Spectral order of operators and range projections

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Abstract

We study the effect algebra (i.e. the positive part of the unit ball of an operator algebra) and its relation to the projection lattice from the perspective of the spectral order. A spectral orthomorphism is a map between effect algebras which preserves the spectral order and orthogonality of elements. We show that if the spectral orthomorphism preserves the multiples of the unit, then it is a restriction of a Jordan homomorphism between the corresponding algebras. This is an optimal extension of the Dye’s theorem on orthomorphisms of the projection lattices to larger structures containing the projections. Moreover, results on automatic countable additivity of spectral homomorphisms are proved. Further, we study the order properties of the range projection map, assigning to each positive contraction in a JBW algebra its range projection. It is proved that this map preserves infima of positive contractions in the spectral (respectively standard order) if, and only if, the projection lattice of the algebra in question is a modular lattice.

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1. Introduction and preliminaries

Suppose that $x$ and $y$ are bounded self-adjoint operators acting on a Hilbert space $H$. We say that $x$ is less than $y$ (in symbols $x \leq y$) if the same holds for the corresponding quadratic forms:

$$(x\xi, \xi) \leq (y\xi, \xi) \quad \text{for all } \xi \in H.$$  

The order, $\leq$, will be referred to as the standard operator order. The set $B(H)_{sa}$ of all self-adjoint operators acting on $H$ endowed with the standard order forms a partially ordered set which is...
far from being a lattice. Indeed, it was proved by Kadison [20] that the supremum, $x \vee y$, and infimum $x \wedge y$ in the partially ordered set $(B(H)_{sa}, \preceq)$ exists if, and only if, either $x \preceq y$ or $y \preceq x$. A related classical result of Sherman [31] says that the self-adjoint part of a $C^*$-algebra $A$ acting on $H$ is a lattice with respect to the standard order precisely when $A$ is abelian.

A less known, but natural order, $\preceq_S$, on self-adjoint operators was introduced by Olson [29]. Assume that $(E^x_\lambda)_{\lambda \in \mathbb{R}}$ and $(E^y_\lambda)_{\lambda \in \mathbb{R}}$ are the spectral families of the self-adjoint operators $x$ and $y$, respectively. We say that $x$ is less than $y$ in the spectral order (in the symbols $x \preceq_S y$) if

$$E^x_\lambda \geq E^y_\lambda \quad \text{for each } \lambda \in \mathbb{R}.$$ 

A nice characterization of the spectral order in terms of the moments was obtained by Olson [29]: If $x$ and $y$ are positive, then $x \preceq_S y$ if, and only if, $x^n \preceq y^n$ for all $n = 1, 2, \ldots$. In particular, $x \preceq_S y$ implies $x \leq y$. It is also clear that both order relations coincide on condition that $x$ and $y$ commute or in case when both $x$ and $y$ are projections. (However, the commutativity of $x$ and $y$ is not necessary for the comparability $x \preceq_S y$—see e.g. [15].) In contrast to the standard order of operators, the spectral order has the advantage of organizing the structure of operators into a boundedly complete lattice. Indeed, it can be proved (see [15,29]) that any upper bounded collection $(x_\alpha)_{\alpha \in I}$ of self-adjoint operators has a supremum relative to the spectral order; it is that self-adjoint operator $y$ whose spectral family is given by

$$E^y_\lambda = \bigwedge_{\alpha \in I} E^{x_\alpha}_\lambda, \quad \lambda \in \mathbb{R}. $$

(Here $\bigwedge$ denotes the infimum in the projection lattice.) It is well known [20,22] that the structure $P(H)$ of projections acting on $H$ equipped with the standard order is a complete lattice which is never a sublattice of $(B(H)_{sa}, \preceq)$. (Except for the trivial case $\dim H = 1$.) On the other hand, $(P(H), \preceq)$ is always a complete sublattice of $(B(H)_{sa}, \preceq_S)$ and so the spectral lattice operations extend that in the projection lattice. The spectral order has proved to be useful for solving several open problems of spectral theory and has been studied in the context of von Neumann algebras and matrix algebras in [1–3,24,29].

Besides its mathematical meaning, the spectral order has the following natural interpretation in mathematical foundations of quantum theory. The system of observables in quantum mechanics is given by the self-adjoint part $\mathcal{M}_{sa}$ of the von Neumann algebra $\mathcal{M}$ [4,16,32]. Denote by $P[x \leq \lambda]$ the probability that a measurement of the observable $x$ detects its value in the interval $(-\infty, \lambda]$. For the observables $x$, $y \in \mathcal{M}_{sa}$ the relation $x \preceq_S y$ means that, for each $\lambda \in \mathbb{R},$

$$P[x \leq \lambda] \geq P[y \leq \lambda]$$

in each state of the system. (The distribution functions are pointwise ordered.) As we know, for commuting i.e. simultaneously measurable observables $x$ and $y$ the spectral order coincides with the usual one. The same holds when $x$ and $y$ are projections (so-called yes–no observables). However, in general case the spectral order behaves differently. Notably, $\preceq_S$ is not translation invariant in general.

