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Fundamental solutions, transition densities and the integration of Lie symmetries

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A B S T R A C T

In this paper we present some new applications of Lie symmetry analysis to problems in stochastic calculus. The major focus is on using Lie symmetries of parabolic PDEs to obtain fundamental solutions and transition densities. The method we use relies upon the fact that Lie symmetries can be integrated with respect to the group parameter. We obtain new results which show that for PDEs with nontrivial Lie symmetry algebras, the Lie symmetries naturally yield Fourier and Laplace transforms of fundamental solutions, and we derive explicit formulas for such transforms in terms of the coefficients of the PDE.

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1. Introduction

A Lie group symmetry of a differential equation is a transformation which maps solutions to solutions. Lie symmetries allow complex solutions to be constructed from trivial solutions. Indeed crucially, with a Lie group symmetry, we obtain a continuous family of solutions, parameterized by the group variable. The books [17] and [1] contain both an introduction to Lie symmetry groups and numerous applications.

The current paper is a part of a program which explores the connections between Lie symmetry analysis and harmonic analysis. Here we will illustrate the links between the two areas by focusing on the application of Lie symmetries to the construction of fundamental solutions.

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The current author and his collaborators have developed the methodology used here in several publications, such as [7–9]. The idea is to produce an integral transform of a fundamental solution by applying a single Lie symmetry to a trivial solution. For this to be useful, the transform must have a known inversion integral. Ideally, the transform should be one of the most well studied, such as the Fourier or Laplace transform, since extensive tables for these transforms exist and there is a rich theory which we can exploit to obtain further information about the fundamental solution, such as its asymptotic behaviour.

In [7], Craddock and Lennox studied equations of the form

$$u_t = \sigma x^{\gamma} u_{xx} + f(x)u_x - \mu x^r u, \tag{1.1}$$

where $\sigma > 0$, μ, γ and r are constants. They proved that it is always possible to obtain integral transforms of fundamental solutions of (1.1) by symmetry, when the drift f satisfies any one of a family of Riccati equations. However, Laplace transforms could only be obtained for certain subclasses of the problems they studied. For the remaining classes, integral transforms involving Whittaker functions were needed. Although these so-called Whittaker transforms have known inversion integrals, explicit inversion is usually not possible as few Whittaker transforms have been calculated.

In this paper we significantly improve on previous results. We consider a class of equations of the form $u_t = \sigma x^{\gamma} u_{xx} + f(x)u_x - g(x)u$. Suppose that g is fixed and given and $\gamma \neq 2$. We show that if $h(x) = x^{1-\gamma} f(x)$ is a solution of any one of three families of Riccati equations, then by using the full group of symmetries, it is *always possible* to obtain a generalized Laplace transform or Fourier transform of fundamental solutions of the PDE, purely by symmetry. Our method has natural advantages over many other techniques and we will show how it can actually be used to rectify a shortcoming of the method of reduction to canonical form. We will apply the theory to a number of examples and problems in stochastic analysis.

The fact that we may compute Laplace and Fourier transforms, purely through a Lie algebra calculation, suggests a close relationship between Lie symmetry analysis and harmonic analysis. The symmetry group itself gives us these transforms absolutely explicitly as functions of the coefficients of the derivatives in the PDE. This is a consequence of the relationship between the representations of the underlying Lie group and the symmetry transformations, which was first developed in [3] and [4]. We believe that the connections between Lie symmetry analysis and harmonic analysis needs to be explored further. It is hoped that this work will stimulate further investigations in this direction.

1.0.1. Integrating symmetries

Suppose that we have a linear PDE

$$P\big(x, D^{\alpha}\big)u = \sum_{|\alpha| \leqslant n} a_{\alpha}(x) D^{\alpha} u, \quad x \in \Omega \subseteq \mathbb{R}^m, \tag{1.2}$$

with $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_i \in \mathbb{N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}$.

Let G be a one parameter group of symmetries, generated by some vector field $\mathbf{v} = \sum_{i=1}^m \xi(x)\partial_{x_i} + \phi(x,u)\partial_u$, where $\partial_x = \frac{\partial}{\partial x}$, etc. Denote the action of G on a solution u by $\sigma(\exp \epsilon \mathbf{v})u(x) = U_{\epsilon}(x)$. By the Lie symmetry property, there is an interval $I \subseteq \mathbb{R}$ containing zero such that $U_{\epsilon}(x)$ is a *continuous* one parameter family of solutions of (1.2), for all $\epsilon \in I$. Continuity and linearity imply the following.

Lemma 1.1. *Suppose that $U_{\epsilon}(x)$ is a continuous one parameter family of solutions of the PDE* (1.2)*, which holds for $\epsilon \in I \subseteq \mathbb{R}$. Suppose further that $\varphi : I \to \mathbb{R}$ is a function with sufficiently rapid decay. Then*

$$u(x) = \int_I \varphi(\epsilon) U_{\epsilon}(x)\, d\epsilon \tag{1.3}$$

is a solution of the PDE (1.2)*. Further, if the PDE is time dependent and $U_{\epsilon}(x,t)$ is the family of symmetry so-*

lutions, then $u(x,t) = \int_I \varphi(\epsilon) U_\epsilon(x,t)\, d\epsilon$ *and* $u(x,0) = \int_I \varphi(\epsilon) U_\epsilon(x,0)\, d\epsilon$. *Further,* $\frac{d^n U_\epsilon(x)}{d\epsilon^n}$ *is also a solution for all* $n = 0, 1, 2, 3, \ldots$.

Proof. To prove that u is a solution, we simply differentiate under the integral sign. If I is unbounded, we may take φ to have compact support to achieve convergence of the integral. The final claim follows from the fact that the PDE does not depend on ϵ, so the order of differentiation may be reversed. □

This simple result lies at the heart of the results we obtain here.

1.1. Fundamental solutions as transition densities

A fundamental solution for the Cauchy problem

$$\begin{aligned} u_t &= a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u, \quad x \in \Omega \subseteq \mathbb{R}, \\ u(x,0) &= \phi(x) \end{aligned} \tag{1.4}$$

is a kernel $p(x, y, t)$ with the property that: (i) for each fixed y, $p(x, y, t)$ is a solution of (1.4) on $\Omega \times (0, T]$ for some $T > 0$; and (ii) $u(x,t) = \int_\Omega \phi(y) p(x, y, t)\, dy$ is a solution of the given Cauchy problem for appropriate initial data ϕ. In general a fundamental solution will be a distribution in the sense of Schwartz. For an introduction to the theory of fundamental solutions for parabolic problems, see chapter one of the book by Friedman [11].

Fundamental solutions play an important role in probability theory. Consider an Itô diffusion $X = \{X_t\colon t \geqslant 0\}$ which satisfies the stochastic differential equation (SDE)

$$dX_t = b(X_t, t)\, dt + \sigma(X_t, t)\, dW_t, \qquad X_0 = x, \tag{1.5}$$

in which $W = \{W_t\colon t \geqslant 0\}$ is a standard Wiener process. The existence and uniqueness of solutions of (1.5) depends on the coefficient functions b, σ. See [13] for conditions guaranteeing a unique strong solution to (1.5). Assume that b and σ are such that (1.5) has a unique strong solution. Then the expectations

$$u(x,t) = \mathbb{E}_x\big[\phi(X_t)\big] \stackrel{\text{def}}{=} \mathbb{E}\big[\phi(X_t)\big|X_0 = x\big] \tag{1.6}$$

are solutions of the Cauchy problem

$$\begin{aligned} u_t &= \frac{1}{2}\sigma^2(x,t)u_{xx} + b(x,t)u_x, \\ u(x,0) &= \phi(x). \end{aligned} \tag{1.7}$$

The PDE (1.7) is known as the Kolmogorov forward equation. See [13] for background on stochastic calculus. Thus if $p(x, y, t)$ is the appropriate fundamental solution of (1.7) then we may compute the given expectations according to $\mathbb{E}_x[\phi(X_t)] = \int_\Omega \phi(y) p(x, y, t)\, dy$.

In this context, the fundamental solution is known as the probability transition density for the process. Obviously we also require that $\int_\Omega p(x, y, t)\, dy = 1$.

Recall that fundamental solutions are not unique. The PDE (1.7) may have many fundamental solutions, only one of which will be the transition density. One of the strengths of the methods of this paper, is that they will always produce a fundamental solution which is a probability density. There are many other methods for which this is not the case.

To illustrate, recall Lie's result that any PDE of the form (1.4) which has a four-dimensional Lie algebra of symmetries can be reduced to an equation of the form

$$u_t = u_{xx} - \frac{A}{x^2}u, \tag{1.8}$$

where A is a constant. (See the paper [12] for a detailed discussion of this reduction method.) This PDE has a well-known fundamental solution

$$Q(x,y,t)=\frac{\sqrt{xy}}{2t}\exp\left(-\frac{x^2+y^2}{4t}\right)I_{\frac{1}{2}\sqrt{1+4A}}\left(\frac{xy}{2t}\right). \tag{1.9}$$

Thus a common solution strategy is to reduce an equation of the form (1.4) with a four-dimensional Lie algebra of symmetries to the form (1.8) and construct from Q a fundamental solution of the original PDE. This is often referred to as the *method of reduction to canonical form*.

