T}$ is also a quasi-normal zero. For any triangular set $\mathbb{T}$ and polynomial P, suppose there exists a non-negative integer d such that $\mathrm{prem}(P^d, \mathbb{T}) = 0$. It is easy to see that $\mathrm{Zero}(\{P\}) \supseteq \mathrm{Zero}(\mathbb{T}/\mathrm{ini}(\mathbb{T}))$ from the following *pseudo-remainder formula*:

$$\left(\prod_{i=1}^{s} I_i^{d_i}\right) P^d = \sum_{j=1}^{s} Q_j f_j,$$

where each d_i is a non-negative integer and $Q_j \in \mathbf{K}[\mathbf{x}]$ for all j. Most algorithms of zero decomposition for polynomial sets or systems depend more or less upon the above algebraic fact. By the analytic method, Zhang et al. (1991) established the theory of the weakly nondegenerate condition of regular sets in $\mathbf{K}[\mathbf{x}]$. One further proved that if zero $\mathbf{z}_0$ is a quasi-normal zero of the regular set $\mathbb{T}$, then $\mathbf{z}_0 \in \mathrm{Zero}(\{P\})$, no matter whether $\prod_{i=1}^{s} I_i$ vanishes at $\mathbf{z}_0$ or not; consequently, $\mathrm{Zero}(\mathbb{T}) = \mathrm{Zero}(\mathrm{sat}(\mathbb{T}))$ if $\mathbb{T}$ is a strong regular set (see Li, 2001; Li et al., 2002, for details). Using these results, we can avoid excluding some zeros which should not be excluded in zero decomposition, and reduce the branches of the decomposition tree. A persuasive example is Example 2.1 given in Li et al. (2002), where the polynomial set $\mathbb{P}$ was decomposed into seven fine triangular systems $[\mathbb{T}_1, \mathbb{U}_1], \ldots, [\mathbb{T}_7, \mathbb{U}_7]$, and $\{\mathbb{T}_1, \ldots, \mathbb{T}_7\}$ is also a characteristic (or regular) series of $\mathbb{P}$. In fact, we only need to consider $\mathrm{ini}(\mathbb{T}_2)$ or $\mathbb{U}_2$, and all the others $\mathrm{ini}(\mathbb{T}_i)$ or $\mathbb{U}_i$ do not have to be discussed at all.

Furthermore, by the algorithm Dec presented in Li et al. (2002), one can decompose any polynomial system $[\mathbb{P}, \mathbb{Q}]$ into a finite set Φ of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ by dint of algorithm Reg such that

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{\mathbb{T} \in \Phi} \mathrm{Proj}_{\mathbf{x}} \mathrm{Zero}(\mathbb{T}),$$

where t is a new variable and $\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T})$ the *projection* (Li et al., 2002) of $\mathrm{Zero}(\mathbb{T})$ onto $\mathbf{x} = (x_1, \ldots, x_n)$. At the same time, $\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) \cap \mathrm{Zero}(\{Q\}) = \emptyset$ for any $\mathbb{T} \in \Phi$ and $Q \in \mathbb{Q}$.

In Section 2, some properties of regular sets are established using the theory of weakly nondegenerate condition. Based on algorithm Reg, we present an algorithm SReg for decomposing any polynomial set in $\mathbf{K}[\mathbf{x}]$ into a *strong regular series* in $\mathbf{K}[\mathbf{x}, t]$. Furthermore, two new methods are proposed according to algorithm SReg; using one of them we can decompose any polynomial system into a *strong regular series* even though its efficiency may be low in most cases, and with the other we can decompose any algebraic variety into *equidimensional* components and remove redundant components without computing Gröbner bases.

In Section 3, we present a complete algorithm for decomposing any polynomial system. This algorithm, called SRD, is developed from algorithm Dec given in Li et al. (2002). It can decompose any polynomial system $[\mathbb{P}, \mathbb{Q}]$ into two finite sets Υ_1 and Υ_2 of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ such that

- $$\mathrm{Zero}(\mathbb{P}) = \bigcup_{\mathbb{T} \in \Upsilon_1 \cup \Upsilon_2} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}); \qquad \mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{\mathbb{T} \in \Upsilon_1} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T});$$
- for any $\mathbb{T} \in \Upsilon_1$ and $Q \in \mathbb{Q}$, $\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) \cap \mathrm{Zero}(\{Q\}) = \emptyset$;
- for any $\mathbb{T} \in \Upsilon_2$, there exists some $Q_0 \in \mathbb{Q}$ such that

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) \subseteq \mathrm{Zero}(\{Q_0\}).$$

Moreover, we propose an alternative method for decomposing any quasi-algebraic variety into *equidimensional* components and removing redundant components without computing Gröbner bases. Some examples are given to illustrate the performance of our methods.

2. Decomposing polynomial sets into strong regular series

In this section we first introduce some properties of regular sets and present the algorithm SReg for decomposing any polynomial set into *strong regular series*. A further two new methods are proposed.

2.1. Some properties of regular sets and algorithm SReg

For any triangular set $\mathbb{T}$, we denote $\mathrm{ldeg}(\mathbb{T}) \triangleq \prod_{f \in \mathbb{T}} \mathrm{ldeg}(f)$.

Theorem 2.1.1. *Let* $\mathbb{T} = [f_1, \ldots, f_s]$ *be a regular set and* P *a polynomial in* $\mathbf{K}[\mathbf{x}]$. *Then the following properties are equivalent:*

a. $\mathrm{Zero}(\mathbb{T}/\mathrm{ini}(\mathbb{T})) \subseteq \mathrm{Zero}(\{P\})$;
b. *for any quasi-normal zero* z_0 *of* $\mathbb{T}$, $z_0 \in \mathrm{Zero}(\{P\})$;
c. *there exists an integer* $0 < d \leq \mathrm{ldeg}(\mathbb{T})$ *such that* $\mathrm{prem}(P^d, \mathbb{T}) = 0$.

Proof. c $\Longrightarrow$ b. Theorem 2.1 in Li et al. (2002).
b $\Longrightarrow$ a. It is obvious.
a $\Longrightarrow$ c. It is analogous to the proof of Theorem 5.1.9 given in Wang (2001), and we omit the details. $\square$

Directly applying the theory of weakly nondegenerate condition to algorithm Reg, one can simplify the zero decomposition of form (2) as follows.

Theorem 2.1.2. *Let* $\mathbb{P}$ *be a polynomial set in* $\mathbf{K}[\mathbf{x}]$, *and* $\mathrm{Reg}(\mathbb{P}) = \{\mathbb{T}_1, \ldots, \mathbb{T}_e\}$ *with*

$$\mathbb{T}_i = [f_{i,1}, \ldots, f_{i,s_i}]$$

for any $1 \le i \le e$. *Then*

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i),$$

where

$$\mathbb{U}_i = \{\mathrm{ini}(f_{i,j}) : \mathrm{Zero}(\mathbb{T}_i^{\{j\}} \cup \mathbb{C}_{f_{i,j}}) \neq \emptyset, 1 \le j \le s_i\}$$

for any $1 \le i \le e$.

Proof. From algorithm Reg, we know that

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathrm{sat}(\mathbb{T}_i)) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathrm{ini}(\mathbb{T}_i)).$$

It is obvious that

$$\mathrm{Zero}(\mathbb{T}_i/\mathrm{ini}(\mathbb{T}_i)) \subseteq \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i)$$

by the construction of $\mathbb{U}_i$ for each i. Thus

$$\mathrm{Zero}(\mathbb{P}) \subseteq \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i).$$

On the other hand, it is easy to see that any $\mathbf{z} \in \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i)$ is also a quasi-normal zero of $\mathbb{T}_i$. As $\{\mathbb{T}_1, \ldots, \mathbb{T}_e\}$ is a regular series of $\mathbb{P}$, there exists an integer $d > 0$ such that $\mathrm{prem}(P^d, \mathbb{T}_i) = 0$ for each $P \in \mathbb{P}$ and $1 \le i \le e$. It follows that $\mathrm{Zero}(\mathbb{P}) \supseteq \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i)$ for each i. This implies that

$$\mathrm{Zero}(\mathbb{P}) \supseteq \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i).$$

Therefore,

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathrm{sat}(\mathbb{T}_i)) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathrm{ini}(\mathbb{T}_i)) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i).$$

The proof is complete. $\square$

Remark 2.1.1. In practical computation, one can simply determine that $\mathrm{Zero}(\mathbb{T}_i^{\{j\}} \cup \mathbb{C}_{f_{i,j}}) = \emptyset$ in Theorem 2.1.2 according as $\mathbb{C}_{f_{i,j}} \cap \mathbf{K} \neq \emptyset$ or

$$\{\mathrm{res}(c, \mathbb{T}) : c \in \mathbb{C}_{f_{i,j}}\} \cap (\mathbf{K}\backslash\{0\}) \neq \emptyset,$$

even though the obtained $\mathbb{U}_i$ may be bigger than the $\mathbb{U}_i$ in Theorem 2.1.2 in most cases.