The projection lattice $P(\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$ which constitutes a complete orthomodular lattice was proposed by G. Birkhoff and J. von Neumann as the “logic of quantum system” in the basic work [5] which stimulated much of the research in the quantum measure theory (see e.g. [16,18,30,32]). Later on, the positive part of the unit ball, $\mathcal{E}(\mathcal{M})$, of a von Neumann algebra $\mathcal{M}$, become relevant to studying quantum measurement (see e.g. [4,10]). $\mathcal{E}(\mathcal{M})$ is called the effect algebra. The aim of this note is to study the effect algebra $\mathcal{E}(\mathcal{M})$ and its relation to the projection lattice $P(\mathcal{M})$ from the point of view of the spectral order. Main goal
of Section 2 is to determine those maps between the effect algebras which preserve the spectral order and the orthogonality of elements. (Operators are called orthogonal if their range projections have zero product.) We show that such maps, termed the spectral orthomorphisms, may be nonlinear. However, our main result says that, under mild condition of preserving the scales, any spectral orthomorphism is a restriction of Jordan \(*\)-homomorphism between the underlying algebras (compare [21]). As a consequence, such spectral orthomorphisms correspond to orthomorphisms of the projection lattices. Moreover, we demonstrate that any map between the effect algebras that preserves the multiples of the unit, orthogonality, and suprema of two projections, extends to a \(\sigma\)-additive Jordan \(*\)-homomorphism. It means that for preserving countable suprema in the spectral lattice it is sufficient and necessary to preserve suprema of two projections only. These results complement Dye’s theorem on automorphisms of the projection lattices as orthomodular structures [8,14] and may be thought of as ramifications of Gleason’s theorem in the light of [6–8]. Let us also remark that the automorphisms of \(E(M)\) preserving the standard order and some other relevant structural properties have been studied extensively in the literature. A classical result in this direction due to G. Ludwig [26] says that any bijection of the algebra of all \(1\)-valued states \(\varphi: M \rightarrow M\) which preserves the standard order in both directions and the orthocomplementation \(a \mapsto 1 - a\) is induced by a unitary or an antunitary operator. For further interesting development in this direction see e.g. [11,12,17,27,28].

Since the spectral orthomorphisms are related to the Jordan structure of operator algebras, it is natural to study the spectral order in the context of Jordan operator algebras. Let us remark that the results in Section 2 remain valid for JBW algebras (nonassociative generalization of von Neumann algebras) as well. However, since the proofs are similar we prefer clarity to maximal generality and formulate the characterization of orthomorphisms only in the context of von Neumann algebras. Unlike this, the results in Section 3 concerning the order properties of the range projection maps require essentially new arguments in the nonassociative case. For this reason in this part of the paper we study the spectral order in a more general setting of Jordan operator algebras. As a preparatory material we extend Olson’s characterization of the spectral order to JBW algebras. Besides, as a generalization of the result due to Kato [24] we show that the spectral order supremum, \(x \vee \sigma y\), of positive elements \(x, y\) in the JBW algebra can be computed by the following limit in the strong topology \(x \vee \sigma y = \lim_{n\to\infty} (x^n + y^n)^{1/n}\). Main part of Section 3 concerns the map \(r: E(M) \rightarrow P(M)\) which sends a positive contraction \(x\) in a JBW algebra \(M\) to its range projection \(r(x) = \lim_{n\to\infty} x^{1/n}\). It is a natural map which connects the effect algebra and the projection lattice. We prove that \(P(M)\) is a modular lattice if, and only if, the range projection map \(r\) preserves infima (if they exist) with respect to the standard order. Moreover, it turns out that \(P(M)\) is a modular lattice exactly when \(r: (E(M), \leq_S) \rightarrow (P(M), \leq)\) is a complete lattice orthomorphism. This is a generalization of the corresponding results for von Neumann algebra obtained in [13,15]. The Jordan context introduces new difficulties.

2. Spectral orthomorphisms

Let us now introduce basic concepts and fix the notation. Let \(M\) be a von Neumann algebra. By the symbol \(M_{sa}\) we shall denote the self-adjoint part of \(M\) and by \(E(M)\) the positive part of the unit ball of \(M\). In other words, \(E(M) = \{x^*x \mid x \in M, \|x\| \leq 1\}\). By \(1_M\) we shall denote the unit in the algebra \(M\). Alternatively, we can write \(E(M) = \{x \in M \mid 0 \leq x \leq 1_M\}\). The set \(E(M)\) will be called the effect algebra of \(M\). The projection lattice, \(P(M)\), of \(M\) is the set \(P(M) = \{p \in M_{sa} \mid p^2 = p\}\). If \(x \in M_{sa}\) and \(\lambda \in \mathbb{R}\) we denote by \(E_\lambda x\) the spectral projection of \(x\) corresponding to the interval \((-\infty, \lambda]\). For \(x \in M_{sa}\) we define the range projection, \(r(x),\)
as the smallest projection $p$ in $\mathcal{M}$ for which $xp = x$. Two elements $x$ and $y$ in $\mathcal{M}_{sa}$ are called orthogonal (written $x \perp y$) if $r(x) r(y) = 0$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras. A Jordan $\ast$-homomorphism $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is a linear map such that for each $x \in \mathcal{M}_{sa}$, $\pi(x) \in \mathcal{M}_{sa}$ and $\pi(x^2) = \pi(x)^2$. By the symbol $\leq_S$ we shall denote the spectral order on $\mathcal{M}_{sa}$ introduced above. The infimum and supremum of two elements $x, y \in \mathcal{E}(\mathcal{M})$ in the spectral order will be denoted by $x \sqcup y$ and $x \sqcap y$, respectively. The supremum and infimum of these elements in the ordered structure $(\mathcal{E}(\mathcal{M}), \leq)$ will be denoted by $x \vee y$ and $x \wedge y$, respectively. The latter lattice operations are not defined for all pairs of elements. Nevertheless, if $p$ and $q$ are projections, then $p \vee q = p \sqcup q$ and $p \wedge q = p \sqcap q$. Moreover $p \vee q$ and $p \wedge q$ coincide with the supremum and the infimum in the projections lattice $(P(\mathcal{M}), \leq)$.