This works if all we seek is a fundamental solution. But if we require the fundamental solution to also be a density, then this strategy will in general fail. As an example, suppose that we wish to find the transition density for the diffusion $X=\{X_t\colon\ t\geqslant 0\}$ satisfying the SDE

$$dX_t=\frac{2aX_t}{2+aX_t}\,dt+\sqrt{2X_t}\,dW_t,\qquad X_0=x>0,\quad a>0. \tag{1.10}$$

Observe that the drift is Lipschitz continuous for positive a and $\sqrt{x}$ is Hölder continuous, with Hölder constant $\frac{1}{2}$. Thus by the Yamada–Watanabe Theorem (Theorem 5.5 of [15]), the SDE has a unique strong solution. We require a fundamental solution of

$$u_t=xu_{xx}+\frac{2ax}{2+ax}u_x. \tag{1.11}$$

This can be reduced to (1.8) with $A=\frac{3}{4}$ by making the change of variables $x\to\sqrt{x}$ and $t\to 4t$, then eliminating the first derivative term by letting $u=e^{\psi(x)}\tilde{u}(x,t)$ for a suitable choice of ψ. We may then obtain a fundamental solution of (1.11) by applying the change of variables to $Q(x,y,t)$. From which we conclude that

$$q(x,y,t)=\frac{1}{t}\frac{2+ay}{(2+ax)}\sqrt{\frac{x}{y}}e^{-\frac{(x+y)}{t}}I_1\left(\frac{2\sqrt{xy}}{t}\right) \tag{1.12}$$

is a fundamental solution of (1.11). This conclusion is correct. The problem is that (1.12) is *not* the transition density for the diffusion. To see this, observe that

$$l(x,t)=\int_0^\infty q(x,y,t)\,dy=1-\frac{e^{-\frac{x}{2t}}}{2+ax}\neq 1.$$

Thus q is not a probability density. Notice that $l(x,t)$ is a solution of the Cauchy problem for (1.11), on $(0,\infty)$ with $u(x,0^+)=1$ and that it has bounded second and first derivatives in x and t. However l is not a solution of the Cauchy problem on $[0,\infty)$, since $l(0,t)=\frac{1}{2}\neq 1$.

In [9], it was shown that the transition density is actually

$$p(x,y,t)=\frac{e^{-\frac{(x+y)}{t}}}{(2+ax)t}\left[\sqrt{\frac{x}{y}}(2+ay)I_1\left(\frac{2\sqrt{xy}}{t}\right)+t\delta(y)\right]. \tag{1.13}$$

This produces solutions of the Cauchy problem for (1.11) on $[0,\infty)$ for bounded initial data.

Since (1.9) is not itself a probability density, there is no a priori reason to expect that we will obtain a density from it if we make a change of variables. In order to obtain a density, it is often necessary to include additional terms which involve generalized functions, such as the Dirac delta that occurs in (1.13). The obvious question is how do we know what these extra terms are?

Exactly the same problem arises with a number of other techniques, such as the method of group invariant solutions. See [8] for a discussion of this. An advantage of our method is that the required generalized function terms appear naturally.

2. Generalized Laplace transforms of fundamental solutions

We begin with a definition.

Definition 2.1. Let $f:[0,\infty)\to\mathbb{R}$ be Lebesgue integrable and be of suitably slow growth. The generalized Laplace transform of f is the function

$$F_\gamma(\lambda)=\int_0^\infty f(y)e^{-\lambda y^{2-\gamma}}\,dy, \tag{2.1}$$

where $\lambda>0$ and $\gamma\neq 2$.

An explicit inversion theorem for this transform may be found in [14]. However the transform is usually inverted by reducing it to a Laplace transform by setting $z=y^{2-\gamma}$. We need the integral resulting from (2.1) to be convergent under this change of variables, but for the transforms we consider, this is not a problem.

In this paper we will be concerned with PDEs of the form

$$u_t=\sigma x^\gamma u_{xx}+f(x)u_x-g(x)u,\quad x\geqslant 0, \tag{2.2}$$

for $\sigma>0$, $\gamma\neq 2$. Similar results can be proved in the case where $\gamma=2$, but this is usually best handled by letting $y=\ln x$ and reducing to $\gamma=0$. However see [8] for an explicit result when $\gamma=2$.

Eq. (2.2) always has time translation symmetries and we may multiply solutions by a constant. In our context, we term these trivial symmetries. We require richer, nontrivial (at least four-dimensional) symmetry groups. For $\gamma\neq 2$, introduce $h(x)=x^{1-\gamma}f(x)$. It can be shown (see [8]) that the Lie algebra of symmetries of (2.2) is nontrivial if and only if for a given g, h satisfies any one of the following Riccati equations:

$$\sigma xh'-\sigma h+\frac{1}{2}h^2+2\sigma x^{2-\gamma}g(x)=2\sigma Ax^{2-\gamma}+B,$$

$$\sigma xh'-\sigma h+\frac{1}{2}h^2+2\sigma x^{2-\gamma}g(x)=\frac{Ax^{4-2\gamma}}{2(2-\gamma)^2}+\frac{Bx^{2-\gamma}}{2-\gamma}+C,$$

$$\sigma xh'-\sigma h+\frac{1}{2}h^2+2\sigma x^{2-\gamma}g(x)=\frac{Ax^{4-2\gamma}}{2(2-\gamma)^2}+\frac{Bx^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma}+\frac{Cx^{2-\gamma}}{2-\gamma}-\kappa,$$

with $\kappa=\frac{\gamma}{8}(\gamma-4)\sigma^2$. The constant factors multiplying A, B and C above are included to simplify our later notation.

Remark 2.2. Clearly we could fix f in advance and these equations would then give us conditions on g which guarantee the existence of nontrivial symmetries.

For the first Riccati equation Craddock and Lennox proved in [8] the following result.

Theorem 2.3. *Suppose that $\gamma\neq 2$ and $h(x)=x^{1-\gamma}f(x)$ is a solution of the Riccati equation*

$$\sigma xh'-\sigma h+\frac{1}{2}h^2+2\sigma x^{2-\gamma}g(x)=2\sigma Ax^{2-\gamma}+B. \tag{2.3}$$

Then the PDE (2.2) *has a symmetry solution of the form*

$$\overline{U}_\epsilon(x,t) = \frac{1}{(1+4\epsilon t)^{\frac{1-\gamma}{2-\gamma}}} \exp\left\{\frac{-4\epsilon(x^{2-\gamma} + A\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+4\epsilon t)}\right\}$$
$$\times \exp\left\{\frac{1}{2\sigma}\left(F\left(\frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F(x)\right)\right\} u\left(\frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}}, \frac{t}{1+4\epsilon t}\right),$$

where $F'(x) = f(x)/x^\gamma$ and u is a solution of the PDE. That is, for ϵ sufficiently small, U_ϵ is a solution of (2.2) whenever u is. If $u(x,t) = u_0(x)$ with u_0 an analytic, stationary solution then there is a fundamental solution $p(x,y,t)$ of (2.2) such that

$$\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0(y) p(x,y,t)\, dy = U_\lambda(x,t). \tag{2.4}$$

Here $U_\lambda(x,t) = \overline{U}_{\frac{1}{4}\sigma(2-\gamma)^2\lambda}$. Further, if $u_0 = 1$, then $\int_0^\infty p(x,y,t)\,dy = 1$.

Thus we can find fundamental solutions by inverting a generalized Laplace transform. These generalized Laplace transforms can always be explicitly inverted, see [10]. Inversion of the transforms which arise from Theorem 2.3 will frequently involve the use of distributions. The following result is useful for the purpose of inverting the Laplace transforms that we will encounter.

Proposition 2.4. *When n is a nonnegative integer we have*

$$\mathcal{L}^{-1}\left[\lambda^n e^{\frac{k}{\lambda}}\right] = \sum_{l=0}^{n} \frac{k^l}{l!}\delta^{(n-l)}(y) + \left(\frac{k}{y}\right)^{\frac{n+1}{2}} I_{n+1}(2\sqrt{ky}), \tag{2.5}$$

where $\mathcal{L}$ is the Laplace transform, $\delta(y)$ is the Dirac delta and I_n is a modified Bessel function of the first kind.

For a proof, together with more general Laplace transforms of distributions, see [10].

2.0.1. Reduction to canonical form revisited

Using Theorem 2.3 we reconsider the method of reduction to canonical form.

Example 2.1. Consider the PDE

$$u_t = u_{xx} - \frac{A}{x^2}u, \quad x > 0, \tag{2.6}$$

and for simplicity let $A > -\frac{3}{4}$. Stationary solutions of the PDE are $u_0(x) = x^{\frac{1+\sqrt{1+4A}}{2}}$ and $u_1(x) = x^{\frac{1-\sqrt{1+4A}}{2}}$. Using Theorem 2.3 and u_0 we know that there is a fundamental solution Q with

$$\int_0^\infty e^{-\lambda y^2} u_0(y) Q(x,y,t)\, dy = \frac{u_0(x)}{(1+4\lambda t)^{1+\frac{\sqrt{1+4A}}{2}}} \exp\left(-\frac{\lambda x^2}{1+4\lambda t}\right).$$

Letting $z = y^2$ converts this to a Laplace transform and inverting this Laplace transform gives the fundamental solution (1.9). The calculations are quite elementary. But what if we take the solution u_1? We now seek a fundamental solution $\overline{p}(x,y,t)$ such that

$$\int_0^\infty e^{-\lambda y^2} u_1(y) \overline{p}(x,y,t)\, dy = u_1(x)(1+4\lambda t)^{\frac{-2+\sqrt{1+4A}}{2}} \exp\left(-\frac{\lambda x^2}{1+4\lambda t}\right).$$

Let us fix $A=\frac{3}{4}$. We let $z=y^2$ in the generalized Laplace transform, perform the inversion with the aid of Proposition 2.4 and obtain a second fundamental solution of (2.6) for this choice of A given by

$$\bar{p}(x,y,t)=2e^{\frac{-(x^2+y^2)}{4t}}y\sqrt{\frac{y}{x}}\left(\frac{xI_1(\frac{xy}{2t})}{4ty}+\delta(y^2)\right). \tag{2.7}$$

Now recall that in Section 1.1 we considered the Itô diffusion satisfying the SDE

$$dX_t=\frac{2aX_t}{2+aX_t}\,dt+\sqrt{2X_t}\,dW_t, \qquad X_0=x>0, \quad a>0. \tag{2.8}$$

The corresponding Kolmogorov forward equation

$$u_t=xu_{xx}+\frac{2ax}{2+ax}u_x \tag{2.9}$$

can be reduced to the canonical form

$$u_t=u_{xx}-\frac{3}{4x^2}u_x \tag{2.10}$$

by a change of variables. This PDE has a known fundamental solution given by (1.9). From this we earlier found a fundamental solution of (2.9), but this fundamental solution was not a transition density.