For any polynomial set $\mathbb{P} \subset \mathbf{K}[\mathbf{x}]$, one cannot guarantee that all regular sets of $\mathrm{Reg}(\mathbb{P})$ are strong regular sets even if most of them are indeed so in practical computation. In order to obtain a *strong regular series* of $\mathbb{P}$ defined in the following theorem, we add a new variable t with an ordering for variables $x_1 \prec \cdots \prec x_n \prec t$, and decompose $\mathbb{P}$ in $\mathbf{K}[x_1, \ldots, x_n, t]$ (or $\mathbf{K}[\mathbf{x}, t]$). By the following result, one can compute a strong regular series of not only a polynomial set $\mathbb{P}$ in $\mathbf{K}[\mathbf{x}]$, but also some special polynomial set in $\mathbf{K}[\mathbf{x}, t]$.

Hereinafter, only a special class of polynomial sets in $\mathbf{K}[\mathbf{x}, t]$ is considered: for each polynomial set $\mathbb{P}$ in the class, either $\mathbb{P} \subset \mathbf{K}[\mathbf{x}]$ or $\mathbb{P} = \mathbb{P}_0 \cup \{\mu t - 1\} \subset \mathbf{K}[\mathbf{x}, t]$ with $\mathbb{P}_0 \cup \{\mu\} \subset \mathbf{K}[\mathbf{x}]$.

Theorem 2.1.3. *Let $\mathbb{P}$ be a polynomial set in $\mathbf{K}[\mathbf{x}, t]$. Then one can obtain a finite set Ψ, denoted by $\mathrm{SReg}(\mathbb{P})$, of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ such that*

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{P}) = \bigcup_{\mathbb{T}\in\Psi} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T});$$

the set Ψ is called a strong regular series of $\mathbb{P}$.

Proof. Consider first the case $\mathbb{P} \subset \mathbf{K}[\mathbf{x}]$. Compute $\mathrm{Reg}(\mathbb{P}) = \{\mathbb{T}_1, \ldots, \mathbb{T}_e\}$. With the notation introduced in Theorem 2.1.2, one can obtain $\mathbb{U}_i$ for $1 \le i \le e$ such that

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i/\mathbb{U}_i).$$

Set $\mathbb{T}_i^* = \mathbb{T}_i$ if $\mathbb{U}_i = \emptyset$; otherwise, set

$$\mathbb{T}_i^* = \mathbb{T}_i \cup \left[\left(\prod_{c\in\mathbb{U}_i} c\right) t - 1\right]$$

for $1 \le i \le e$. Then Theorem 2.1.3 holds true with $\Psi = \{\mathbb{T}_1^*, \ldots, \mathbb{T}_e^*\}$.

Next consider the case $\mathbb{P} = \mathbb{P}_0 \cup \{\mu t - 1\} \subset \mathbf{K}[\mathbf{x}, t]$ with $\mathbb{P}_0 \cup \{\mu\} \subset \mathbf{K}[\mathbf{x}]$.

Compute similarly $\mathrm{Reg}(\mathbb{P}) = \{\bar{\mathbb{T}}_1, \ldots, \bar{\mathbb{T}}_{\bar{e}}\}$ with

$$\bar{\mathbb{T}}_i = [\bar{f}_{i,1}, \ldots, \bar{f}_{i,\bar{s}_i}, \bar{\mu}_{i,0}t - 1]$$

for $1 \le i \le \bar{e}$. It follows from Theorem 2.1.2 that

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{i=1}^{\bar{e}} \mathrm{Zero}(\bar{\mathbb{T}}_i/\bar{\mathbb{U}}_i),$$

where

$$\bar{\mathbb{U}}_i = \{\mathrm{ini}(\bar{f}_{i,j}) : \mathrm{Zero}(\bar{\mathbb{T}}_i^{\{j\}} \cup \mathbb{C}_{\bar{f}_{i,j}}) \neq \emptyset, 1 \leq j \leq \bar{s}_i\}.$$

Set

$$\bar{\mathbb{T}}_i^* = \left[\bar{f}_{i,1}, \ldots, \bar{f}_{i,\bar{s}_i}, \left(\prod_{c \in \bar{\mathbb{U}}_i} c\right) \bar{\mu}_{i,0} t - 1\right].$$

It is easy to verify that

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{P}) = \bigcup_{\mathbb{T} \in \Psi} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}),$$

where $\Psi = \{\bar{\mathbb{T}}_1^*, \ldots, \bar{\mathbb{T}}_{\bar{e}}^*\}$. The proof is complete. □

Example 2.1.1. Let $\mathbb{P} = \{P_1, P_2, P_3\}$ with $P_1 = x_1x_4^2 + x_4^2 - x_1x_2x_4 - x_2x_4 + x_1x_2 + 3x_2$, $P_2 = x_1x_4 + x_3 - x_1x_2$, $P_3 = x_3x_4 - 2x_2^2 - x_1x_2 - 1$ and $\mathbb{T}^* = \{2x_1x_2 - x_1^2 + 2x_2 - x_1, (2x_1 + 2)x_2^2 - 2x_1x_2 + x_1 + 1, x_1 + 2x_1x_2^2 + x_1^2x_2 + x_3^2 - x_1x_2x_3, x_1x_4 + x_3 - x_1x_2, x_1t - 1\}$. $\mathbb{P}$ has been considered in Wang (2001), and $\mathbb{T}^*$ is a polynomial set in $\mathbf{K}[\mathbf{x}, t]$. Under the variable ordering $x_1 \prec x_2 \prec x_3 \prec x_4$, one can compute $\mathrm{SReg}(\mathbb{P}) = \{\mathbb{T}_1, \mathbb{T}_2\}$ and $\mathrm{SReg}(\mathbb{T}^*) = \{\mathbb{T}_{2,2}, \mathbb{T}_{2,3}\}$ according to Theorem 2.1.3, where

$$\begin{aligned}
\mathbb{T}_1 &= [x_1, 2x_2^2 + 1, x_3, x_4^2 - x_2x_4 + 3x_2];\\
\mathbb{T}_2 &= [(2x_1 + 2)x_2^2 - 2x_1x_2 + x_1 + 1, x_3^2 - x_1x_2x_3 + x_1 + 2x_1x_2^2 + x_1^2x_2,\\
&\qquad x_1x_4 + x_3 - x_1x_2, x_1t - 1];\\
\mathbb{T}_{2,2} &= [x_1^3 - x_1^2 + 2x_1 + 2, -2(x_1 + 1)x_2 + x_1^2 + x_1, -2x_3^2 + x_1^2x_3 + 2x_1\\
&\qquad -2x_1^2 + 4, -2x_1x_4 + x_1^2 - 2x_3, x_1t - 1];\\
\mathbb{T}_{2,3} &= [x_1 + 1, x_2, x_3^2 - 1, x_4 - x_3, t + 1].
\end{aligned}$$

We have

$$\mathrm{Zero}(\mathbb{P}) = \mathrm{Zero}(\mathbb{T}_1) \cup \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_2)$$

and

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}^*) = \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_{2,2}) \cup \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_{2,3}).$$

Remark 2.1.2. One can check that $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_{2,2}$ and $\mathbb{T}_{2,3}$ in Example 2.1.1 are strong regular sets by Proposition 2.1 in Li et al. (2002). Sometimes, such as the case for the above $\mathbb{T}_1$ or $\mathbb{T}_{2,3}$, we can simply determine that a regular set $\mathbb{T}$ is a strong regular set if $\mathbb{C}_f \cap \mathbf{K} \neq \emptyset$ for each $f \in \mathbb{T}$.