2.1. Definition. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras. A map $\varphi : (\mathcal{E}(\mathcal{M}_1), \leq_S) \to (\mathcal{E}(\mathcal{M}_2), \leq_S)$ is a spectral orthomorphism if the following conditions are satisfied for all $x, y \in \mathcal{E}(\mathcal{M})$:

(i) $\varphi(x) \leq_S \varphi(y)$ whenever $x \leq_S y$,
(ii) $\varphi(x) \perp \varphi(y)$ and $\varphi(x \sqcup y) = \varphi(x) \sqcup \varphi(y)$ whenever $x \perp y$.

Similarly, a map $\varphi : P(\mathcal{M}_1) \to P(\mathcal{M}_2)$ is an orthomorphism if the following holds for all $p, q \in P(\mathcal{M}_1)$:

$$\varphi(p) \perp \varphi(q) \quad \text{and} \quad \varphi(p \vee q) = \varphi(p) \vee \varphi(q) \quad \text{whenever} \quad p \perp q \in P(\mathcal{M}_1).$$

Moreover, in both cases, $\varphi$ is called an orthoisomorphism if $\varphi$ is one-to-one, surjective and both $\varphi$ and $\varphi^{-1}$ are orthomorphisms.

In case when, in addition to the conditions above, $\varphi$ preserves suprema and infima of finitely many elements in the corresponding lattices, $\varphi$ is called a lattice orthomorphism.

Let us remark that if $x, y \in \mathcal{E}(\mathcal{M})$ are orthogonal, then $x \sqcup y = x + y$. So the condition (ii) in the previous definition amounts to $\varphi(x + y) = \varphi(x) + \varphi(y)$.

Famous Dye’s theorem [14] and its generalization given by L.J. Bunce and J.D.M. Wright [8] tells us that nearly all orthomorphisms between the projection lattices extend to Jordan $\ast$-homomorphisms. Next example shows that it is not the case of the spectral lattices.

2.2. Example. Let $\mathcal{M}$ be a von Neumann algebra. The map $\varphi : \mathcal{E}(\mathcal{M}) \to \mathcal{E}(\mathcal{M})$ defined as $\varphi(a) = a^2$ is an orthoisomorphism.

Proof. It is clear that $\varphi$ is a one-to-one map mapping $\mathcal{E}(\mathcal{M})$ onto $\mathcal{E}(\mathcal{M})$. As $r(a^2) = r(a)$, the map $\varphi$ preserves orthogonality of elements in both directions. Suppose that $a \leq_S b$. By Olson’s result discussed in the introduction $(a^2)^n \leq (b^2)^n$ for each $n$ and so $a^2 \leq_S b^2$. On the other hand, $a^2 \leq_S b^2$ is equivalent to $a^{2n} \leq b^{2n}$ for each $n$. Since the square root is operator monotone this is equivalent to $a^n \leq b^n$, for each $n$, and so $a \leq_S b$. □

Another example of a nonlinear orthomorphism acting on $\mathcal{E}(\mathcal{M})$ is the map that assigns to an operator its range projection. Such maps may even be lattice orthomorphisms and will be studied in the next section.
2.3. Theorem. Let $\mathcal{M}_1$ be a von Neumann algebra with no Type $I_2$ direct summand. Suppose that $\mathcal{M}_2$ is another von Neumann algebra and $\varphi : \mathcal{E}(\mathcal{M}_1) \rightarrow \mathcal{E}(\mathcal{M}_2)$ an orthomorphism satisfying the following condition

$$\varphi(\lambda 1_{\mathcal{M}_1}) = \lambda 1_{\mathcal{M}_2}, \quad \text{for each } 0 \leq \lambda \leq 1.$$ 

Then there is a unique Jordan $\ast$-homomorphism $\pi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ which extends $\varphi$.

Proof. First we show that $\varphi$ maps the projection lattice $P(\mathcal{M}_1)$ into $P(\mathcal{M}_2)$. For this fix $p \in P(\mathcal{M}_1)$. The orthogonal decomposition $1_{\mathcal{M}_1} = p + (1_{\mathcal{M}_1} - p)$ is preserved by $\varphi$. Hence, by the assumption,

$$\varphi(1_{\mathcal{M}_1}) = 1_{\mathcal{M}_2} = \varphi(p) + \varphi(1_{\mathcal{M}_1} - p),$$

where $\varphi(p)$ and $\varphi(1_{\mathcal{M}_1} - p)$ are orthogonal positive contractions in $\mathcal{M}_2$. On multiplying both sides by $\varphi(p)$ we obtain