However we now have a second fundamental solution of Eq. (2.10). If we use the fundamental solution (2.7) we deduce that

$$p(x,y,t)=\frac{e^{-\frac{(x+y)}{t}}}{(2+ax)t}\left[\sqrt{\frac{x}{y}}(2+ay)I_1\left(\frac{2\sqrt{xy}}{t}\right)+t\delta(y)\right] \tag{2.11}$$

is also a fundamental solution of (2.9) and this, as noted above, is the transition density for the diffusion.

Thus we failed to produce a transition density when we first considered this example, because we did not use the appropriate fundamental solution of (2.10). The point is that in order to obtain a transition density in general via the method of reduction to canonical form, it is necessary to consider all fundamental solutions of the canonical form and choose the one which leads to the desired density. This however can be quite a laborious exercise. Our results will give the correct result much more efficiently.

If the drift in (2.2) satisfies either of the remaining two Riccati equations, Craddock and Lennox proved in [7] that fundamental solutions can be obtained by inverting a Whittaker transform. Unfortunately Whittaker transforms are difficult to invert, so we would prefer a result which gives fundamental solutions in terms of generalized Laplace transforms. Fortunately this is possible. This discovery is one of the main contributions of this paper. The first result is the following.

Theorem 2.5. *Consider the PDE*

$$u_t=\sigma x^{\gamma}u_{xx}+f(x)u_x-g(x)u, \quad \gamma\neq 2,\ x\geqslant 0, \tag{2.12}$$

and suppose that g and $h(x)=x^{1-\gamma}f(x)$ satisfy

$$\sigma xh'-\sigma h+\frac{1}{2}h^2+2\sigma x^{2-\gamma}g(x)=\frac{A}{2(2-\gamma)^2}x^{4-2\gamma}+\frac{B}{2-\gamma}x^{2-\gamma}+C,$$

where $A > 0$, B and C are arbitrary constants. Let u_0 be a stationary, analytic solution of (2.12)*. Then* (2.12) *has a solution of the form*

$$\begin{aligned}\overline{U}_\epsilon(x,t) &= \big(1+2\epsilon^2\big(\cosh(\sqrt{A}t)-1\big)+2\epsilon\sinh(\sqrt{A}t)\big)^{-c}\\ &\quad\times\left|\frac{\cosh(\frac{\sqrt{A}t}{2})+(1+2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}{\cosh(\frac{\sqrt{A}t}{2})-(1-2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}\right|^{\frac{B}{2\sigma\sqrt{A}(2-\gamma)}} e^{-\frac{1}{2\sigma}F(x)-\frac{Bt}{\sigma(2-\gamma)}}\\ &\quad\times\exp\left\{\frac{-\sqrt{A}\epsilon x^{2-\gamma}(\cosh(\sqrt{A}t)+\epsilon\sinh(\sqrt{A}t))}{\sigma(2-\gamma)^2(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))}\right\}\\ &\quad\times\exp\left\{\frac{1}{2\sigma}F\left(\frac{x}{(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))^{\frac{1}{2-\gamma}}}\right)\right\}\\ &\quad\times u_0\left(\frac{x}{(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))^{\frac{1}{2-\gamma}}}\right),\end{aligned}\tag{2.13}$$

where $F'(x) = f(x)/x^\gamma$ and $c = \frac{(1-\gamma)}{2-\gamma}$. Further, there exists a fundamental solution $p(x,y,t)$ of (2.12) *such that*

$$\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0(y)p(x,y,t)\,dy = U_\lambda(x,t),\tag{2.14}$$

where $U_\lambda(x,t) = \overline{U}_{\frac{\sigma(2-\gamma)^2\lambda}{\sqrt{A}}}(x,t)$.

Proof. Applying Lie's algorithm shows that if f satisfies the given condition, then there are infinitesimal symmetries

$$\begin{aligned}\mathbf{v}_1 &= \frac{\sqrt{A}x}{2-\gamma}e^{\sqrt{A}t}\partial_x + e^{\sqrt{A}t}\partial_t - \left(\frac{Ax^{2-\gamma}}{2\sigma(2-\gamma)^2}+\frac{\sqrt{A}x^{1-\gamma}}{2\sigma(2-\gamma)}f(x)\right)e^{\sqrt{A}t}u\partial_u - \alpha u e^{\sqrt{A}t}\partial_u,\\ \mathbf{v}_2 &= -\frac{\sqrt{A}x}{2-\gamma}e^{-\sqrt{A}t}\partial_x + e^{-\sqrt{A}t}\partial_t - \left(\frac{Ax^{2-\gamma}}{2\sigma(2-\gamma)^2}-\frac{\sqrt{A}x^{1-\gamma}}{2\sigma(2-\gamma)}f(x)+\beta\right)e^{\sqrt{-A}t}u\partial_u,\\ \mathbf{v}_3 &= \partial_t,\qquad \mathbf{v}_4 = u\partial_u.\end{aligned}$$

Here $\alpha = \frac{1-\gamma}{2(2-\gamma)}\sqrt{A}+\frac{B}{2\sigma(2-\gamma)}$, $\beta = -\frac{1-\gamma}{2(2-\gamma)}\sqrt{A}+\frac{B}{2\sigma(2-\gamma)}$. The key is to find the right symmetry to generate a generalized Laplace transform. This turns out to be given by using $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ to produce the infinitesimal symmetry

$$\mathbf{v} = \frac{2\sqrt{A}x}{2-\gamma}\sinh(\sqrt{A}t)\partial_x + 2\big(\cosh(\sqrt{A}t)-1\big)\partial_t - \frac{1}{\sigma}g(x,t)u\partial_u,$$

where

$$g(x,t) = \left(\frac{Ax^{2-\gamma}}{(2-\gamma)^2}+\frac{B}{2-\gamma}\right)\cosh(\sqrt{A}t)+\sqrt{A}\sinh(\sqrt{A}t)\left(\frac{x^{1-\gamma}}{2-\gamma}f(x)+\sigma c\right).$$

Exponentiation of this symmetry shows that $\overline{U}_\epsilon(x,t)$ is a solution of (2.12). The change of parameters $\epsilon \to \frac{\sigma(2-\gamma)^2\lambda}{\sqrt{A}}$ shows that $U_\lambda(x,0) = e^{-\lambda x^{2-\gamma}} u_0(x)$. The basic principle is that if $p(x,y,t)$ is a fundamental solution of (2.12), then we should have

$$\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0(y) p(x,y,t)\, dy = U_\lambda(x,t). \tag{2.15}$$

That is, $U_\lambda(x,t)$ is the generalized Laplace transform of $u_0 p$.

To prove this, we first observe that U_λ is a generalized Laplace transform if it is a Laplace transform. This follows from the obvious change of variables reducing it to a Laplace transform. Any function which can be written as a product of Laplace transforms is a Laplace transform. The drift functions can be expressed in terms of hypergeometric functions, which are analytic. So a straightforward argument shows that U_λ can be written as the product of functions analytic in $\frac{1}{\lambda}$ and any function analytic in $\frac{1}{\lambda}$ is automatically a Laplace transform. Thus U_λ is a generalized Laplace transform of some distribution $u_0 p$. (Chapter Ten of [20] contains the most general conditions under which a distribution may be written as a Laplace transform.)

Now suppose that (2.15) holds for some distribution p. We prove that p is a fundamental solution of the PDE. The fact that $U_\lambda(x,t)$ is a solution and $U_\lambda(x,t) = \int_0^\infty e^{-\lambda y^{2-\gamma}} u_0(y) p(x,y,t)\, dy$ implies that p is a solution for each fixed y. Now integrate a test function $\varphi(\lambda)$ with sufficiently rapid decay against U_λ. By Lemma 1.1 the function $u(x,t) = \int_0^\infty U_\lambda(x,t)\varphi(\lambda)\, d\lambda$ is a solution of (2.2). We also have

$$u(x,0) = \int_0^\infty U_\lambda(x,0)\varphi(\lambda)\, d\lambda = \int_0^\infty u_0(x) e^{-\lambda x^{2-\gamma}} \varphi(\lambda)\, d\lambda = u_0(x)\Phi_\gamma(x),$$

where Φ_γ is the generalized Laplace transform of φ. Next observe that by Fubini's Theorem

$$\begin{aligned}\int_0^\infty u_0(y)\Phi_\gamma(y) p(x,y,t)\, dy &= \int_0^\infty \int_0^\infty u_0(y)\varphi(\lambda) p(x,y,t) e^{-\lambda y^{2-\gamma}}\, d\lambda\, dy \\ &= \int_0^\infty \int_0^\infty u_0(y)\varphi(\lambda) p(x,y,t) e^{-\lambda y^{2-\gamma}}\, dy\, d\lambda \\ &= \int_0^\infty \varphi(\lambda) U_\lambda(x,t)\, dx = u(x,t).\end{aligned}$$

But $u(x,0) = u_0(x)\Phi_\gamma(x)$. This means that if we integrate initial data $u_0\Phi_\gamma$ against p, the resulting function solves the Cauchy problem for (2.12), with this initial data, which proves that p is a fundamental solution. □

Corollary 2.6. *Suppose that the conditions of the previous theorem hold and that $g = 0$. Take the stationary solution $u_0 = 1$. The resulting fundamental solution p has the property that $\int_0^\infty p(x,y,t)\, dy = 1$.*

Proof. We know that $\int_0^\infty e^{-\lambda y^{2-\gamma}} p(x,y,t)\, dy = U_\lambda(x,t)$. Observe that $U_0(x,t) = 1$, since

$$\left| \frac{\cosh(\frac{\sqrt{A}t}{2}) + \sinh(\frac{\sqrt{A}t}{2})}{\cosh(\frac{\sqrt{A}t}{2}) - \sinh(\frac{\sqrt{A}t}{2})} \right|^{\frac{B}{2\sigma\sqrt{A}(2-\gamma)}} e^{-\frac{Bt}{\sigma(2-\gamma)}} = 1. \qquad \square$$

Remark 2.7. The extension of these results to the case $A<0$ is achieved by replacing $\cosh(\sqrt{A}t)$ with $\cos(\sqrt{|A|}t)$, etc. The reader may check the details.