Remark 2.1.3. Algorithm SReg is developed from RegToStr presented in Li et al. (2002); by the latter, one can only decompose any regular set $\mathbb{T}$ in $\mathbf{K}[\mathbf{x}, t]$ into a finite set of strong regular sets, which may not constitute a strong regular series of $\mathbb{T}$. In other words, we have

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) \supseteq \bigcup_{\mathbb{T}^* \in \mathrm{RegToStr}(\mathbb{T})} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}^*)$$

rather than

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) = \bigcup_{\mathbb{T}^* \in \mathrm{SReg}(\mathbb{T})} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}^*).$$

We proceed to decompose any polynomial system in $\mathbf{K}[\mathbf{x}]$ according to algorithm SReg.

Proposition 2.1.1. *Let* $[\mathbb{P}, \mathbb{Q}]$ *be a polynomial system in* $\mathbf{K}[\mathbf{x}]$ *with* $\mathbb{Q} \neq \emptyset$. *There exists an algorithm, by which one can compute a finite set* Ψ_0 *of strong regular sets in* $\mathbf{K}[\mathbf{x}, t]$ *such that*

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{\mathbb{T} \in \Psi_0} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T});$$

the set Ψ_0 *is called a strong regular series of* $[\mathbb{P}, \mathbb{Q}]$.

Proof. Set

$$\mathbb{P}_0 = \mathbb{P} \cup \left\{ 1 - \left(\prod_{Q \in \mathbb{Q}} Q \right) t \right\}.$$

Compute $\Psi_0 = \mathrm{SReg}(\mathbb{P}_0)$ according to Theorem 2.1.3. It is very easy to see that Ψ_0 is a strong regular series of $[\mathbb{P}, \mathbb{Q}]$. The proof is complete. □

2.2. Unmixed decomposition for polynomial sets

Let $\mathcal{V}$ be a collection of points in an n-dimensional affine space $\mathbf{A}^n_{\tilde{\mathbf{K}}}$ with coordinates $\mathbf{x}$ over $\tilde{\mathbf{K}}$. $\mathcal{V}$ is called an (affine) *algebraic variety*, or simply a *variety*, if there is a polynomial set $\mathbb{P} \subset \mathbf{K}[\mathbf{x}]$ such that $\mathcal{V} = \mathrm{Zero}(\mathbb{P})$. $\mathbb{P}$ is called the *defining set* of $\mathcal{V}$. An algebraic variety is said to be *unmixed* or *equidimensional* if all its *irredundant irreducible* (Wang, 2001) components have the same *dimension* (Wang, 2001); see Wang (2001) for details.

Decomposing given algebraic varieties into equidimensional components has various applications in modern geometry engineering. Some successful methods for decomposing any algebraic variety into unmixed components are proposed and have been implemented in Wang (2001). Actually, algorithm SReg has provided an equidimensional decomposition for any algebraic variety, but there may exist many redundant components in the output.

The following *expanding pseudo-remainder* (Li, 2001; Li et al., 2002) of a polynomial P with respect to any triangular set $\mathbb{T}$ is applied for removing redundant components in this section.

Definition 2.2.1. Let $\mathbb{T} = [f_1, \ldots, f_s, \mu_0 t - 1]$ or $[f_1, \ldots, f_s]$ be any triangular set in $\mathbf{K}[\mathbf{x}, t]$ and P a nonzero polynomial in $\mathbf{K}[\mathbf{x}]$. One can form a sequence of nonzero polynomials $P_{-1}, P_0, P_1, \ldots, P_{m-1}, P_m$, with $P_{-1} = P$ and $P_0 = \mathrm{prem}(P_{-1}, \mathbb{T})$ such that

$$P_i = \mathrm{prem}(\mathrm{prem}(f_{\mathrm{cls}(P_{i-1},\mathbb{T})}, P_{i-1}, y_{\mathrm{cls}(P_{i-1},\mathbb{T})}), \mathbb{T}), \qquad i = 1, \ldots, m,$$

and either $P_m \in \mathbf{K}[\mathbf{u}] \backslash \{0\}$ or

$$\mathrm{prem}(\mathrm{prem}(f_{\mathrm{cls}(P_m,\mathbb{T})}, P_m, y_{\mathrm{cls}(P_m,\mathbb{T})}), \mathbb{T}) = 0.$$

P_m is called the *expanding pseudo-remainder* of P with respect to $\mathbb{T}$, denoted simply by $\mathrm{Eprem}(P, \mathbb{T})$.

Example 2.2.1. Continued from Example 2.1.1, let G_1 and G_2, considered in Wang (2001), be two polynomials in $\mathbf{K}[x_1, \ldots, x_4]$ as follows:

$$G_1 = x_1x_4^2 + x_2x_3 - 3x_1x_2^2 + 3x_1x_2 - x_1,$$
$$G_2 = 2x_2x_4 + x_3 - 2x_1x_2^2 - 2x_2 - 1.$$

By the above definition, we know that

$$\mathrm{Eprem}(G_1, \mathbb{T}_1) = \mathrm{Eprem}(G_1, \mathbb{T}_2) = G_1$$

because $\mathrm{prem}(G_1, \mathbb{T}_1) = \mathrm{prem}(G_1, \mathbb{T}_2) = 0$. We proceed to compute $\mathrm{Eprem}(G_2, \mathbb{T}_1)$ and $\mathrm{Eprem}(G_2, \mathbb{T}_2)$ respectively. One can get $\mathrm{Eprem}(G_2, \mathbb{T}_1) = P_0$ from the sequence P_{-1}, P_0 as follows:

$$P_{-1} = G_2,$$
$$P_0 = \mathrm{prem}(P_{-1}, \mathbb{T}_1) = 4x_2x_4 - 2 - 4x_2,$$

as $\mathrm{prem}(\mathrm{prem}(\mathbb{T}_1[4], P_0, x_4), \mathbb{T}_1) = 0$. Secondly, $\mathrm{Eprem}(G_2, \mathbb{T}_2) = P_0^*$ from the sequence P_{-1}^*, P_0^* as follows:

$$P_{-1}^* = G_2,$$
$$P_0^* = \mathrm{prem}(P_{-1}^*, \mathbb{T}_2) = (-2x_1^2 + 4x_1x_2 - 2x_1 + 4x_2)x_3 + 2x_1^2 + 4x_1x_2 + 4x_1 - 2x_1^3 + 4x_1^3x_2,$$

as $\mathrm{prem}(\mathrm{prem}(\mathbb{T}_2[2], P_0^*, x_3), \mathbb{T}_1) = 0$.

Remark 2.2.1. $\mathrm{Eprem}(P, \mathbb{T})$ is different from $\mathrm{res}(P, \mathbb{T})$. P must be a nonzero polynomial in Definition 2.2.1, and $\mathrm{Eprem}(P, \mathbb{T})$ is also a nonzero polynomial at all times. For example, $\mathrm{Eprem}(G_1, \mathbb{T}_1) = G_1$ and $\mathrm{Eprem}(G_2, \mathbb{T}_1) = 4x_2x_4 - 2 - 4x_2$, but $\mathrm{res}(G_1, \mathbb{T}_1) = \mathrm{res}(G_2, \mathbb{T}_1) = 0$ in Example 2.2.1. On the other hand, we find that $\mathrm{Eprem}(P, \mathbb{T})$ is simpler than $\mathrm{res}(P, \mathbb{T})$ in practical computation when $\mathrm{res}(P, \mathbb{T}) \neq 0$ and $|\mathbb{T}| > 1$. For example, $\mathrm{res}(G_1 + u, \mathbb{T}_1) = 16u^4$ and $\mathrm{res}(G_1 + u + x_2, \mathbb{T}_1) = 16u^4 + 16u^2 + 4$, but $\mathrm{Eprem}(G_1 + u, \mathbb{T}_1) = 2u$ and $\mathrm{Eprem}(G_1 + u + x_2, \mathbb{T}_1) = 4 + 8u^2$. In fact, we are mainly concerned with the case $\mathrm{res}(P, \mathbb{T}) = 0$, because $\mathrm{Zero}(\{P\} \cup \mathbb{T}) \neq \emptyset$. We shall see how to split the strong regular set $\mathbb{T}$ with respect to P by dint of $\mathrm{Eprem}(P, \mathbb{T})$.