$$\varphi(p) = \varphi(p)^2$$

and so $\varphi(p) \in P(\mathcal{M}_2)$. Moreover, we see that

$$\varphi(1_{\mathcal{M}_1} - p) = 1_{\mathcal{M}_2} - \varphi(p).$$

In other words, $\varphi$ induces an orthomorphism between the lattices $P(\mathcal{M}_1)$ and $P(\mathcal{M}_2)$. By the generalized Dye’s theorem [8] there is a Jordan $\ast$-homomorphism $\pi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ which coincides with $\varphi$ on $P(\mathcal{M}_1)$. Our goal is to show that $\varphi$ is equal to $\pi$ on the whole of $\mathcal{E}(\mathcal{M}_1)$. Similarly as above, if $0 \leq \lambda \leq 1$ and $p \in P(\mathcal{M}_1)$, then

$$\lambda 1_{\mathcal{M}_1} = \lambda p + \lambda (1_{\mathcal{M}_1} - p)$$

is an orthogonal decomposition which is transferred by $\varphi$ to

$$\varphi(\lambda 1_{\mathcal{M}_1}) = \lambda 1_{\mathcal{M}_2} = \varphi(\lambda p) + \varphi(\lambda (1_{\mathcal{M}_1} - p)),$$

with $\varphi(\lambda p)$ and $\varphi(\lambda (1_{\mathcal{M}_1} - p))$ orthogonal. On multiplying (1) by $\varphi(p)$, one obtains

$$\lambda \varphi(p) = \varphi(\lambda p)\varphi(p).$$

However, as $\varphi(\lambda p) \leq \varphi(p)$ and $\varphi(p)$ is a projection, we conclude that

$$\lambda \varphi(p) = \varphi(\lambda p)\varphi(p) = \varphi(\lambda p).$$

This homogeneity of $\varphi$ gives immediately its linearity with respect to commuting elements with the finite spectrum. More precisely,

$$\varphi(\lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n) = \lambda_1 \varphi(p_1) + \lambda_2 \varphi(p_2) + \cdots + \lambda_n \varphi(p_n),$$

provided that $p_1, \ldots, p_n$ are orthogonal projections and $\lambda_1, \ldots, \lambda_n \in [0, 1]$. Let us now pick $x \in \mathcal{E}(\mathcal{M}_1)$. For each $n$ we take approximating elementary operators

$$f_n = \sum_{k=0}^{n-1} \frac{k}{n} (E_{\frac{k+1}{n}}^x - E_{\frac{k}{n}}^x),$$

$$g_n = \sum_{k=0}^{n-1} \frac{k+1}{n} (E_{\frac{k+1}{n}}^x - E_{\frac{k}{n}}^x)$$

where $E_{\frac{k}{n}}^x$ is the projection onto the subspace of $\mathcal{M}_1$ spanned by $x$ and its $n$-th multiple.
It is clear that $f_n \leq_S x \leq_S g_n$. As $\varphi$ preserves the spectral order we have that $\varphi(f_n) \leq \varphi(x) \leq \varphi(g_n)$ and, in turn,

\[ 0 \leq \varphi(x) - \varphi(f_n) \leq \varphi(g_n) - \varphi(f_n). \]  

By the previous reasoning

\[ \varphi(g_n) - \varphi(f_n) = \sum_{k=0}^{n-1} \frac{1}{n} \varphi\left( E_{k+1}^X - E_k^X \right) \leq \frac{1}{n} M_{S_2}. \]

It then follows from (2) that

\[ \| \varphi(x) - \varphi(f_n) \| \leq \frac{1}{n}. \]

In other words, $\varphi(f_n) \to \varphi(x)$ in norm. Since $\varphi(f_n) = \pi(f_n)$ for each $n$, we conclude that $\varphi(x) = \pi(x)$. \qed

As Example 2.2 clearly demonstrates the assumption on preserving the scales cannot be omitted in Theorem 2.3. Let us note that the assumption (i) in the definition of the spectral order is not necessary for validity of Theorem 2.3. By inspection of the proof it can be seen that this condition may be relaxed to $\varphi(\lambda p) \leq_S \varphi(x)$ whenever $0 \leq \lambda \leq 1$ and $p$ is a projection such that $\lambda p \leq_S x$.

A map $\varphi : E(M_1) \to E(M_2)$ between the effect algebras of von Neumann algebras is called \( \sigma \)-additive if

\[ \varphi\left( \sum_{n=1}^{\infty} p_n \right) = \sum_{n=1}^{\infty} \varphi(p_n), \]

whenever $(p_n) \subset P(M_1)$ is a sequence of pairwise orthogonal projections. (The sums above are supposed to converge in the strong operator topology.) It was proved by L.J. Bunce and the author in [9] that any lattice homomorphism between von Neumann projection lattices is \( \sigma \)-additive (except for the obvious obstacle given by Type $I_2$ algebra and abelian algebra). This result together with the previous theorem implies that the same is true for lattice orthomorphisms between effect algebras endowed with the spectral order. Moreover, it turns out that for an orthomorphism $\varphi$ between the effect algebras to be a spectral lattice orthomorphism it is necessary and sufficient to preserve suprema of the projections only.

2.4. Corollary. Let $M_2$ be a von Neumann algebra. Suppose that $M_1$ is a von Neumann algebra not containing Type $I_2$ direct summand and nonzero abelian direct summand. Let $\varphi : E(M_1) \to E(M_2)$ be a spectral orthomorphism such that

\[ \varphi(\lambda 1_{M_1}) = \lambda 1_{M_2} \quad \text{for each } 0 \leq \lambda \leq 1. \]

The following conditions are equivalent:

(i) $\varphi(\bigcup_{n=1}^{\infty} x_n) = \bigcup_{n=1}^{\infty} \varphi(x_n)$, whenever $(x_n) \subset E(M_1)$.