For the final class of Riccati equations, we are able to establish results similar to Theorem 2.5, which give Laplace transforms under a change of variables.

Theorem 2.8. *Suppose that $\gamma \neq 2$ and $h(x)=x^{1-\gamma}f(x)$ and g satisfy $\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma} g(x) = \frac{Ax^{4-2\gamma}}{2(2-\gamma)^2} + \frac{Bx^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{Cx^{2-\gamma}}{2-\gamma} - \kappa$, where $\kappa = \frac{\gamma}{8}(\gamma-4)\sigma^2$, $\gamma\neq 2$ and $A>0$. Let u_0 be an analytic stationary solution of the PDE* (2.2)*. Define the following constants: $a=\frac{C}{2\sigma(2-\gamma)}$, $b=\frac{(1-\gamma)\sqrt{A}}{2(2-\gamma)}$, $k=\frac{2(2-\gamma)B}{3\sqrt{A}}$, $d=\frac{B^2}{9A\sigma}$, $l=\frac{B\gamma}{3Ak}$ and $s=\frac{a+d}{\sqrt{A}}-\frac{\sqrt{A}k^2}{2\sigma(2-\gamma)^2}$. Let*

$$X(\epsilon,x,t)=\left(\frac{x^{1-\frac{\gamma}{2}}+k}{\sqrt{1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t)}}-k\right)^{\frac{2}{2-\gamma}},$$

and $F'(x)=\frac{f(x)}{x^\gamma}$. Then Eq. (2.12) *has a solution of the form*

$$\begin{aligned}\overline{U}_\epsilon(x,t)&=\frac{x^l(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))^{-\frac{2b}{\sqrt{A}}}}{(k+kx^{\frac{\gamma}{2}}(1-\sqrt{1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t)}))^l}\\&\quad\times\left|\frac{\cosh(\frac{\sqrt{A}t}{2})+(1+2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}{\cosh(\frac{\sqrt{A}t}{2})-(1-2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}\right|^s e^{\frac{\sqrt{A}k^2}{\sigma(2-\gamma)^2}-2s\sqrt{A}t}\\&\quad\times\exp\left\{\frac{-\sqrt{A}\epsilon(x^{1-\frac{\gamma}{2}}+k)^2(\cosh(\sqrt{A}t)+\epsilon\sinh(\sqrt{A}t))}{\sigma(2-\gamma)^2(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))}\right\}\\&\quad\times\exp\left\{\frac{1}{2\sigma}\big(F\big(X(\epsilon,x,t)-F(x)\big)\big)\right\}u_0\big(X(\epsilon,x,t)\big).\end{aligned}$$

Further, (2.12) *has a fundamental solution $p(x,y,t)$ such that*

$$\int_0^\infty e^{-\lambda(y^{2-\gamma}+2ky^{1-\frac{\gamma}{2}})}u_0(y)p(x,y,t)\,dy=U_\lambda(x,t), \tag{2.16}$$

in which $U_\lambda(x,t)=\overline{U}_{\frac{\sigma(2-\gamma)^2\lambda}{\sqrt{A}}}(x,t)$.

Proof. The proof is similar to the previous result. It is a matter of finding the right symmetry, which turns out to be

$$\mathbf{v}=\left(\frac{x}{2-\gamma}+\frac{2B}{3A}x^{\frac{\gamma}{2}}\right)\sinh(\sqrt{A}t)\partial_x+\frac{(\cosh(\sqrt{A}t)-1)}{\sqrt{A}}\partial_t+\frac{N(x,t)}{2}u\partial_u,$$

where

$$\begin{aligned}N(x,t)&=-\frac{\sqrt{A}(x^{1-\frac{\gamma}{2}}+k)^2}{\sigma(2-\gamma)^2}\cosh(\sqrt{A}t)+\frac{\gamma B}{A}x^{\frac{\gamma}{2}-1}\sinh(\sqrt{A}t)\\&\quad-\frac{f(x)}{\sigma x^\gamma}\left(\frac{x}{2-\gamma}+\frac{2B}{3A}x^{\frac{\gamma}{2}}\right)\sinh(\sqrt{A}t)-2s\cosh(\sqrt{A}t)-\frac{2b}{\sqrt{A}}\sinh(\sqrt{A}t).\end{aligned}$$

Exponentiation of the symmetry produces the solution $\overline{U}_\epsilon(x,t)$. The rest of the proof proceeds as before. □

Remark 2.9. This theorem involves a somewhat different form of generalized Laplace transform. Note however that the transform $\Phi(\lambda) = \int_0^\infty e^{-\lambda(y^{2-\gamma}+2ky^{1-\frac{\gamma}{2}})}\phi(y)\,dy$ reduces to a Laplace transform when we make the substitution $z = y^{2-\gamma} + 2ky^{1-\frac{\gamma}{2}}$. (This gives a quadratic for $y^{1-\frac{\gamma}{2}}$ in terms of z and we take the positive root.)

When $A < 0$ a similar result holds, with $\cosh(\sqrt{A}t)$ replaced by $\cos(\sqrt{|A|}t)$, etc. When $A = 0$, the generalized Laplace transform takes on a simpler form. The case where $\gamma = 1, B = 0$ case can be found in [9]. In addition, if $g = 0$, then taking $u_0 = 1$ yields a fundamental solution $p(x,y,t)$ with $\int_0^\infty p(x,y,t)\,dy = 1$.

These Laplace transforms are less tractable than the previous two cases and so a detailed analysis of the inversion process will be given elsewhere. The drifts covered by this case include some very important examples, such as the so-called double square root model of Longstaff, [16] which is an alternative to the Cox–Ingersoll–Ross model for interest rates dynamics. Thus this result is potentially very useful in financial modelling.

2.1. When is a fundamental solution a transition density?

We have already seen that a fundamental solution of the Kolmogorov forward equation is not necessarily a transition density. We also require that the fundamental solution integrates to one. However, even this is not enough for the fundamental solution to be the desired density, as the following new result for squared Bessel processes shows.

Proposition 2.10. *Consider a squared Bessel process $X = \{X_t\colon t \geqslant 0\}$ of dimension $2n$. The Kolmogorov forward equation is*

$$u_t = 2xu_{xx} + 2nu_x. \tag{2.17}$$

There are two linearly independent stationary solutions $u_0(x) = 1$ and $u_1(x) = x^{1-n}$. For the stationary solution u_0, the inverse Laplace transform of $U_\lambda(t,x)$ coming from Theorem 2.3 yields the transition density for a squared Bessel process of dimension $2n$. If $n = 2, 3, 4, \ldots$ and we use the second stationary solution u_1 to construct $U_\lambda(t,x)$, then the inverse Laplace transform of the symmetry solution produces a second fundamental solution $q(t,x,y)$ with $\int_0^\infty q(t,x,y)\,dy = 1$. Moreover, if $\mathbb{E}_q$ denotes expectation taken with respect to this fundamental solution, then

$$\mathbb{E}_q\big[(X_t)^{1-n}\big|X_0 = x\big] = x^{1-n}.$$

Proof. Taking $u_0(x) = 1$ gives $U_\lambda(x,t) = \frac{1}{(1+2\lambda t)^n}\exp(-\frac{\lambda x}{1+2\lambda t})$, and inversion of this Laplace transform gives the well-known transition density for a squared Bessel process of dimension $2n$.

To prove the second claim, we use the fact that if $u_1(x) = x^{1-n}$ then $U_\lambda(x,t) = x^{1-n}(1 + 2t\lambda)^{n-2}e^{-\frac{\lambda x}{1+2\lambda t}}$. The fundamental solution will be

$$q(x,y,t) = (2t)^{n-2}\left(\frac{y}{x}\right)^{n-1} e^{-\frac{x+y}{2t}}\mathcal{L}^{-1}\big[\lambda^{n-2}e^{k/\lambda}\big]. \tag{2.18}$$

Here $k = \frac{x}{(2t)^2}$. Since $n-2$ is a nonnegative integer, we have

$$\mathcal{L}^{-1}\big[\lambda^{n-2}e^{k/\lambda}\big] = \sum_{l=0}^{n-2}\frac{k^l}{l!}\delta^{(n-2-l)}(y) + \left(\frac{k}{y}\right)^{\frac{n-1}{2}} I_{n-1}(2\sqrt{ky}). \tag{2.19}$$

From which

$$q(x,y,t)=\frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{n-1}{2}}e^{-\frac{x+y}{2t}}I_{n-1}\left(\frac{2\sqrt{xy}}{t}\right)+(2t)^{n-2}\left(\frac{y}{x}\right)^{n-1}\sum_{l=0}^{n-2}\frac{x^l\delta^{(n-2-l)}(y)}{(2t)^{2l}l!}.$$

Now

$$\int_0^\infty (2t)^{n-2}\left(\frac{y}{x}\right)^{n-1}\sum_{l=0}^{n-2}\frac{x^l}{(2t)^{2l}l!}\delta^{(n-2-l)}(y)\,dy=0,$$

since the Dirac delta functions and their derivatives select the value of the test function y^{n-1} and its derivatives at zero. Also

$$\int_0^\infty \frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{n-1}{2}}e^{-\frac{x+y}{2t}}I_{n-1}\left(\frac{2\sqrt{xy}}{t}\right)dy=1,$$

as the integrand is simply the transition density of a squared Bessel process of dimension $2n$. To complete the proof we note that $U_0(x,t)=u_1(x)$ and

$$U_\lambda(x,t)=\int_0^\infty e^{-\lambda y}u_1(y)q(t,x,y)\,dy. \tag{2.20}$$

Which implies that $\int_0^\infty u_1(y)q(x,y,t)\,dy=u_1(x)$. $\square$

Thus we have two fundamental solutions which integrate to one, but they cannot both be the transition density, as the transition density for squared Bessel processes is known to be unique. (See [18] for a detailed exposition of the theory of squared Bessel processes.) The existence of these additional fundamental solutions does not seem to have been previously observed.