We are ready to present a new method for decomposing any algebraic variety, by which one can simply remove redundant components according to the result given in Chou and Gao (1990), Theorem 2.1.1 and Definition 2.2.1. Compared to other algorithms, ours does not involve any computation of Gröbner bases. In fact, some varieties in the output of our method lie in an $n + 1$-dimensional affine space and one needs to consider their projections only.

Algorithm UnmVarDecA. $\Phi \leftarrow \mathrm{UnmVarDecA}(\mathbb{P})$. Given a nonempty polynomial set $\mathbb{P} \subset \mathbf{K}[\mathbf{x}]$, this algorithm computes a strong regular series $\Phi = \{\mathbb{T}_1, \ldots, \mathbb{T}_e, \bar{\mathbb{T}}_1, \ldots, \bar{\mathbb{T}}_{\bar{e}}\}$

such that each $\mathbb{T}_i$ defines an unmixed algebraic variety in $\mathbf{A}^n_{\tilde{\mathbf{K}}}$, each $\bar{\mathbb{T}}_{\bar{i}}$ defines an unmixed algebraic variety in $\mathbf{A}^{n+1}_{\tilde{\mathbf{K}}}$, and the following decomposition holds:

$$\mathrm{Zero}(\mathbb{P}) = \left(\bigcup_{i=1}^{e} \mathrm{Zero}(\mathbb{T}_i)\right) \cup \left(\bigcup_{\bar{i}=1}^{\bar{e}} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\bar{\mathbb{T}}_{\bar{i}})\right),$$

where $\mathbb{T}_i \backslash \mathbf{K}[\mathbf{x}] = \emptyset$ and $\bar{\mathbb{T}}_{\bar{i}} \backslash \mathbf{K}[\mathbf{x}] \neq \emptyset$ for any $1 \leq i \leq e$, $1 \leq \bar{i} \leq \bar{e}$.

- U1. Compute $\Phi \leftarrow \mathrm{SReg}(\mathbb{P})$.
- U2. Let $\mathbb{T}$ be an element of Φ. If $|\mathbb{T}| > |\mathbb{P}|$ with $\mathbb{T} \subset \mathbf{K}[\mathbf{x}]$, or $|\mathbb{T}| - 1 > |\mathbb{P}|$ with $\mathbb{T}\backslash\mathbf{K}[\mathbf{x}] \neq \emptyset$, then set $\Phi \leftarrow \Phi\backslash\{\mathbb{T}\}$.
- U3. While $\exists\ \mathbb{T}, \mathbb{T}^* \in \Phi$ such that $\mathrm{prem}(\hat{T}^{d^*}, \mathbb{T}^*) = 0$ with $d^* = \mathrm{ldeg}(\mathbb{T}^*)$ for any $\hat{T} \in \mathbb{T}$ with $\deg(\hat{T}, t) = 0$, and one of the following conditions holds true, to: Set $\Phi \leftarrow \Phi\backslash\{\mathbb{T}^*\}$.

  - $\mathbb{T}, \mathbb{T}^* \subset \mathbf{K}[\mathbf{x}]$;
  - $\mathbb{T} \subset \mathbf{K}[\mathbf{x}]$ and $\mathbb{T}^*\backslash\mathbf{K}[\mathbf{x}] \neq \emptyset$;
  - $\mathbb{T} = \mathbb{T}_0 \cup [\mu t - 1]$, $\mathbb{T}^* \subset \mathbf{K}[\mathbf{x}]$ and $\mathbb{T}^*$ is strongly independent of $\mathrm{Eprem}(\mu, \mathbb{T}_0)$;
  - $\mathbb{T} = \mathbb{T}_0 \cup [\mu t - 1]$, $\mathbb{T}^* = \mathbb{T}^*_0 \cup [\mu^* t - 1]$ and there exists an integer $d_0 > 0$ such that $\mathrm{Eprem}(\mu^*, \mathbb{T}^*_0)$ divides $(\mathrm{Eprem}(\mu, \mathbb{T}_0))^{d_0}$.

Example 2.2.2. Continued from Example 2.1.1, one can see that $\mathrm{UnmVarDecA}(\mathbb{P}) = \{\mathbb{T}_2\}$, and $\mathrm{Zero}(\mathbb{P}) = \mathrm{Proj}_{\mathbf{x}}(\mathbb{T}_2)$, and $\mathbb{T}_1$ is removed because $|\mathbb{T}_1| - 1 > |\mathbb{P}|$.

Remark 2.2.2. If $\mathrm{Reg}(\mathbb{P})$ is computed by using the algorithm RegSer, then set $\Phi_0 = \mathrm{RegSer}(\mathbb{P}, \emptyset)$. Let μ be the product of all the polynomials in $\mathbb{U}$ with $[\mathbb{T}, \mathbb{U}] \in \Phi_0$ and set

$$\Phi^* \leftarrow \{\mathbb{T} \cup [\mu t - 1] : [\mathbb{T}, \mathbb{U}] \in \Phi_0\}.$$

After replacing $\mathrm{SReg}(\mathbb{P})$ with Φ^* in step U1, algorithm UnmVarDecA remains true.

3. Decomposing polynomial systems

We can compute, from any polynomial system $[\mathbb{P}, \mathbb{Q}]$ in $\mathbf{K}[\mathbf{x}]$, a strong regular series of $[\mathbb{P}, \mathbb{Q}]$ according to Proposition 2.1.1. But the proposed method has several disadvantages, because some components of $\mathbb{P}$ which are sometimes useful are removed. On the other hand, the efficiency of this method is rather low when $|\mathbb{Q}|$ is an appreciably large number. In this section, we present a complete algorithm to decompose any polynomial system $[\mathbb{P}, \mathbb{Q}]$ in $\mathbf{K}[\mathbf{x}]$ into two finite sets of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ which have some good properties using the following algorithm RSplit. In addition, a new method for decomposing any quasi-algebraic variety into *equidimensional* components and removing redundant components is given.

3.1. *Algorithm RSplit*

Given a polynomial set $\mathbb{P}_0$ in $\mathbf{K}[\mathbf{x}, t]$ and a polynomial P in $\mathbf{K}[\mathbf{x}]$, by the following algorithm RSplit, we can split $\mathbb{P}_0$ into two finite sets of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ with respect to P.

Theorem 3.1.1. *Let* $\mathbb{T}$ *be a strong regular set with* $\mathbb{T} \in \mathrm{SReg}(\mathbb{P}_0)$*, where* $\mathbb{P}_0$ *is a polynomial set in* $\mathbf{K}[\mathbf{x}, t]$*, and* P *a polynomial in* $\mathbf{K}[\mathbf{x}]\backslash\mathbf{K}[\mathbf{u}]$ *such that* $\mathrm{prem}(P, \mathbb{T}) \neq 0$*. If* $\mathrm{cls}(P^*, \mathbb{T}) = k$ $(1 \leq k \leq s)$ *with* $P^* = \mathrm{Eprem}(P, \mathbb{T})$ *and* $\mathbb{T}$ *is strongly independent of* $\mathrm{ini}(P^*)$*, then* $\mathbb{T}$ *can be split into two strong regular sets* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *in* $\mathbf{K}[\mathbf{x}, t]$ *with respect to* P *such that*

$$\mathrm{Zero}(\mathbb{T}) = \mathrm{Zero}(\mathbb{T}_1) \cup \mathrm{Zero}(\mathbb{T}_2).$$

Proof. Consider the case in which $\mathbb{T} = [f_1, \ldots, f_s, \mu_0 t - 1]$ with the above notation. It follows from Definition 2.2.1 that

$$I_0^{q_0} f_k = f_{k,1} f_{k,2} + R; \qquad \mathrm{prem}(R, [f_1, \ldots, f_{k-1}]) = 0, \tag{3}$$

where $f_{k,1} = P^*$, $I_0 = \mathrm{ini}(f_{k,1})$, $f_{k,2}$, $R \in \mathbf{K}[\mathbf{u}, y_1, \ldots, y_k]$ and q_0 is some non-negative integer. Two triangular sets $\mathbb{T}_1$ and $\mathbb{T}_2$ may be obtained by substituting f_k in $\mathbb{T}$ for $f_{k,1}$ and $f_{k,2}$ respectively.