(ii) $\varphi(\bigvee_{n=1}^{\infty} p_n) = \bigvee_{n=1}^{\infty} \varphi(p_n)$, whenever $(p_n) \subset P(M_1)$.

(iii) $\varphi$ is \( \sigma \)-additive.

(iv) $\varphi(\sum_{n=1}^{\infty} x_n) = \sum_{n=1}^{\infty} \varphi(x_n)$, whenever $(x_n) \subset E(M_1)$ is a sequence consisting of pairwise orthogonal elements. (The sums are considered in the strong operator topology.)
(v) \( \phi(x \sqcup y) = \phi(x) \sqcup \phi(y) \) for all \( x, y \in \mathcal{E}(\mathcal{M}_1) \).

(vi) \( \phi(p \lor q) = \phi(p) \lor \phi(q) \) for all \( p, q \in \mathcal{P}(\mathcal{M}_1) \).

**Proof.** By Theorem 2.3, \( \phi \) is a restriction of a Jordan \( \ast \)-homomorphism \( \pi \). According to [9], the condition (vi) is equivalent to \( \sigma \)-additivity of \( \pi \). Except for the trivial implications it remains to prove that any \( \sigma \)-additive Jordan \( \ast \)-homomorphism \( \pi \) preserves countable suprema in the effect algebra. To this end, suppose that \( \pi \) is \( \sigma \)-additive. Then \( \pi \) preserves suprema and infima of monotone sequences of projections and thereby suprema and infima of all countable families of projections. Consider a sequence \( (x_n) \subset \mathcal{E}(\mathcal{M}) \) and put \( z = \bigsqcup_{n=1}^\infty x_n \). Then, for each \( \lambda \in \mathbb{R} \),

\[
E_{\lambda}^x = \bigcap_{n=1}^\infty E_{\lambda}^{x_n}
\]

and so

\[
\pi\left(E_{\lambda}^x\right) = \bigcap_{n=1}^\infty \pi\left(E_{\lambda}^{x_n}\right).
\]  

(3)

Let \( (f_n) \) be an increasing sequence of continuous functions such that \( f_n \not\searrow \chi_{[0,\lambda]} \), where \( 0 \leq \lambda \leq 1 \). Then, thanks to \( \sigma \)-additivity of \( \pi \)

\[
\pi\left(f_n(x)\right) \not\searrow \pi\left(E_{\lambda}^x\right).
\]

However, \( \pi \) commutes with continuous function calculus and so

\[
\pi\left(f_n(x)\right) = \pi\left(\pi(x)\right) \not\searrow \chi_{[0,\lambda]}\left(\pi(x)\right) = E_{\lambda}^{\pi(x)}.
\]

In other words, \( \pi\left(E_{\lambda}^x\right) = E_{\lambda}^{\pi(x)} \) and so \( \pi \) preserves the spectral projections. This fact together with (3) yields

\[
\pi\left(\bigsqcup_{n=1}^\infty x_n\right) = \bigcap_{n=1}^\infty \pi\left(x_n\right)
\]

as required. \( \square \)

3. Range projections

We recall a few concepts of the theory of Jordan operator algebras. (For all unmentioned details on Jordan algebras we refer the reader to the monograph [19].) The JB algebra is a real Banach algebra \( A \) with a product, \( \circ \), such that the following conditions hold for all \( a, b \in A \): (i) \( a \circ b = b \circ a \), (ii) \( a \circ (a^2 \circ b) = a^2 \circ (a \circ b) \), (iii) \( \|a^2\| = \|a\|^2 \), (iv) \( \|a^2 + b^2\| \geq \|a\|^2 \). The real space \( B(H)_{sa} \) with the anticommutant product \( x \circ y = \frac{1}{2}(xy + yx) \) is a JB algebra. A JC algebra is defined as a normed closed Jordan subalgebra of this algebra. The symbol \( M_3^8 \) shall stand for the JB algebra of all 3 by 3 matrices over the Cayley numbers. Given a JB algebra \( A \), we shall denote by \( A^+ \) its positive part, \( A^+ = \{a^2 \mid a \in A\} \). \( A^+ \) is a positive cone introducing the standard order, \( \leq \), on \( A \) by defining \( a \leq b \) if \( b - a \in A^+ \). The elements \( a \) and \( b \) of a JB algebra \( A \) are called operator commuting if \( a \circ (b \circ x) = b \circ (a \circ x) \) for all \( x \in A \). If \( a \) and \( b \) operator commute, we shall simply write \( ab \) instead of \( a \circ b \). The center, \( Z(A) \), of a JB algebra \( A \) is the set of all elements of \( A \) which operator commutes with all other elements. Given \( a \in A \), the symbol \( U_a \) (\( a \in A \)) shall be reserved for the map acting on a JB algebra \( A \) by \( U_a(b) = 2a \circ (a \circ b) - a^2 \circ b \). This map is positive i.e. \( U_a(b) \geq 0 \) whenever \( b \geq 0 \).
A JBW algebra is a JB algebra which is simultaneously a dual Banach space. An important example of a JBW algebra is a JW algebra which is a JB algebra isomorphic to a JC algebra closed in the weak operator topology. By a Jordan homomorphism we mean a linear map between JB algebras which preserves the Jordan product. The strong topology on $M$ is locally convex topology given by the system of seminorms $a \rightarrow q(a^2)^{1/2}$, where $q$ is a positive functional from the predual of $M$.