Example 2.2. Take $n=3$. Then the transition density for the squared Bessel process of dimension 6 is

$$p(x,y,t)=\frac{1}{2t}\frac{y}{x}e^{-\frac{x+y}{2t}}I_2\left(\frac{\sqrt{xy}}{t}\right). \tag{2.21}$$

From the stationary solution $u_0(x)=x^{-2}$ we obtain a second fundamental solution

$$p_2(x,y,t)=p(x,y,t)+e^{-\frac{x+y}{2t}}\left(\frac{y^2}{2tx}\delta(y)+2t\left(\frac{y}{x}\right)^2\delta'(y)\right), \tag{2.22}$$

and $\int_0^\infty p_2(t,x,y)\,dy=1$. It acts on test functions which have finite derivative at zero. Notice however that the solution of the Cauchy problem $u_t=2xu_{xx}+6u_x$, $u(x,0)=\phi(x)$ given by this fundamental solution will not be continuous as $x\to 0$.

So how can we tell that a fundamental solution is in fact the desired density? A reasonably straightforward one in the $\gamma=1$ case is the following. Similar results can be proved for any γ.

Proposition 2.11. *Let $X = \{X_t\colon t \geqslant 0\}$ be an Itô diffusion which is the unique strong solution of*

$$X_t = X_0 + \int_0^t f(X_s)\,ds + \int_0^t \sqrt{2\sigma X_t}\,dW_t, \tag{2.23}$$

where $W = \{W_t\colon t \geqslant 0\}$ is a standard Wiener process. Suppose further that f is measurable and there exist constants $K > 0$, $a > 0$ such that $|f(x)| \leqslant Ke^{ax}$ for all x. Then there exists a $T > 0$ such that $u(x;t,\lambda) = \mathbb{E}_x[e^{-\lambda X_t}]$ is the unique solution of the first-order PDE

$$\frac{\partial u}{\partial t} + \lambda^2\sigma\frac{\partial u}{\partial\lambda} + \lambda\mathbb{E}_x\big[f(X_t)e^{-\lambda X_t}\big] = 0, \tag{2.24}$$

subject to $u(x;0,\lambda) = e^{-\lambda x}$, for $0 \leqslant t < T$, $\lambda > a$.

Proof. The Itô formula gives

$$e^{-\lambda X_t} = e^{-\lambda x} + \int_0^t e^{-\lambda X_s}\big(\lambda^2\sigma X_s - \lambda f(X_s)\big)\,ds - \lambda M_t, \tag{2.25}$$

where $M_t = \int_0^t e^{-\lambda X_s}\sqrt{2\sigma X_s}\,dW_s$ is a local martingale. Obviously we have

$$\mathbb{E}_x\left[\int_0^t \big(\sqrt{X_s}e^{-\lambda X_s}\big)^2\,ds\right] = \mathbb{E}_x\left[\int_0^t X_s e^{-2\lambda X_s}\,ds\right] \leqslant \mathbb{E}_x\left[\int_0^t \frac{ds}{2e\lambda}\right] < \infty,$$

so M_t is a martingale, from which $\mathbb{E}_x[M_t] = M_0 = 0$. Taking expectations in (2.25) therefore gives

$$\mathbb{E}_x\big[e^{-\lambda X_t}\big] - \lambda^2\sigma\int_0^t \mathbb{E}_x\big[X_s e^{-\lambda X_s}\big]\,ds = e^{-\lambda x} - \lambda\int_0^t \mathbb{E}_x\big[f(X_s)e^{-\lambda X_s}\big]\,ds. \tag{2.26}$$

Further, $\mathbb{E}_x[X_s e^{-\lambda X_s}] = -\frac{\partial}{\partial\lambda}\mathbb{E}_x[e^{-\lambda X_s}]$. Differentiation of (2.26) with respect to t gives (2.24). Now $f(x)e^{-\lambda x}$ is bounded and measurable for $\lambda > a$. So $\mathbb{E}_x[f(X_t)e^{-\lambda X_t}]$ is continuous in t. (For example, Proposition 15.49 of Breiman [2].) For each fixed t it is analytic in λ. The coefficients of the PDE are analytic as is the initial data. The uniqueness of the solution follows from the Cauchy–Kovalevskaya Theorem for first-order systems. (See Trèves' book [19] for a proof of the Cauchy–Kovalevskaya Theorem.) $\square$

Example 2.3. Let $X = \{X_t\colon t \geqslant 0\}$ be the squared Bessel process satisfying the SDE $dX_t = n\,dt + 2\sqrt{X_t}\,dW_t$. Then Eq. (2.24) implies that $u(x;t,\lambda) = \mathbb{E}_x[e^{-\lambda X_t}]$ is the unique solution of the first-order PDE $u_t + 2\lambda^2 u_\lambda + \lambda n u = 0$, $u(x;0,\lambda) = e^{-\lambda x}$. This PDE is easily solved by the method of characteristics, giving $\mathbb{E}_x[e^{-\lambda X_t}] = \frac{1}{(1+2\lambda t)^{\frac{n}{2}}}\exp(-\frac{\lambda x}{1+2\lambda t})$. This is of course the same result we obtained from Theorem 2.3 with $u_0 = 1$.

3. Some applications and examples

Now that we are able to determine when a fundamental solution is a transition density, we are in a position to give some examples. Of course the transition densities will always arise from taking $u_0 = 1$ as the stationary solution. This is obvious in practice, though rather more difficult to prove in general. The virtue of our results is that they are extremely easy to use. Despite the apparently complicated expressions for $\overline{U}_\epsilon$ in Theorem 2.5, their Laplace transforms can easily be inverted.

Example 3.1. We will now obtain the transition density for the Cox–Ingersoll–Ross process of interest rate modelling from Theorem 2.5. That is, we will obtain the transition density for the process $X = \{X_t\colon\ t \geqslant 0\}$ where

$$dX_t = (a - bX_t)\,dt + \sqrt{2\sigma X_t}\,dW_t, \qquad X_0 = x. \tag{3.1}$$

We assume that a is positive and real for convenience. The drift is a solution of $\sigma x f' - \sigma f + \frac{1}{2} f^2 = \frac{1}{2}Ax^2 + Bx + C$ with $A = b^2$, $B = -ab$, $C = \frac{1}{2}a^2 - a\sigma$. We require a fundamental solution of

$$u_t = \sigma x u_{xx} + (a - bx)u_x, \tag{3.2}$$

which is positive and integrates to one. Using Theorem 2.5, with $u_0(x) = 1$, $F(x) = a\ln x - bx$ and after some cancellations, we arrive at

$$\int_0^\infty e^{-\lambda y} p(x, y, t)\,dy = \frac{b^{\frac{a}{\sigma}} e^{\frac{ab}{\sigma}t}}{(\lambda\sigma(e^{bt} - 1) + be^{bt})^{\frac{a}{\sigma}}} \exp\left\{\frac{-\lambda bx}{\lambda\sigma(e^{bt} - 1) + be^{bt}}\right\}. \tag{3.3}$$

We easily invert this Laplace transform to obtain

$$p(x, y, t) = \frac{be^{b(\frac{a}{\sigma}+1)t}}{\sigma(e^{bt} - 1)} \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left\{\frac{-b(x + e^{bt}y)}{\sigma(e^{bt} - 1)}\right\} I_\nu\left(\frac{b\sqrt{xy}}{\sigma\sinh(\frac{bt}{2})}\right), \tag{3.4}$$

with $\nu = \frac{a}{\sigma} - 1$. This is the transition density for the CIR process. We can check this using Proposition 2.11.

Example 3.2. Consider now the diffusion $X = \{X_t\colon\ t \geqslant 0\}$ satisfying the SDE

$$dX_t = 2X_t\tanh(X_t) + \sqrt{2X_t}\,dW_t, \qquad X_0 = x > 0. \tag{3.5}$$

Then the transition density is a fundamental solution of $u_t = xu_{xx} + 2\tanh(x)u_x$. The drift satisfies $xf' - f + \frac{1}{2}f^2 = 2x^2$. So $A = 4$, $B = C = 0$. An application of Theorem 2.5 with $u_0 = 1$ reveals the existence of a fundamental solution $p(x, y, t)$ such that

$$\int_0^\infty e^{-\lambda y} p(x, y, t)\,dy = \frac{\cosh(\frac{x}{1+\lambda\sinh(t)(2\cosh(t)+\lambda\sinh(t))})}{\cosh(x)} \exp\left\{-\frac{\lambda x(\cosh(2t) + \lambda\cosh(t)\sinh(t))}{1 + \lambda\sinh(t)(2\cosh(t) + \lambda\sinh(t))}\right\}.$$

Inversion of this Laplace transform requires a certain amount of work, which we omit for the sake of brevity. The result is that

$$p(x, y, t) = \frac{\exp\{-\frac{x+y}{\tanh(t)}\}}{\sinh(t)} \frac{\cosh(y)}{\cosh(x)} \left(\sqrt{\frac{x}{y}} I_1\left(\frac{2\sqrt{xy}}{\sinh t}\right) + \sinh(t)\delta(y)\right).$$

This is the transition density for X satisfying (3.5).