For any $\bar{\mathbf{z}} = (\bar{\mathbf{u}}, \bar{y}_1, \ldots, \bar{y}_s, \bar{t}) \in \mathrm{Zero}(\mathbb{T})$, as $\mathbb{T}$ is a strong regular set, it is easy to see that $\bar{\mathbf{z}}^{\{k-1\}} = (\bar{\mathbf{u}}, \bar{y}_1, \ldots, \bar{y}_{k-1})$ is also a quasi-normal zero of $[f_1, \ldots, f_{k-1}]$ in $\mathbf{K}[\mathbf{u}, y_1, \ldots, y_{k-1}]$, which is a regular set. Thereby, we have $R(\bar{\mathbf{z}}) = 0$. It follows from (3) that $f_{k,1}(\bar{\mathbf{z}}) f_{k,2}(\bar{\mathbf{z}}) = 0$. Thus we have $\mathrm{Zero}(\mathbb{T}) \subseteq \mathrm{Zero}(\mathbb{T}_1) \cup \mathrm{Zero}(\mathbb{T}_2)$.

Now consider any $\mathbf{z}_0 \in \mathrm{Zero}(\mathbb{T}_1) \cup \mathrm{Zero}(\mathbb{T}_2)$. As $\mathbb{T} \in \mathrm{SReg}(\mathbb{P}_0)$, $\mathbf{z}_0^{\{k-1\}}$ is a quasi-normal zero of the regular set $[f_1, \ldots, f_{k-1}]$ by the construction of $\mathrm{SReg}(\mathbb{P}_0)$ in Theorem 2.1.3. This implies $R(\mathbf{z}_0) = 0$. Note that $\mathbb{T}$ is strongly independent of I_0, so $I_0(\mathbf{z}_0) = I_0(\mathbf{z}_0^{\{k-1\}}) \neq 0$. It follows from (3) that $f_k(\mathbf{z}_0) = 0$. Thus $\mathrm{Zero}(\mathbb{T}) \supseteq \mathrm{Zero}(\mathbb{T}_1) \cup \mathrm{Zero}(\mathbb{T}_2)$. Therefore

$$\mathrm{Zero}(\mathbb{T}) = \mathrm{Zero}(\mathbb{T}_1) \cup \mathrm{Zero}(\mathbb{T}_2).$$

We shall show that $\mathbb{T}_1$ and $\mathbb{T}_2$ are both regular sets. The fact that $\mathrm{res}(I_0, \mathbb{T}) \neq 0$ implies that $[f_1, \ldots, f_{k-1}, f_{k,1}]$ is a regular set. As $\mathrm{ini}(f_{k,1})\mathrm{ini}(f_{k,2}) = I_0^{q_0}\mathrm{ini}(f_k)$, one can easily see that $\mathrm{res}(\mathrm{ini}(f_{k,2}), \mathbb{T}^{\{k-1\}}) \neq 0$. Thus $[f_1, \ldots, f_{k-1}, f_{k,2}]$ is also a regular set. If $k = s$, then $\mathbb{T}_1$ and $\mathbb{T}_2$ are both regular sets. Now consider $k < s$: for any

$$\mathbf{z}^{\{k\}} = (\mathbf{u}, \eta_1, \ldots, \eta_k) \in \bigcup_{i=1}^{2} \mathrm{RegZero}([f_1, \ldots, f_{k-1}, f_{k,i}]),$$

it is easy to see that $R(\mathbf{z}^{\{k\}}, y_k) = R(\mathbf{z}^{\{k-1\}}, y_k) \equiv 0$, and $I_0(\mathbf{z}^{\{k\}}) = I_0(\mathbf{z}^{\{k-1\}}) \neq 0$ by Proposition 5.1.4 given in Wang (2001). Hence

$$\mathrm{RegZero}([f_1, \ldots, f_{k-1}, f_k]) \supseteq \bigcup_{i=1}^{2} \mathrm{RegZero}([f_1, \ldots, f_{k-1}, f_{k,i}]).$$

We proceed to show that $[f_1, \ldots, f_{k-1}, f_{k,i}, f_{k+1}]$ is a regular set for each $i = 1, 2$. Suppose that

$$\mathrm{res}(\mathrm{ini}(f_{k+1}), [f_1, \ldots, f_{k-1}, f_{k,i}, f_{k+1}]) = \mathrm{res}(\mathrm{ini}(f_{k+1}), [f_1, \ldots, f_{k-1}, f_{k,i}]) = 0$$

for some i. By Proposition 5.1.5 given in Wang (2001), $\text{ini}(f_{k+1})(\mathbf{z}^{\{k\}}) = 0$ for any

$$\mathbf{z}^{\{k\}} \in \text{RegZero}([f_1, \ldots, f_{k-1}, f_{k,i}]).$$

This contradicts the fact that $[f_1, \ldots, f_{k-1}, f_k, f_{k+1}]$ is a regular set. Thus,

$$[f_1, \ldots, f_{k-1}, f_{k,i}, f_{k+1}]$$

is a regular set for each $i = 1, 2$. By induction, one can see that $\mathbb{T}_1$ and $\mathbb{T}_2$ are both regular sets.

Finally, we are ready to show that $\mathbb{T}_1$ and $\mathbb{T}_2$ are both strong regular sets in $\mathbf{K}[\mathbf{x}, t]$. It is easy to see that $\mathbb{T}_1$ is a strong regular set. We shall prove that $\mathbb{T}_2$ is also a strong regular set. For any $\mathbf{z} \in \text{Zero}(\mathbb{T}_2)$, it follows from the above result that $\mathbf{z}$ is also a quasi-normal zero of $\mathbb{T}$. Suppose that $\mathbf{z}$ is not a quasi-normal zero of $\mathbb{T}_2$; it induces that $\mathbf{z}^{\{k-1\}} \in \text{Zero}(\mathbb{C}_{f_{k,2}})$. Plunging $\mathbf{z}^{\{k-1\}}$ into (3), we get

$$I_0^{q_0}(\mathbf{z}^{\{k-1\}})\, f_k(\mathbf{z}^{\{k-1\}}, y_k) = f_{k,1}(\mathbf{z}^{\{k-1\}}, y_k)\, f_{k,2}(\mathbf{z}^{\{k-1\}}, y_k) + R(\mathbf{z}^{\{k-1\}}, y_k).$$

Since $I_0^{q_0}(\mathbf{z}^{\{k-1\}}) \neq 0$, we have $f_k(\mathbf{z}^{\{k-1\}}, y_k) \equiv 0$, namely, $\mathbf{z}$ is not a quasi-normal zero of $\mathbb{T}$. This is impossible; hence $\mathbb{T}_2$ is a strong regular set.

The case in which $\mathbb{T} = [f_1, \ldots, f_s]$ may be proved similarly, and we omit the details. The proof is complete. □

Example 3.1.1 (Continued from Examples 2.1.1 and 2.2.1). Since

$$\text{ini}(\text{Eprem}(G_2, \mathbb{T}_1)) = 2x_2 \qquad \text{and} \qquad \text{res}(2x_2, \mathbb{T}_1) = 16,$$

by Theorem 3.1.1, $\mathbb{T}_1$ can be split up into two strong regular sets $\mathbb{T}_{1,1}$ and $\mathbb{T}_{1,2}$ in $\mathbf{K}[\mathbf{x}, t]$ with respect to G_2, where

$$\mathbb{T}_{1,1} = [x_1, 2x_2^2 + 1, x_3, 2x_2x_4 - 1 - 2x_2];$$
$$\mathbb{T}_{1,2} = [x_1, 2x_2^2 + 1, x_3, x_2x_4 + x_2 + 1].$$

Thus, $\text{Zero}(\mathbb{T}_1) = \text{Zero}(\mathbb{T}_{1,1}) \cup \text{Zero}(\mathbb{T}_{1,2})$.

Theorem 3.1.2. *Let $\mathbb{T}$ be a strong regular set in $\mathbf{K}[\mathbf{x}, t]$ and P a polynomial in $\mathbf{K}[\mathbf{x}]$ such that* $\text{prem}(P, \mathbb{T}) \neq 0$. *If $P^* \in \mathbf{K}[\mathbf{u}]\backslash\mathbf{K}$ or* $\text{Zero}(\mathbb{T} \cup \{\text{ini}(P^*)\}) \neq \emptyset$ *with* $P^* = \text{Eprem}(P, \mathbb{T})$, *then $\mathbb{T}$ can be further split up into a set $\Psi_{\mathbb{T}}$ (with $|\Psi_{\mathbb{T}}| > 1$) of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ with respect to P such that*

$$\text{Proj}_{\mathbf{x}}\text{Zero}(\mathbb{T}) = \bigcup_{\mathbb{T}^* \in \Psi_{\mathbb{T}}} \text{Proj}_{\mathbf{x}}\text{Zero}(\mathbb{T}^*).$$

Proof. We first consider the case with $P^* \in \mathbf{K}[\mathbf{u}]\backslash\mathbf{K}$, or $\text{ini}(P^*) \in \mathbf{K}[\mathbf{u}]\backslash\mathbf{K}$ and $\text{cls}(P^*, \mathbb{T}) = k$ $(1 \le k \le s)$.