In the sequel let $M$ stand for a JBW algebra. A projection in $M$ is an idempotent. By the symbol $P(M)$ we shall denote the set of all projections in $M$. Similarly to the associative case, the structure $(P(M), \leq, 0, 1_M, \perp)$ is an orthomodular complete lattice with orthocomplementation $p^\perp = 1_M - p$. The effect algebra of $M$ is defined as the set $\mathcal{E}(M) = \{x^2 \mid x \in M, \|x\| \leq 1\} = \{x \in M \mid 0 \leq x \leq 1_M\}$. A symmetry in $M$ is an element $s$ such that $s^2 = 1$. We say that projections $p$ and $q$ are exchangeable by a symmetry $s$ (in symbols $p \sim_1 q$) if $U_s(p) = q$.

Given $x \in M$, the smallest JBW subalgebra of $M$, $JBW(x)$, containing $x$ is the self-adjoint part of a von Neumann algebra. This allows us to define the spectral projections, $E^x_\lambda (\lambda \in \mathbb{R})$, of $x$ and thereby the spectral order, $\leq_S$, on $M$ in the same way as in the case of von Neumann algebras. By the same argument as in the associative case, $(M, \leq_S)$ is a boundedly complete lattice containing $(P(M), \leq)$ as a complete sublattice. In particular, $(\mathcal{E}(M), \leq_S)$ is a complete lattice.

We shall first observe that Olson’s characterization of the spectral order transfers to the Jordan case.

### 3.1. Proposition
Let $x$ and $y$ be positive elements in $M$. Then $x \leq_S y$ if, and only if, $x^n \leq_S y^n$ for all $n = 1, 2, \ldots$.

**Proof.** Without loss of generality we may assume that $x$ and $y$ are positive elements generating $M$. In this case $M$ has to be a JW algebra. We believe this fact to be known. However, since we have not been able to find any appropriate reference, we shall give the full argument here. By the structure theory of JBW algebras $M$ can be written as $M = M_{ex} \oplus M_{sp}$, where $M_{sp}$ is a JW algebra and $M_{ex}$ is either zero or it is isomorphic to the algebra $C(X, M_3^8)$ of all real continuous functions mapping a hyperstonean space $X$ into the exceptional algebra $M_3^8$. Let $z$ be the central element in $M$ such that $zM = M_{ex}$. Suppose that $z \neq 0$. As $zx$ and $zy$ generate $M_{ex}$, we see that $zx(\omega)$ and $zy(\omega)$ generate $M_3^8$ for each $\omega \in X$. However, the algebra generated by $zx(\omega)$ and $zy(\omega)$ is a JC algebra by [33], which is a contradiction. Therefore, $M = M_{sp}$ and we can view $M$ as an operator algebra acting on a complex Hilbert space $H$. Now the arguments in the proof of [29, Theorem 3] can be applied. Indeed, if $x$ in $B(H)$ is positive, then by [25]

$$E^x_\lambda (H) = \{ \xi \in H \mid \|x^n \xi\| \leq \lambda^n \|\xi\| \text{ for all } n = 1, 2, \ldots \}. $$

From this, if $x^n \leq_S y^n$ for each $n$, then $x \leq_S y$. The proof of the reverse implication is the same as in [29, Theorem 3].

It is well known that the supremum, $e \vee f$, of the projections $e$ and $f$ in the projection lattice $P(M)$ is the strong operator limit of the sequence $((e + f)^{1/n})$. This formula was generalized to the positive operators on a Hilbert space by Kato [24]. He showed that the spectral order supremum, $x \sqcup y$, of the positive operators $x$ and $y$ acting on a Hilbert space $H$ is the strong operator limit of the sequence $((x^n + y^n)^{1/n})$. (For further ramifications and strengthening of this result see [11].) This formula can be readily generalized to JBW algebras. Indeed, if $x, y \in \mathcal{E}(M)$, then the spectral order supremum, $x \sqcup y$, of $x, y \in \mathcal{E}(M)$ lies in the subalgebra generated
by $x$ and $y$ which is a JW algebra (see e.g. the proof of Proposition 3.1) representable on some Hilbert space $H$. Hence, by Kato’s result [24] we obtain the following proposition:

3.2. Proposition. Let $x, y \in \mathcal{E}(M)$. Then

$$x \sqcup y = \lim_{n \to \infty} \left( x^n + y^n \right)^{1/n}$$

in the strong topology.

A natural map connecting the projection lattice $P(M)$ and the effect algebra $\mathcal{E}(M)$ is the map assigning to each element in $\mathcal{E}(M)$ its range projection $r(x)$. Let $x \in M$. The range projection, $r(x)$, of $x$ is the smallest projection in $P(M)$ with the property $r(x) \circ x = x$. If $x \geq 0$ then $r(x) = 1 - E_x^0$. If $M = C(X)$ is a function algebra, then for $f \in C(X)$ the projection $r(f)$ corresponds to the support of $f$. In this view, the range projection localizes given quantum observable. If $x \in \mathcal{E}(M)$, then $r(x)$ is the smallest projection majorizing $x$ and $r(x) = \lim_n x^{1/n}$ in the strong topology.