An important application of our results is to the calculation of Laplace transforms of joint densities. We present some examples.

Example 3.3. We will find a fundamental solution of the PDE

$$u_t = xu_{xx} + \left(\frac{1}{2} + \sqrt{x}\right)u_x - \frac{\mu}{x}u, \quad x \geqslant 0,\ 0 \leqslant \mu < \frac{15}{16},$$

which at $\mu = 0$ reduces to the transition density of the process $X = \{X_t\colon t \geqslant 0\}$, where $dX_t = (\frac{1}{2} + \sqrt{X_t})\,dt + \sqrt{2X_t}\,dW_t$. To obtain such a fundamental solution, we require a stationary solution $u_0^{\mu}(x)$ with the property that $u_0^0(x) = 1$. Such a stationary solution is not difficult to find. It is $u_0^{\mu}(x) = x^{\frac{1}{4}}e^{-\sqrt{x}}(I_\alpha(\sqrt{x}) + I_{-\alpha}(\sqrt{x}))$, where $\alpha = \frac{1}{2}\sqrt{1+16\mu}$. Applying Theorem 2.3 we have to invert the Laplace transform,

$$\int_0^\infty e^{-\lambda y}u_0^{\mu}(y)p(x,y,t)\,dy = \frac{e^{-\sqrt{x}}x^{\frac{1}{4}}e^{-\frac{\lambda(x+\frac{1}{4}t^2)}{1+\lambda t}}}{1+\lambda t}\operatorname{Bes}(x,y,t), \tag{3.6}$$

where $\operatorname{Bes}(x,y,t) = I_\alpha(\frac{\sqrt{x}}{1+\lambda t}) + I_{-\alpha}(\frac{\sqrt{x}}{1+\lambda t})$. This is quite straightforward with the aid of the inversion formula

$$\mathcal{L}^{-1}\left[\frac{1}{\lambda}\exp\left(\frac{m^2+n^2}{\lambda}\right)I_d\left(\frac{2mn}{\lambda}\right)\right] = I_d(2m\sqrt{y})I_d(2n\sqrt{y}), \quad d > -1.$$

Inverting the Laplace transform gives the fundamental solution

$$p(x,y,t) = \frac{e^{\sqrt{y}-\sqrt{x}}}{t}\left(\frac{x}{y}\right)^{\frac{1}{4}}e^{-\frac{x+y}{t}-\frac{1}{4}t}P(x,y,t), \tag{3.7}$$

where $P(x,y,t) = \frac{I_\alpha(\frac{\sqrt{xy}}{t})I_\alpha(\sqrt{y})+I_{-\alpha}(\frac{\sqrt{xy}}{t})I_{-\alpha}(\sqrt{y})}{I_\alpha(\sqrt{y})+I_{-\alpha}(\sqrt{y})}$. For $\mu \geqslant \frac{15}{16}$ this fundamental solution is not integrable near zero. Taking $\mu = 0$ gives the transition density for X (see [9] for the density), which means that we can use it to calculate $\mathbb{E}_x[e^{-\lambda X_t-\mu\int_0^t \frac{ds}{X_s}}]$ via the Feynman–Kac formula. That is

$$\mathbb{E}_x\left[e^{-\lambda X_t-\mu\int_0^t \frac{ds}{X_s}}\right] = \int_0^\infty e^{-\lambda y}p(x,y,t)\,dy, \tag{3.8}$$

where p is given by (3.7). See [15] for the Feynman–Kac formula. However it does not seem possible to evaluate this integral in closed form.

Example 3.4. Here we will consider the process $X = \{X_t\colon t \geqslant 0\}$ where

$$dX_t = \left(\frac{1}{2} + \sqrt{X_t}\tanh(\sqrt{X_t})\right)dt + \sqrt{2X_t}\,dW_t.$$

We wish to calculate the expectation $\mathbb{E}_x[e^{-\lambda X_t-\mu\int_0^t \frac{ds}{X_s}}]$. To this end we need a fundamental solution of $u_t = xu_{xx} + (\frac{1}{2} + \sqrt{x}\tanh(\sqrt{x}))u_x - \frac{\mu}{x}u$. A stationary solution which reduces to $u = 1$ at $\mu = 0$ is provided by $u_0(x) = \frac{x^{\frac{1}{4}}}{\cosh(\sqrt{x})}I_{-\alpha}(\sqrt{x})$, where $\alpha = \frac{\sqrt{1+16\mu}}{2}$. This leads to

$$\int_0^\infty u_0(y)e^{-\lambda y}p(x,y,t)\,dy = \frac{x^{\frac{1}{4}}I_{-\alpha}(\frac{\sqrt{x}}{1+\lambda t})}{(1+\lambda t)\cosh(\sqrt{x})}\exp\left(-\frac{\lambda(x+\frac{1}{4}t^2)}{1+\lambda t}\right).$$

Inversion of the Laplace transform gives the fundamental solution

$$p(x,y,t)=\frac{1}{t}\left(\frac{x}{y}\right)^{\frac{1}{4}}\frac{\cosh(\sqrt{y})}{\cosh(\sqrt{x})}\exp\left(-\frac{x+y}{t}-\frac{1}{4}t\right)I_{-\alpha}\left(\frac{2\sqrt{xy}}{t}\right).$$

This reduces to the transition density of the process at $\mu=0$. (See [9] for the derivation of the transition density.) This fundamental solution is integrable near zero for $0\leqslant\mu<\frac{1}{2}$. We may calculate by the Feynman–Kac formula $\mathbb{E}_x[e^{-\lambda X_t-\mu\int_0^t\frac{ds}{X_s}}]=\int_0^\infty e^{-\lambda y}p(x,y,t)\,dy$ but this integral again does not seem to be easy to evaluate analytically. Of course it is possible to evaluate it numerically.

We leave it to the reader to show that for $X=\{X_t\colon t\geqslant 0\}$ where

$$dX_t=\left(\frac{1}{2}+\sqrt{X_t}\coth(\sqrt{X_t})\right)dt+\sqrt{2X_t}\,dW_t,$$

we may solve the equation $u_t=xu_{xx}+(\frac{1}{2}+\sqrt{x}\coth(\sqrt{x}))u_x-\frac{\mu}{x}u$ to obtain the fundamental solution

$$p(x,y,t)=\frac{1}{t}\left(\frac{x}{y}\right)^{\frac{1}{4}}\frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})}\exp\left(-\frac{x+y}{t}-\frac{1}{4}t\right)I_{\alpha}\left(\frac{2\sqrt{xy}}{t}\right),$$

where $\alpha=\frac{\sqrt{1+16\mu}}{2}$. This reduces to the transition density of the process at $\mu=0$. It is integrable at $y=0$ for all $\mu>-\frac{1}{16}$.

The range of μ values in these examples for which the fundamental solution is integrable at $y=0$ depends on the properties of the process. For μ outside the given ranges we may find other fundamental solutions, and investigate their properties. Some will contain distributional terms. A full study of this is beyond the scope of the current paper.

Example 3.5. Let $X=\{X_t\colon t\geqslant 0\}$ be a squared Bessel process, where $dX_t=n\,dt+2\sqrt{X_t}\,dW_t$, $X_0=x$. The joint density of $(X_t,\int_0^t\frac{ds}{X_s})$ arises in the pricing of Asian options and other problems, see [6]. To obtain its Laplace transform we require a fundamental solution for the PDE $u_t=2xu_{xx}+nu_x-\frac{\mu}{x}u$. From Theorem 2.3 the reader may check that the stationary solution $u_0(x)=x^d$, where $d=\frac{1}{4}(2-n+\sqrt{(n-2)^2+8\mu})$, leads to the Laplace transform $\int_0^\infty y^d p(x,y,t)e^{-\lambda y}\,dy=\frac{x^d}{(1+2\lambda t)^{2d+n/2}}\exp\{\frac{-\lambda x}{1+2\lambda t}\}$. Inversion gives the fundamental solution

$$p(x,y,t)=\frac{1}{2t}\left(\frac{x}{y}\right)^{\frac{1}{2}(1-\frac{n}{2})}I_{2d+\frac{n}{2}-1}\left(\frac{\sqrt{xy}}{t}\right)\exp\left\{-\frac{(x+y)}{2t}\right\}. \tag{3.9}$$

Now

$$\begin{aligned}\mathbb{E}_x\left[e^{-\lambda X_t-\mu\int_0^t\frac{ds}{X_s}}\right]&=\int_0^\infty e^{-\lambda y}p(x,y,t)\,dy\\&=e^{-\frac{x}{2t}}\left(\frac{x}{2t}\right)^d\frac{\Gamma(\alpha)\,{}_1F_1(\alpha,\beta,\frac{x}{2t+4t^2\lambda})}{\Gamma(\beta)(1+2\lambda t)^\alpha},\end{aligned}$$

with $\alpha=d+\frac{n}{2}$, $\beta=2d+\frac{n}{2}$. Here ${}_1F_1$ is Kummer's confluent hypergeometric function. This is the Laplace transform of the joint density of $(X_t,\int_0^t\frac{ds}{X_s})$. See [8] for more on this example and applications of symmetries to the calculation of joint densities.