Set $c_0 = P^*$ if $P^* \in \mathbf{K}[\mathbf{u}]\backslash\mathbf{K}$, or $c_0 = \text{ini}(P^*)$ otherwise. Put

$$\mathbb{T}_1 = \begin{cases} [f_1, \ldots, f_s, c_0 t - 1] & \text{if } \mathbb{T} = [f_1, \ldots, f_s], \\ [f_1, \ldots, f_s, c_0\mu_0 t - 1] & \text{if } \mathbb{T} = [f_1, \ldots, f_s, \mu_0 t - 1]; \end{cases}$$

and

$$\mathbb{T}_2 = \begin{cases} \{c_0, f_1, \ldots, f_s\} & \text{if } \mathbb{T} = [f_1, \ldots, f_s], \\ \{c_0, f_1, \ldots, f_s, \mu_0 t - 1\} & \text{if } \mathbb{T} = [f_1, \ldots, f_s, \mu_0 t - 1]. \end{cases}$$

$\mathbb{T}_1$ is obviously a strong regular set in $\mathbf{K}[\mathbf{x}, t]$, but $\mathbb{T}_2$ may not be a triangular set. By Theorem 2.1.3, one can compute $\mathrm{SReg}(\mathbb{T}_2)$ such that

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_2) = \bigcup_{\mathbb{T}^* \in \mathrm{SReg}(\mathbb{T}_2)} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}^*).$$

Set

$$\Psi_{\mathbb{T}} \leftarrow \{\mathbb{T}_1\} \cup \mathrm{SReg}(\mathbb{T}_2).$$

This implies that

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) = \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_1) \cup \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_2) = \bigcup_{\mathbb{T}^* \in \Psi_{\mathbb{T}}} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}^*).$$

Now, we consider the case with $\mathrm{ini}(P^*) \notin \mathbf{K}[\mathbf{u}]\backslash\mathbf{K}$ and $\mathrm{cls}(P^*, \mathbb{T}) = k$ $(1 \le k \le s)$.

Set $c_0 = \mathrm{ini}(P^*)$. $\mathbb{T}_1$ and $\mathbb{T}_2$ are similarly defined as above. Note that $\mathbb{T}_1$ may not be a regular set at all. By Theorem 2.1.3, one can compute $\mathrm{SReg}(\mathbb{T}_1)$ and $\mathrm{SReg}(\mathbb{T}_2)$ respectively, such that

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_i) = \bigcup_{\mathbb{T}^* \in \mathrm{SReg}(\mathbb{T}_i)} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}^*)$$

for $i = 1, 2$. Set

$$\Psi_{\mathbb{T}} \leftarrow \mathrm{SReg}(\mathbb{T}_1) \cup \mathrm{SReg}(\mathbb{T}_2).$$

This implies that

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) = \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_1) \cup \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_2) = \bigcup_{\mathbb{T}^* \in \Psi_{\mathbb{T}}} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}^*).$$

The proof is complete. □

Example 3.1.2 (Continued from Examples 2.1.1 and 2.2.1). Since

$$\mathrm{res}(\mathrm{ini}(\mathrm{Eprem}(G_2, \mathbb{T}_2)), \mathbb{T}_2) = (x_1 + 1)^2(4x_1 + 4 - 2x_1^2 + 2x_1^3),$$

by Theorem 3.1.2, $\mathbb{T}_2$ can be split up into a set $\Psi_{\mathbb{T}_2} = \{\mathbb{T}_{2,1}\} \cup \mathrm{SReg}(\mathbb{T}^*) = \{\mathbb{T}_{2,1}, \mathbb{T}_{2,2}, \mathbb{T}_{2,3}\}$ of strong regular sets in $\mathbf{K}[x_1, \ldots, x_4, t]$ with respect to G_2 such that

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_2) = \bigcup_{i=1}^{3} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}_{2,i}),$$

where

$$\mathbb{T}_{2,1} = [(2x_1 + 2)x_2^2 - 2x_1x_2 + x_1 + 1, x_3^2 - x_1x_2x_3 + x_1 + 2x_1x_2^2 + x_1^2x_2,$$
$$x_1x_4 + x_3 - x_1x_2, x_1(2x_1x_2 - x_1^2 + 2x_2 - x_1)t - 1],$$

$\mathbb{T}^*$, $\mathbb{T}_{2,2}$ and $\mathbb{T}_{2,3}$ have already been given in Example 2.1.1.

Let $\mathbb{P}_0$ be a polynomial set in $\mathbf{K}[\mathbf{x}, t]$ and P a polynomial in $\mathbf{K}[\mathbf{x}]$. By the following algorithm, one can decompose $\mathbb{P}_0$ into two finite sets Φ_1 and Φ_2 of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ such that

$$\mathrm{Zero}(\{P\}) \supseteq \bigcup_{\mathbb{T}\in\Phi_1} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T});$$

$$\mathrm{Zero}(\{P\}) \cap \bigcup_{\mathbb{T}\in\Phi_2} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) = \emptyset.$$

The basic operation in RSplit is to split some strong regular sets with respect to P recursively by Theorem 3.1.1 in most cases or by Theorem 3.1.2.

Algorithm RSplit ($[\Phi_1, \Phi_2] \leftarrow \mathrm{RSplit}(\mathbb{P}_0, P)$). Given a polynomial set $\mathbb{P}_0$ in the above-mentioned class in $\mathbf{K}[\mathbf{x}, t]$ and a polynomial P in $\mathbf{K}[\mathbf{x}]$, this algorithm computes two finite sets Φ_1 and Φ_2 of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ such that

$$\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{P}_0) = \bigcup_{\mathbb{T}\in\Phi_1\cup\Phi_2} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}),$$

and there exists an integer $d > 0$ such that $\mathrm{prem}(P^d, \mathbb{T}) = 0$ for any $\mathbb{T} \in \Phi_1$ and $\mathbb{T}$ is strongly independent of P for any $\mathbb{T} \in \Phi_2$.

SP1. Compute $\mathrm{SReg}(\mathbb{P}_0)$ according to Theorem 2.1.3 and set $\Phi_1 \leftarrow \emptyset$, $\Phi_2 \leftarrow \emptyset$ and $\Psi \leftarrow \mathrm{SReg}(\mathbb{P}_0)$.
SP2. While $\Psi \neq \emptyset$, do:
SP2.1. Let $\mathbb{T}$ be an element of Ψ, and set $\Psi \leftarrow \Psi\setminus\{\mathbb{T}\}$. If there exists an integer $0 < d_0 \leq \mathrm{ldeg}(\mathbb{T})$ such that $\mathrm{prem}(P^{d_0}, \mathbb{T}) = 0$, then set $\Phi_1 \leftarrow \Phi_1 \cup \{\mathbb{T}\}$, and go to SP2.
SP2.2. If $\mathbb{T}$ is strongly independent of P, then set $\Phi_2 \leftarrow \Phi_2 \cup \{\mathbb{T}\}$ and go to SP2.
SP2.3. If the condition of Theorem 3.1.1 holds, then $\mathbb{T}$ can be split up into two strong regular sets $\mathbb{T}_1$ and $\mathbb{T}_2$ with respect to P by Theorem 3.1.1. Set $\Psi \leftarrow \Psi \cup \{\mathbb{T}_1, \mathbb{T}_2\}$ and go to SP2.
SP2.4. Now the condition of Theorem 3.1.1 does not hold, so $\mathbb{T}$ can be split up into a set $\Psi_{\mathbb{T}}$ of strong regular sets with respect to P by Theorem 3.1.2. Set $\Psi \leftarrow \Psi \cup \Psi_{\mathbb{T}}$.