Obviously, the range projection is monotone: $x \leq y$ implies $r(x) \leq r(y)$.

It was proved in [13] that the range projection on a von Neumann algebra $\mathcal{M}$ preserves the infima of elements in $(\mathcal{E}(\mathcal{M}), \leq)$ if, and only if, $\mathcal{M}$ is finite in the Murray–von Neumann classification. (We say that the range projection preserves the infima if $r(x \wedge y) = r(x) \wedge r(y)$ whenever $x$ and $y$ are elements in $\mathcal{E}(\mathcal{M})$ for which the infimum, $x \wedge y$, exists.) Further, de Groote showed in [15] that the range projection map is a spectral lattice homomorphism from $(\mathcal{E}(\mathcal{M}), \leq_S)$ onto the projection lattice $(P(M), \leq)$ if, and only if, $\mathcal{M}$ is finite. We now extend these results to the context of Jordan algebras and provide characterizations of modular JBW algebras in terms of the range projection maps. (A JBW algebra is called modular if its projection lattice is a modular lattice.)

3.3. Theorem. Let $M$ be a JBW algebra. The following conditions are equivalent:

(i) $M$ is modular.
(ii) The range projection preserves infima of elements in $(\mathcal{E}(\mathcal{M}), \leq)$.
(iii) The range projection is a (complete) lattice orthomorphism of $(\mathcal{E}(\mathcal{M}), \leq_S)$ onto $P(M)$.

Proof. First we prove the equivalence of (i) and (ii). Suppose that $M$ is modular. Let $x$ and $y$ be elements in $\mathcal{E}(M)$ for which there is an infimum, $x \wedge y$, in $(\mathcal{E}(\mathcal{M}), \leq)$. There are sequences of the projections $(p_n) \subset JBW(x)$ and $(q_n) \subset JBW(y)$ such that

$$x = \sum_n \frac{1}{2^n} p_n \quad \text{and} \quad y = \sum_n \frac{1}{2^n} q_n.$$ 

Then, for each $k$,

$$x \geq \sum_{n=1}^k \frac{1}{2^n} p_n \geq \frac{1}{2^k} (p_1 + p_2 + \cdots + p_k) \geq \frac{1}{2^k} \bigvee_{n=1}^k p_n.$$ 

Similarly, for each integer $l$,

$$y \geq \frac{1}{2^l} \bigvee_{n=1}^l q_n.$$ 

Therefore, there is a $\lambda > 0$ such that
\[
\lambda \cdot \left[ \bigvee_{i=1}^{k} p_i \right] \wedge \left( \bigvee_{j=1}^{l} q_j \right)
\]
is a lower bound for both $x$ and $y$. Whence,
\[
r(x \wedge y) \geq \left( \bigvee_{i=1}^{k} p_i \right) \wedge \left( \bigvee_{j=1}^{l} q_j \right).
\]
(4)

The lattice $P(M)$ is a complete orthomodular lattice. We shall now apply theorem of Kaplansky [23] which says that every orthocomplemented modular lattice is a continuous geometry. It yields that $e_n \not\rightarrow e$ in $P(M)$ implies $e_n \wedge f \not\rightarrow e \wedge f$ whenever $(e_n), e,$ and $f$ are projections. Making use of this fact we obtain from (4)
\[
r(x \wedge y) \geq \left( \bigvee_{i=1}^{\infty} p_i \right) \wedge \left( \bigvee_{j=1}^{\infty} q_j \right)
\]
for each $l$, and finally,
\[
r(x \wedge y) \geq \left( \bigvee_{i=1}^{\infty} p_i \right) \wedge \left( \bigvee_{j=1}^{\infty} q_j \right).
\]

However,
\[
r(x) = \bigvee_{n=1}^{\infty} p_n \quad \text{and} \quad r(y) = \bigvee_{n=1}^{\infty} q_n,
\]
and so $r(x \wedge y) \geq r(x) \wedge r(y)$. The converse inequality being trivial, we conclude the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let us suppose that $M$ is not modular and show that the range projection map does not preserve the infima of positive contractions. Without loss of generality we can assume that $M$ has no nonzero modular direct summand. It implies that $M$ has zero exceptional part and it is so a JW algebra. Therefore, we shall identify $M$ with the unital subalgebra of the self-adjoint part of $B(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space, endowed with the anticommutant product $x \circ y = \frac{1}{2}(xy + yx)$. By [19, Theorem 7.3.6, p. 172] there is a sequence, $(p_n)$, of mutually orthogonal projections in $M$ such that
\[
p_1 \sim_1 p_2 \sim_1 p_3 \sim_1 \cdots.
\]

We can assume that $\sum_i p_i = 1_M$. Let $N$ be the JW subalgebra generated by the sequence $(p_n)$ and the corresponding symmetries exchanging them. Then $N$ is isomorphic to a Type I JW factor corresponding to the real flip system (for details see [19, Chapter 7.5]). Equivalently, $N$ is isomorphic to the algebra $B(H)_{sym}$ of bounded symmetric operators acting on a real Hilbert space $H$. By identifying $N$ with $B(H)_{sym}$, the sequence $(p_n)$ becomes a sequence of one-dimensional projections.