Example 3.6. Let us now consider a two-dimensional problem. We will solve

$$\begin{aligned} u_t &= u_{xx} + u_{yy} - \frac{A}{x^2+y^2}u, \quad (x,y)\in\mathbb{R}^2,\ A>0, \\ u(x,y,0) &= f(x,y). \end{aligned} \tag{3.10}$$

We convert the problem to polar coordinates. So we let $u(x,y,t) = U(\sqrt{x^2+y^2}, \tan^{-1}(\frac{y}{x}), t)$, where $U(r,\theta,t)$ satisfies

$$U_t = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} - \frac{A}{r^2}U, \tag{3.11}$$

with $U(r,\theta,0) = f(r\cos\theta, r\sin\theta) = F(r,\theta)$. Taking the Fourier transform in θ, where $\widehat{f}(n) = \int_0^{2\pi} f(\theta)e^{-in\theta}\,d\theta$, this becomes

$$\widehat{U}_t = \widehat{U}_{rr} + \frac{1}{r}\widehat{U}_r - \frac{n^2+A}{r^2}\widehat{U}. \tag{3.12}$$

With initial data $\widehat{U}(r,n,0) = \widehat{F}(r,n,0)$, this has solution

$$\widehat{U}(r,n,t) = \int_0^\infty \widehat{F}(\rho,n,0)p(r,\rho,n,t)\,d\rho,$$

with $p(r,\rho,n,t)$ a fundamental solution of (3.12). Stationary solutions are $u_0(r) = r^{\sqrt{n^2+A}}$ and $u_1(r) = r^{-\sqrt{n^2+A}}$. Applying Theorem 2.3 with u_0 we find that there is a fundamental solution of (3.12) such that

$$\int_0^\infty e^{-\lambda\rho^2}u_0(\rho)p(r,\rho,n,t)\,d\rho = \frac{r^{\sqrt{n^2+A}}}{(1+4\lambda t)^{1+\sqrt{n^2+A}}}\exp\left(-\frac{\lambda r^2}{1+4\lambda t}\right).$$

The change of variables $\rho^2 = z$ converts this to a Laplace transform and inversion gives

$$p(r,\rho,n,t) = \frac{1}{2t}\exp\left(-\frac{r^2+\rho^2}{4t}\right)I_{\sqrt{n^2+A}}\left(\frac{r\rho}{2t}\right). \tag{3.13}$$

Formally at least, this gives us the Fourier series expansion of the solution of (3.11)

$$U(r,\theta,t) = \sum_{n\in\mathbb{Z}}\int_0^\infty \widehat{F}(\rho,n)\frac{1}{4\pi t}e^{-\frac{r^2+\rho^2}{4t}}I_{\sqrt{n^2+A}}\left(\frac{r\rho}{2t}\right)e^{in\theta}\,d\rho.$$

Convergence of the series obviously depends on $\widehat{F}$.

As in the one-dimensional case, we can find further solutions. If we take $u_1(r) = r^{-\sqrt{n^2+A}}$ as a stationary solution, then we see that there is a fundamental solution of (3.12) such that

$$\int_0^\infty e^{-\lambda\rho^2}u_1(\rho)p(r,\rho,n,t)\,d\rho = \frac{r^{-\sqrt{n^2+A}}}{(1+4\lambda t)^{1-\sqrt{n^2+A}}}\exp\left(-\frac{\lambda r^2}{1+4\lambda t}\right).$$

Inversion of these generalized Laplace transforms will require distributions and the result depends on n and A. These kinds of distributions are discussed in [10]. As we have a different distribution for each n, the fundamental solutions will depend upon infinite sums of right sided distributions, rather than a single distribution as happens in the one-dimensional case.

4. Fourier transforms of fundamental solutions

We have so far dealt only with PDEs defined over the positive half-line. When the domain of the PDE is the whole of $\mathbb{R}$, we can recover fundamental solutions by inverting a Fourier transform. We illustrate the procedure for an interesting subclass of the equations we have looked at.

Consider as motivation the heat equation. This has a well-known symmetry

$$\tilde{u}_\epsilon(x,t) = \frac{1}{\sqrt{1+4\epsilon t}} \exp\left\{-\frac{\epsilon x^2}{1+4\epsilon t}\right\} u\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right). \tag{4.1}$$

Take $u = 1$. The general method we have introduced implies that we should look for a fundamental solution $K(x, y, t)$ with the property that

$$\int_{-\infty}^{\infty} e^{-\epsilon y^2} K(x,y,t)\,dy = \frac{1}{\sqrt{1+4\epsilon t}} \exp\left\{-\frac{\epsilon x^2}{1+4\epsilon t}\right\}. \tag{4.2}$$

It is easy to verify that $K(x,y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}$ is a solution of this equation. However, if we did not know K to begin with, it is not clear how it can be extracted from (4.2). This integral equation does not even have a unique solution, since for $\epsilon > 0$ and any bounded, continuous odd function h we have $\int_{-\infty}^{\infty} e^{-\epsilon y^2}(K(x,y,t) + h(y))\,dy = \int_{-\infty}^{\infty} e^{-\epsilon y^2} K(x,y,t)\,dy$.

A method for extracting the heat kernel from (4.1) using two stationary solutions is given in [7]. However, the heat equation also possesses a symmetry of the form $\tilde{u}_\epsilon(x,t) = e^{-\epsilon x + \epsilon^2 t} u(x - 2\epsilon t, t)$. If we take $\epsilon = i\lambda$ and let $u = 1$, then this symmetry leads to

$$\int_{-\infty}^{\infty} e^{-i\lambda y} K(x,y,t) = e^{-i\lambda x - \lambda^2 t}. \tag{4.3}$$

So we have obtained the Fourier transform of the heat kernel from a Lie group symmetry. The Fourier inversion theorem easily gives the one-dimensional heat kernel.

This method applies to more problems than the heat equation. Whenever we have a PDE

$$u_t = \sigma u_{xx} + f(x)u_x - g(x)u, \quad x \in \mathbb{R}, \tag{4.4}$$

with a six-dimensional Lie algebra of symmetries, we may construct Fourier transforms of fundamental solutions. We have seen conditions on the drift f which guarantee the existence of a nontrivial symmetry group. Proposition 4.1 is a special case of our earlier results, which we have not yet exploited. It guarantees the existence of a six-dimensional symmetry group for equations of the form (4.4) and provides the symmetries that will be used to produce Fourier transforms.

Proposition 4.1. *Let the drift function f in Eq.* (4.4) *satisfy the Riccati equation*

$$\sigma f' + \frac{1}{2} f^2 + 2\sigma g = \frac{1}{2} A x^2 + Bx + C, \tag{4.5}$$

where A, B, C are arbitrary constants. Then Eq. (4.4) *has a six-dimensional Lie algebra of symmetries. Moreover if $A \neq 0$, then it has a symmetry of the form*

$$\tilde{u}_\epsilon(x,t) = e^{-\frac{\sqrt{A}\epsilon}{\sigma}\cosh(\sqrt{A}t)x + \frac{\sqrt{A}\epsilon^2}{2\sigma}\sinh(2\sqrt{A}t) + \frac{B\epsilon}{\sigma\sqrt{A}}(1-\cosh(\sqrt{A}t))}$$
$$\times e^{\frac{1}{2\sigma}(F(x-2\epsilon\sinh(\sqrt{A}t))-F(x))} u\big(x - 2\epsilon\sinh(\sqrt{A}t), t\big). \tag{4.6}$$

If f satisfies the special case

$$\sigma f' + \frac{1}{2}f^2 + 2\sigma g = Ax + B \tag{4.7}$$

then it has a symmetry of the form

$$\tilde{u}_\epsilon(x,t) = e^{-\frac{\epsilon x}{2\sigma} + \frac{\epsilon^2 t}{4\sigma} - \frac{A\epsilon}{4\sigma}t^2 + \frac{1}{2\sigma}(F(x-\epsilon t)-F(x))} u(x-\epsilon t, t). \tag{4.8}$$

In both cases $F'(x) = f(x)$.

Proof. Let $\mathbf{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u$. Lie's method shows that $\mathbf{v}$ generates symmetries if and only if $\xi = \frac{1}{2}x\tau_t + \rho$, $\phi(x,t,u) = \alpha(x,t)u$ where

$$\alpha = -\frac{x^2}{8\sigma}\tau_{tt} - \frac{x}{2\sigma}\rho_t - \frac{1}{4\sigma}\big(xf(x)\big)\tau_t - \frac{1}{2\sigma}\rho + \eta \tag{4.9}$$

and

$$-\frac{x^2}{8\sigma}\tau_{ttt} - \frac{x}{2\sigma}\rho_{tt} + \eta_t = -\frac{1}{4}\tau_{tt} - \frac{1}{2\sigma}\big(\sigma(xf)'' + f(xf)^2 + 2(\sigma xg)'\big)\tau_t$$
$$- \frac{1}{2\sigma}(\sigma f'' + ff' + 2\sigma g')\rho. \tag{4.10}$$

The Lie algebra of symmetries is six-dimensional if and only if ρ is nonzero. This occurs when f satisfies the given Riccati equations. In the first case, we find that there is an infinitesimal symmetry of the form

$$\mathbf{v}_1 = \sinh(\sqrt{A}t)\partial_x - \left(\left(\sqrt{A}x + \frac{B}{\sqrt{A}}\right)\cosh(\sqrt{A}t) + f(x)\sinh(\sqrt{A}t)\right)\frac{u}{2\sigma}\partial_u.$$

Exponentiating $2\mathbf{v}_1$ and multiplying the result by the constant $e^{\frac{B\epsilon}{\sigma\sqrt{A}}}$ produces (4.6). In the second case there is an infinitesimal symmetry

$$\mathbf{v} = t\partial_x - \frac{1}{2\sigma}\left(x + tf(x) + \frac{A}{2}t^2\right)u\partial_u.$$

This leads to (4.8). □

Observe that in both cases $\tilde{u}_\epsilon(x,0) = e^{-c\epsilon x}u(x,0)$ for constant c. So we can seek a fundamental solution with the property that

$$\int_{-\infty}^{\infty} e^{-i\lambda y}u(y,0)p(x,y,t)\,dy = \tilde{u}_{\frac{i\lambda}{c}}(x,t). \tag{4.11}$$

In other words, we can obtain Fourier transforms of fundamental solutions. We present some examples.