Remark 3.1.1. In step SP2.2, one can determine whether $\mathbb{T}$ is strongly independent of P by computing $\mathrm{Eprem}(P, \mathbb{T})$. It is easy to see that $\mathbb{T}$ is strongly independent of P if $\mathrm{Eprem}(P, \mathbb{T}) \in \mathbf{K}\setminus\{0\}$ when $\mathbb{T} \subset \mathbf{K}[\mathbf{x}]$, or there exists an integer $d^* > 0$ such that $\mathrm{Eprem}(P, \mathbb{T})$ divides $(\mathrm{Eprem}(\mu_0, \mathbb{T}_0))^{d^*}$ when $\mathbb{T} = \mathbb{T}_0 \cup [\mu_0 t - 1]$ in $\mathbf{K}[\mathbf{x}, t]$.

Example 3.1.3 (Continued from Examples 2.1.1, 3.1.1 and 3.1.2). By the above algorithm, one can compute

$$\mathrm{RSplit}(\mathbb{T}_i, G_1) = [\{\mathbb{T}_i\}, \emptyset] \quad \text{for } i = 1, 2;$$

$$\mathrm{RSplit}(\mathbb{T}_1, G_2) = [\{\mathbb{T}_{1,1}\}, \{\mathbb{T}_{1,2}\}];$$

$$\mathrm{RSplit}(\mathbb{T}_{2,1}, G_2) = [\{\bar{\mathbb{T}}_{2,2}, \bar{\mathbb{T}}_{2,4}, \bar{\mathbb{T}}_{2,6}\}, \{\bar{\mathbb{T}}_{2,1}, \bar{\mathbb{T}}_{2,3}, \bar{\mathbb{T}}_{2,5}\}];$$

$$\mathrm{RSplit}(\mathbb{T}_{2,2}, G_2) = [\{\mathbb{T}_{2,2}\}, \emptyset];$$

$$\mathrm{RSplit}(\mathbb{T}_{2,3}, G_2) = [\{\mathbb{T}_{2,4}\}, \{\mathbb{T}_{2,5}\}],$$

where

$$\bar{\mathbb{T}}_{2,1} = [\mathbb{T}_{2,1}[1], -(x_1+1)(x_1-2x_2)x_3 - 2x_1^2x_2^2 - x_1^3x_2 - 2x_1x_2^2 + x_1^2x_2 - x_1^2 - 2x_1x_2 - 2x_1 + x_1^3, \mathbb{T}_{2,1}[3], (2x_1x_2 - x_1^2 + 2x_2 - x_1)(5x_1+17)(x_1+1)(x_1^3 - x_1^2 + 2x_1 + 2)x_1t - 1];$$

$$\bar{\mathbb{T}}_{2,2} = [\mathbb{T}_{2,1}[1], -(x_1+1)(x_1-2x_2)x_3 + x_1^2 + 2x_1x_2 + 2x_1 - x_1^3 + 2x_1^3x_2, \mathbb{T}_{2,1}[3], \mathbb{T}_{2,1}[4]];$$

$$\bar{\mathbb{T}}_{2,3} = [5x_1+17, 4x_2-3, (300x_2+510)x_3 + 2244 - 2159x_2, (30x_2+51)x_4 + 81 - 157x_2, (2040x_2+3468)t - 125];$$

$$\bar{\mathbb{T}}_{2,4} = [5x_1+17, 3x_2-2, (300x_2+510)x_3 + 2244 - 2159x_2, (30x_2+51)x_4 + 81 - 157x_2, (2040x_2+3468)t - 125];$$

$$\bar{\mathbb{T}}_{2,5} = [x_1^3 - x_1^2 + 2x_1 + 2, (x_1+1)((4x_1^2 - 2x_1 - 6)x_2 - 3x_1^2 - x_1 - 4), (2x_1x_2 - x_1^2 + 2x_2 - x_1)x_3 - x_1^3x_2 + x_1^3 - 2x_1^2x_2^2 + x_1^2x_2 - x_1^2 - 2x_1x_2^2 - 2x_1x_2 - 2x_1, \mathbb{T}_{2,1}[3], \mathbb{T}_{2,1}[4]];$$

$$\bar{\mathbb{T}}_{2,6} = [\bar{\mathbb{T}}_{2,5}[1], 2(x_1+1)(2x_1-3)x_2 - x_1^2 + 4 + 7x_1, \bar{\mathbb{T}}_{2,5}[3], \mathbb{T}_{2,1}[3], \mathbb{T}_{2,1}[4]];$$

$$\mathbb{T}_{2,4} = [x_1+1, x_2, x_3-1, x_4-x_3, t+1];$$

$$\mathbb{T}_{2,5} = [x_1+1, x_2, x_3+1, x_4-x_3, t+1].$$

Furthermore, one can get

$$\mathrm{RSplit}(\mathbb{T}_2, G_2) = [\{\mathbb{T}_{2,2}, \mathbb{T}_{2,4}, \bar{\mathbb{T}}_{2,2}, \bar{\mathbb{T}}_{2,4}, \bar{\mathbb{T}}_{2,6}\}, \{\bar{\mathbb{T}}_{2,1}, \bar{\mathbb{T}}_{2,3}, \bar{\mathbb{T}}_{2,5}, \mathbb{T}_{2,5}\}];$$
$$\mathrm{RSplit}(\mathbb{P}, G_1) = [\{\mathbb{T}_1, \mathbb{T}_2\}, \emptyset];$$
$$\mathrm{RSplit}(\mathbb{P}, G_2) = [\{\mathbb{T}_{1,1}, \mathbb{T}_{2,2}, \mathbb{T}_{2,4}, \bar{\mathbb{T}}_{2,2}, \bar{\mathbb{T}}_{2,4}, \bar{\mathbb{T}}_{2,6}\}, \{\mathbb{T}_{1,2}, \bar{\mathbb{T}}_{2,1}, \bar{\mathbb{T}}_{2,3}, \bar{\mathbb{T}}_{2,5}, \mathbb{T}_{2,5}\}].$$

Remark 3.1.2. Algorithm RSplit is better than its previous version presented in Li et al. (2002). Using the previous one, we only get

$$\mathrm{Zero}(\mathbb{T}_0) \supseteq \bigcup_{\mathbb{T}\in\Phi_1\cup\Phi_2} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T})$$

with $\mathrm{RSplit}(\mathbb{T}_0, P) = [\Phi_1, \Phi_2]$, where $\mathbb{T}_0$ must be a strong regular set in $\mathbf{K}[\mathbf{x}, t]$.

3.2. Algorithm SRD

For any polynomial system $[\mathbb{P}, \mathbb{Q}]$ in $\mathbf{K}[\mathbf{x}]$, by the following algorithm SRD, one can compute two finite sets of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ which have some good properties. The main thought of algorithm SRD is to collect every strong regular set $\mathbb{T}$ which satisfies $\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) \subseteq \mathrm{Zero}(\mathbb{P})$ and $\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) \cap \mathrm{Zero}(\{Q\}) = \emptyset$ for any $Q \in \mathbb{Q}$, and every strong regular set $\mathbb{T}$ which satisfies $\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) \subseteq \mathrm{Zero}(\mathbb{P})$ and $\mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T}) \subseteq \mathrm{Zero}(\{Q\})$ for some $Q \in \mathbb{Q}$ respectively, through recursive use of algorithm RSplit.

Algorithm SRD. $[\Upsilon_1, \Upsilon_2] \leftarrow \mathrm{SRD}(\mathbb{P}, \mathbb{Q})$. Given a polynomial system $[\mathbb{P}, \mathbb{Q}]$ in $\mathbf{K}[\mathbf{x}]$, this algorithm computes two finite sets Υ_1 and Υ_2 of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ such that

(1) $\Upsilon_1 \cup \Upsilon_2$ is a strong regular series of the polynomial set $\mathbb{P}$, namely,

$$\mathrm{Zero}(\mathbb{P}) = \bigcup_{\mathbb{T}\in \Upsilon_1\cup\Upsilon_2} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T});$$

(2) Υ_1 is a strong regular series of the polynomial system $[\mathbb{P}, \mathbb{Q}]$, namely,

$$\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{\mathbb{T}\in \Upsilon_1} \mathrm{Proj}_{\mathbf{x}}\mathrm{Zero}(\mathbb{T});$$

(3) for any $\mathbb{T} \in \Upsilon_1$, $P \in \mathbb{P}$ and $Q \in \mathbb{Q}$, there exists an integer $d > 0$ such that $\mathrm{prem}(P^d, \mathbb{T}) = 0$ and $\mathbb{T}$ is strongly independent of Q;

(4) for any $\mathbb{T} \in \Upsilon_2$, $P \in \mathbb{P}$, there exist an integer $\bar{d} > 0$ and some $Q_0 \in \mathbb{Q}$ such that $\mathrm{prem}(P^{\bar{d}}, \mathbb{T}) = 0$ and $\mathrm{prem}(Q_0^{\bar{d}}, \mathbb{T}) = 0$.