Let $(\xi_n)$ be an orthonormal basis of $H$ corresponding to $(p_n)$. Let $s_{ij}$ ($i < j$) be a partial symmetry exchanging $p_i$ and $p_j$ (i.e. $s_{ij}p_i s_{ij} = p_j$ and $s_{ij}^2 = p_i + p_j$). The (infinite) matrix $(a_{ij})$ of $s_{ij}$ with respect to the basis $(\xi_n)$ has zero entries except for $a_{ij} = a_{ji} = 1$. Let us now define the elements $v_{ij}$ in $B(\mathcal{H})$ by $v_{ij} = s_{ij} p_i$ for $i < j$. Then $v_{ij}$ is a partial isometry in
$B(\mathcal{H})$ mapping $p_i(\mathcal{H})$ onto $p_j(\mathcal{H})$. Indeed,
\[
v_{ij}v_{ij}^* = p_i s_{ij}^2 p_i = p_i (p_i + p_j) p_j = p_i,
\]
\[
v_{ij}v_{ij}^* = s_{ij} p_i s_{ij} = p_j.
\]
The system $(v_{ij}, v_{ij}^*, p_i)_{i<j}$ is a matrix unit that generates a von Neumann subalgebra $R$ of $B(\mathcal{H})$ isomorphic to the full algebra $B(K)$, where $K$ is a separable infinite-dimensional complex Hilbert space. Therefore, $\mathcal{H}$ can be identified with the tensor product $p_1 \otimes B(K)$ such that $R$ corresponds to the algebra $p_1 \otimes B(K)$. Since, by simple calculations,
\[
v_{ij} + v_{ij}^* = s_{ij} p_i + p_i s_{ij} = s_{ij},
\]
we have that $N \subset p_1 \otimes B(K) = R$. Let $p$ be an atomic projection in $N$ corresponding to a unit vector $h \in H$ such that for each $n$
\[
|\langle h, \xi_n \rangle|^2 = \frac{1}{n^2} \cdot \frac{6}{\pi^2}.
\]
Let us choose (by extension) a normal state $\varrho_n$ on $R$ such that $\varrho_n(a) = \langle a \xi_n, \xi_n \rangle$ for all $a = p(1) \otimes p$.

We are going to prove that $a$ and $b$ have an infimum in $E(M)$ which happens to be zero. For this, suppose that $0 \leq x \leq a, b$ for some $x \in M$. Employing the fact that $p$ is an atomic projection in $B(K)$ we obtain
\[
x \in (p_1 \otimes p) B(\mathcal{H})(p_1 \otimes p) = p_1 B(\mathcal{H}) p_1 \otimes p.
\]
Therefore we can write $x = z \otimes p$, with $z \in B(\mathcal{H})$, and $0 \leq z \leq p_1$. Suppose $z$ is nonzero. Then there is a normal state $\omega$ on $p_1 B(\mathcal{H}) p_1$ such that $\omega(z) > 0$. By forming the product states $\varphi_n = \omega \otimes \varrho_n$ on $p_1 B(\mathcal{H}) p_1 \otimes R$ we obtain:
\[
\varphi_n(x) = \omega(z) \varrho_n(p) = \omega(z) \cdot \frac{1}{n^2} \cdot \frac{6}{\pi^2},
\]
\[
\varphi_n(a) = \frac{1}{n^3}.
\]
However, the condition $x \leq a$ yields
\[
\frac{1}{n^2} \frac{6}{\pi^2} \omega(z) \leq \frac{1}{n^3} \quad \text{for all } n,
\]
which is a contradiction. Hence $z = 0$ and, in turn, $x = 0$. We conclude that $a \wedge b = 0$. On the other hand,
\[
r(a) = 1_M \quad \text{and} \quad r(b) = b,
\]
which gives
\[
r(a) \wedge r(b) = b > r(a \wedge b) = 0.
\]
(i) ⇒ (iii). If \( P(M), \leq \) is modular, then \((P(M), \leq)\) is a continuous geometry. In general, the infimum \( z = x \sqcap y \) is given by the spectral family
\[ E^z_\lambda = \bigwedge_{\lambda' > \lambda} (E^x_{\lambda'} \vee E^y_{\lambda'}). \]
Employing the fact that \( P(M) \) is a continuous geometry the above formula simplifies to
\[ E^z_\lambda = E^x_\lambda \vee E^y_\lambda \quad \text{for all } \lambda \in \mathbb{R}. \]
Therefore,
\[ r(z) = 1 - E^z_0 = 1 - (E^x_0 \vee E^y_0) = (1 - E^x_0) \wedge (1 - E^y_0) = r(x) \wedge r(y). \]
As the range projection always preserves the suprema in the spectral lattice, it is a spectral lattice orthomorphism.

(iii) ⇒ (i). Let us suppose that \( M \) is not modular. As in the proof of the implication (ii) ⇒ (i) there is a Type \( I_{\infty} \) subfactor \( N \) in \( M \) given by the real flip. As the spectral lattice of \( N \) is a (complete) sublattice of that of \( M \) we may suppose that \( M = N \). An easier variant of the construction in the previous part of the proof gives \( a \in E(M) \) and a nonzero projection \( p \) in \( M \) such that \( a \wedge p = 0 \) and \( r(a) = 1 \). Then, of course, \( a \sqcap p = 0 \), while \( r(a) \sqcap r(p) = r(a) \wedge r(p) = p \). In other words, the range projection does not preserve infima in the spectral order. \( \square \)

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