Example 4.1. We will obtain the transition density of the process $X = \{X_t \colon t \geqslant 0\}$ where

$$dX_t = 2aX_t\left(\frac{ce^{2aX_t}-1}{ce^{2aX_t}+1}\right)dt + \sqrt{2}\,dW_t, \tag{4.12}$$

for $a, c \in \mathbb{R}$. The Kolmogorov equation is $u_t = u_{xx} + 2ax(\frac{ce^{2ax}-1}{ce^{2ax}+1})u_x$ and we seek a fundamental solution which integrates to 1. Observe that the drift satisfies $f' + \frac{1}{2}f^2 = 2a^2$. Using the stationary solution $u_0 = 1$ and Proposition 4.1 leads to the symmetry solution

$$U_\lambda(x,t) = e^{-\lambda^2 t - i\lambda(x-2at)}\left(\frac{1+ce^{2ax-4ai\lambda t}}{1+ce^{2ax}}\right). \tag{4.13}$$

Since $U_\lambda(x,0) = e^{-i\lambda x}$ we seek a fundamental solution $p(x,y,t)$ such that $\int_{-\infty}^{\infty} e^{-i\lambda y} p(x,y,t)\,dy = U_\lambda(x,t)$. The Fourier inversion immediately gives the result. The integrals are standard Gaussians and we have

$$\begin{aligned} p(x,y,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda y} e^{-\lambda^2 t - i\lambda(x-2at)}\left(\frac{1+ce^{2ax-4ai\lambda t}}{1+ce^{2ax}}\right)d\lambda \\ &= \frac{K(x-2at-y,t) + ce^{2ax}K(x+2at-y,t)}{1+ce^{2ax}}, \end{aligned} \tag{4.14}$$

where $K(x-y,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}$. The reader can check that this is a fundamental solution and since $\int_{-\infty}^{\infty} p(x,y,t)\,dy = U_0(x,t) = 1$, it is a density. In fact it is the transition density for the process.

An interesting feature of this Fourier transform analysis is that it is not always optimal to use stationary solutions. In the next set of examples, we use nonstationary solutions, for the simple reason that they lead to Fourier transforms which are easier to invert.

Example 4.2. We will obtain the transition density of a *mean reverting Ornstein–Uhlenbeck* process $X = \{X_t \colon t \geqslant 0\}$,

$$dX_t = (a - bX_t)\,dt + \sqrt{2\sigma}\,dW_t, \tag{4.15}$$

with $a \geqslant 0$, $b > 0$. If $a = 0$, then we have the regular Ornstein–Uhlenbeck process. The Kolmogorov forward equation is

$$u_t = \sigma u_{xx} + (a - bx)u_x, \quad x \in \mathbb{R}. \tag{4.16}$$

Observe that (4.16) has a symmetry of the form

$$\tilde{u}_\epsilon(x,t) = e^{\frac{a}{b} - \frac{a\epsilon}{\sigma} + \frac{\sigma}{2b} + \frac{\epsilon(a-bx+b\epsilon\sinh(bt))}{e^{bt}\sigma}} u\big(x - 2\epsilon\sinh(bt), t\big). \tag{4.17}$$

Now $\tilde{u}_\epsilon(x,0) = e^{x - \frac{bx\epsilon}{\sigma}} u(x,0)$. We will let $\epsilon \to \frac{i\sigma\lambda}{b}$. Then we look for a fundamental solution $p(x,y,t)$ and a solution u such that

$$\int_{-\infty}^{\infty} e^{-i\lambda y} u(y,0) e^y p(x,y,t)\,dy = \tilde{u}_{\frac{i\sigma\lambda}{b}}(x,t). \tag{4.18}$$

We could use $u = 1$, but as illustration, instead we seek a solution of (4.16) of the form $u(x,t) = e^{m(t)x+z(t)}$. Substitution into the PDE gives $u(x,t) = e^{-\frac{2e^{bt}(a-bx)+\sigma}{2be^{2bt}}}$. Using this solution and the given symmetry, we seek a fundamental solution such that

$$\int_{-\infty}^{\infty} e^{-i\lambda y} e^{y} p(x,y,t)\,dy = e^{-\frac{\lambda^2\sigma^2\sinh(bt)}{b} + \frac{i\lambda(\sigma+e^{bt}(a-bx-e^{bt}(a+\sigma)))}{be^{2bt}} + r(x,t)}$$

with $r(x,t) = \frac{-2e^{bt}(a-bx)-\sigma}{2be^{2bt}}$. We can now recover p by taking the inverse Fourier transform. The integrals are standard Gaussians and we leave the details to the reader. The result is that

$$p(x,y,t) = \frac{\sqrt{b}e^{\frac{bt}{2}}}{\sqrt{4\pi\sigma\sinh(bt)}} \exp\left(-\frac{e^{-bt}(a(e^{bt}-1)+b(x-e^{bt}y))^2}{4b\sigma\sinh(bt)}\right).$$

Despite the fact that we did not use the stationary solution $u_0 = 1$, this is the transition density for the mean reverting Ornstein–Uhlenbeck process.

Example 4.3. Consider the PDE $u_t = u_{xx} - Bx^2u$, $x \in \mathbb{R}$, with $B > 0$. Now this has a symmetry of the form

$$\tilde{u}_\epsilon(x,t) = e^{-2\sqrt{B}\epsilon\cosh(2\sqrt{B}t)x - 2\sqrt{B}\epsilon^2\sinh(2\sqrt{B}t)} u\big(x - 2\sqrt{B}\epsilon\cosh(2\sqrt{B}t), t\big).$$

Observe that $\tilde{u}_\epsilon(x,0) = e^{-2\sqrt{B}\epsilon x}u(x,0)$. We replace ϵ with $\epsilon = \frac{i\lambda}{2\sqrt{B}}$. We take $u(x,t) = e^{-\frac{1}{2}\sqrt{B}x^2 - \sqrt{B}t}$. Here $u(x,0) = e^{-\frac{1}{2}\sqrt{B}x^2}$. This gives us the Fourier transform

$$\int_{-\infty}^{\infty} e^{-i\lambda y} e^{-\frac{1}{2}\sqrt{B}y^2} p(x,y,t)\,dy = e^{-\frac{(1-e^{-4\sqrt{B}t})\lambda^2 + 4i\sqrt{B}e^{-2\sqrt{B}t}x\lambda + 2B(x^2+2t)}{4\sqrt{B}}}.$$

We apply the inverse Fourier transform. The result is that

$$\begin{aligned} e^{-\frac{1}{2}\sqrt{B}y^2} p(x,y,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda y} e^{-\frac{(1-e^{-4\sqrt{B}t})\lambda^2 + 4i\sqrt{B}e^{-2\sqrt{B}t}x\lambda + 2B(x^2+2t)}{4\sqrt{B}}}\,dy \\ &= \frac{\sqrt[4]{B}e^{-\frac{\sqrt{B}(x^2-4e^{2\sqrt{B}t}yx-2t+e^{4\sqrt{B}t}(x^2+2y^2+2t))}{2(-1+e^{4\sqrt{B}t})}}}{\sqrt{\pi - e^{-4\sqrt{B}t}\pi}}. \end{aligned} \tag{4.19}$$

From which we obtain

$$p(x,y,t) = \frac{\sqrt[4]{B}}{\sqrt{2\pi\sinh(2\sqrt{B}t)}} \exp\left(-\frac{\sqrt{B}(x^2+y^2)}{2\tanh(2\sqrt{B}t)} + \frac{\sqrt{B}xy}{\sinh(2\sqrt{B}t)}\right),$$

which is valid for all $B > 0$. For $B < 0$, we may show by similar calculations that the obvious extension to negative B, namely

$$q(x,y,t) = \frac{\sqrt[4]{|B|}}{\sqrt{2\pi\sin(2\sqrt{|B|}t)}} \exp\left(-\frac{\sqrt{|B|}(x^2+y^2)}{2\tan(2\sqrt{|B|}t)} + \frac{\sqrt{|B|}xy}{\sin(2\sqrt{|B|}t)}\right),$$

is a fundamental solution.

Example 4.4. Consider now the PDE $u_t = u_{xx} - Axu$, $-\infty < x < \infty$. Lie's method shows that if u is a solution, then so is

$$\tilde{u}_\epsilon(x,t) = e^{-\epsilon x + \epsilon^2 t - A\epsilon t^2} u(x - 2\epsilon t, t). \tag{4.20}$$

Further, $u(x,t) = e^{-Axt + \frac{1}{3}A^2t^3}$ is a solution. Then we look for a fundamental solution such that $\int_{-\infty}^{\infty} e^{-i\lambda y} p(x,y,t)\,dy = e^{\frac{A^2t^3}{3} - t\lambda^2 + i(At^2 - x)\lambda - Atx}$. From which

$$\begin{aligned} p(x,y,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda y} e^{\frac{A^2t^3}{3} - t\lambda^2 + i(At^2 - x)\lambda - Atx}\, d\lambda \\ &= \frac{1}{\sqrt{4\pi t}} e^{\frac{A^2t^3}{12} - \frac{1}{2}A(x+y)t - \frac{(x-y)^2}{4t}}. \end{aligned} \tag{4.21}$$

The results and applications of this paper can be extended to larger classes of equations. Although we have concentrated on PDEs in which the coefficient of the second derivative term is a power law, our methods may be applied to any PDE of the form $u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u$ which has at least a four-dimensional Lie algebra of symmetries. For such PDEs we may obtain, up to a change of variables, a Fourier or Laplace transform of a fundamental solution, by applying a single symmetry to a trivial solution. This is discussed in [5]. The process for constructing integral transforms of fundamental solutions for equations lying outside the class considered in the current work is essentially the same as the method employed here.

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