D1. Compute $\mathrm{SReg}(\mathbb{P})$ according to Theorem 2.1.3 and set $\Upsilon_1 \leftarrow \emptyset$, $\Upsilon_2 \leftarrow \emptyset$, $\Psi \leftarrow \mathrm{SReg}(\mathbb{P})$.

D2. While $\Psi \neq \emptyset$, do:

D2.1. Let $\mathbb{T}$ be an element of Ψ, and set $\Psi \leftarrow \Psi\backslash\{\mathbb{T}\}$.

D2.2. For $Q \in \mathbb{Q}$ do:

D2.1.1. If there exists an integer $0 < d \leq \mathrm{ldeg}(\mathbb{T})$ such that $\mathrm{prem}(Q^d, \mathbb{T}) = 0$, then set $\Upsilon_2 \leftarrow \Upsilon_2 \cup \{\mathbb{T}\}$ and go to D2.

D2.1.2. If $\mathbb{T}$ is not strongly independent of Q, then compute $[\Phi_1, \Phi_2] \leftarrow \mathrm{RSplit}(\mathbb{T}, Q)$, and set $\Upsilon_2 \leftarrow \Upsilon_2 \cup \Phi_1$, $\Psi \leftarrow \Psi \cup \Phi_2$.

D2.3. Set $\Upsilon_1 \leftarrow \Upsilon_1 \cup \{\mathbb{T}\}$.

Example 3.2.1 (Continued from Examples 2.1.1 and 3.1.3). By the above algorithm SRD, one can compute

$$\begin{aligned}
&\mathrm{SRD}(\mathbb{P}, \emptyset) = [\{\mathbb{T}_1, \mathbb{T}_2\}, \emptyset];\\
&\mathrm{SRD}(\mathbb{P}, \{G_1\}) = [\{\mathbb{T}_1, \mathbb{T}_2\}, \emptyset];\\
&\mathrm{SRD}(\mathbb{T}_1, \{G_2\}) = [\{\mathbb{T}_{1,1}\}, \{\mathbb{T}_{1,2}\}];\\
&\mathrm{SRD}(\mathbb{T}_2, \{G_2\}) = [\{\mathbb{T}_{2,2}, \mathbb{T}_{2,4}, \bar{\mathbb{T}}_{2,2}, \bar{\mathbb{T}}_{2,4}, \bar{\mathbb{T}}_{2,6}\}, \{\bar{\mathbb{T}}_{2,1}, \bar{\mathbb{T}}_{2,3}, \bar{\mathbb{T}}_{2,5}, \mathbb{T}_{2,5}\}];\\
&\mathrm{SRD}(\mathbb{P}, \{G_2\}) = \mathrm{SRD}(\mathbb{P}, \{G_1, G_2\}) = [\{\mathbb{T}_{1,1}, \mathbb{T}_{2,2}, \mathbb{T}_{2,4}, \bar{\mathbb{T}}_{2,2}, \bar{\mathbb{T}}_{2,4}, \bar{\mathbb{T}}_{2,6}\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \{\mathbb{T}_{1,2}, \bar{\mathbb{T}}_{2,1}, \bar{\mathbb{T}}_{2,3}, \bar{\mathbb{T}}_{2,5}, \mathbb{T}_{2,5}\}].
\end{aligned}$$

Remark 3.2.1. It is easy to see that $\mathrm{SRD}(\mathbb{P}, \emptyset) = [\mathrm{SReg}(\mathbb{P}), \emptyset]$. Algorithm SRD can be implemented by using existing mathematical software. As the operation Eprem and algorithm RSplit are frequently used, the efficiency of algorithm SRD may be influenced in some cases even though its output is better than that of other algorithms. On the other hand, it also depends upon the efficiency of algorithm Reg. All of the examples in this section are computed using the Maple system in an interactive way. In this paper, we focus our attention mainly on the development of theory and algorithms.

3.3. Unmixed decomposition for polynomial systems

$\mathcal{V}$ is called a *quasi-algebraic variety* if there is a polynomial system $[\mathbb{P}, \mathbb{Q}]$ such that $\mathcal{V} = \text{Zero}(\mathbb{P}/\mathbb{Q})$. We present an algorithm similar to UnmVarDecA for decomposing any quasi-algebraic variety into unmixed components and removing redundant components.

Algorithm UnmVarDecB. $[\Upsilon_1, \Upsilon_2] \leftarrow \text{UnmVarDecB}(\mathbb{P}, \mathbb{Q})$. Given a polynomial system $[\mathbb{P}, \mathbb{Q}]$ with $\mathbb{Q} \neq \emptyset$ in $\mathbf{K}[\mathbf{x}]$, this algorithm computes two finite sets Υ_1 and Υ_2 of strong regular sets in $\mathbf{K}[\mathbf{x}, t]$ such that

(1)
$$\text{Zero}(\mathbb{P}) = \bigcup_{\mathbb{T}\in \Upsilon_1 \cup \Upsilon_2} \text{Proj}_{\mathbf{x}}\text{Zero}(\mathbb{T});$$

(2)
$$\text{Zero}(\mathbb{P}/\mathbb{Q}) = \bigcup_{\mathbb{T}\in \Upsilon_1} \text{Proj}_{\mathbf{x}}\text{Zero}(\mathbb{T});$$

(3) for any $\mathbb{T} \in \Upsilon_1$ and $Q \in \mathbb{Q}$, we have
$$\text{Proj}_{\mathbf{x}}\text{Zero}(\mathbb{T}) \cap \text{Zero}(\{Q\}) = \emptyset;$$

(4) for any $\mathbb{T} \in \Upsilon_2$, there exists some $Q_0 \in \mathbb{Q}$ such that
$$\text{Proj}_{\mathbf{x}}\text{Zero}(\mathbb{T}) \subseteq \text{Zero}(\{Q_0\});$$

(5) each $\mathbb{T} \in \Upsilon_1 \cup \Upsilon_2$ defines an unmixed algebraic variety in $\mathbf{A}^n_{\tilde{\mathbf{K}}}$ or $\mathbf{A}^{n+1}_{\tilde{\mathbf{K}}}$.

U1. Compute $[\Upsilon_1, \Upsilon_2] \leftarrow \text{SRD}(\mathbb{P}, \mathbb{Q})$.
U2. Apply step U2 of UnmVarDecA to Υ_1 and Υ_2 respectively.
U3. Apply step U3 of UnmVarDecA to Υ_1 and Υ_2 respectively.

Example 3.3.1. Continued from Example 3.2.1, one can see that

$$\text{UnmVarDecB}(\mathbb{P}, \{G_1, G_2\}) = [\{\bar{\mathbb{T}}_{2,2}\}, \{\bar{\mathbb{T}}_{2,1}\}].$$

We have

$$\text{Zero}(\mathbb{P}) = \text{Proj}_{\mathbf{x}}\text{Zero}(\bar{\mathbb{T}}_{2,2}) \cup \text{Proj}_{\mathbf{x}}\text{Zero}(\bar{\mathbb{T}}_{2,1});$$
$$\text{Zero}(\mathbb{P}/\{G_1, G_2\}) = \text{Proj}_{\mathbf{x}}\text{Zero}(\bar{\mathbb{T}}_{2,2}).$$

At the same time,

$$\text{Proj}_{\mathbf{x}}\text{Zero}(\bar{\mathbb{T}}_{2,2}) \cap \text{Zero}(\{Q_1\}) = \text{Proj}_{\mathbf{x}}\text{Zero}(\bar{\mathbb{T}}_{2,2}) \cap \text{Zero}(\{Q_2\}) = \emptyset$$

and $\text{Proj}_{\mathbf{x}}\text{Zero}(\bar{\mathbb{T}}_{2,1}) \subseteq \text{Zero}(\{Q_2\})$.

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