Sensitivity analysis for nonlinear generalized mixed implicit equilibrium problems with non-monotone set-valued mappings

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Abstract

In this paper, we introduce and study a new class of nonlinear generalized mixed implicit equilibrium problems with non-monotone set-valued mappings. By using Wiener–Hopf equations and the Yosida approximation notion, we prove the existence of solutions and analyze the sensitivity of solutions for this class of nonlinear generalized mixed implicit equilibrium problems in Hilbert spaces. Our results are new and extend, improve and unify some recent results in this field.

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1. Introduction

It is well known that equilibrium problem, which includes variational inequality, optimization problem, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems as special cases, have been studied by many authors (see, for example, [2,4,8,9,16,19,20] and the references therein). As pointed out by Moudafi [13], ‘Up to now no sensitivity analysis and only few iterative methods to solve such problems have been done. It is worth mentioning that the analysis developed here can be applied to set-valued mixed equilibrium problems.’

On the other hand, sensitivity analysis of solutions for variational inequalities have been studied by many authors in quite different methods. By using the projection technique, the implicit function approach and the implicit resolvent equations technique, Agarwal et al. [1], Dafermos [4], Dong et al. [6], Gao et al. [7], Loridan and Morgan [12], Noor and Noor [17] and Robinson [18] studied the sensitivity analysis of solutions for variational inequalities. In 2002, Moudafi [13] considered the sensitivity analysis framework, developed recently for variational inequalities, to mixed equilibrium problems with single-valued mappings, and proposed iterative methods for solving this class of problems.

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Inspired and motivated by the recent works [4,5] and [11–14], in this paper, we introduce and study a new class of nonlinear generalized mixed implicit equilibrium problems with non-monotone set-valued mappings. By using Wiener–Hopf equations and the Yosida approximation notion, we prove the existence of solutions and analyze the sensitivity of solutions for this class of set-valued nonlinear generalized mixed implicit equilibrium problems in Hilbert spaces. Our results are new and unify, improve and generalize many known results in recent literature.

2. Preliminaries

Throughout this paper, let $E$ be a real Hilbert space with dual space endowed with $\| \cdot \|$ and a scalar product $( \cdot , \cdot )$, respectively, $2^E$ denote the family of all the nonempty subsets of $E$. We assume that $CB(E)$ is the family of all the nonempty bounded closed subsets of $E$ and $K$ is a nonempty closed convex subset of $E$.

Let $F : E \times E \to R$ be a given bifunction satisfying $F(x, x) = 0$ for all $x \in E$, $g, S : E \to E$ and $\eta, N : E \times E \to E$ be single-valued mappings and $T : E \to 2^E$ be a non-monotone set-valued mapping. We consider the following problem:

Find $\hat{x} \in K, \hat{u} \in T(\hat{x})$ such that

$$F(g(\hat{x}), y) + (N(\hat{u}, S(\hat{x})), \eta(y, g(\hat{x}))) \geq 0, \quad \forall y \in K,$$

which is called a nonlinear generalized mixed implicit equilibrium problem with non-monotone set-valued mapping.

(1) If $g = I$, the identity mapping, then problem (2.1) becomes the following set-valued strongly nonlinear mixed equilibrium problem:

Find $\hat{x} \in K$ and $\hat{u} \in T(\hat{x})$ such that

$$F(\hat{x}, y) + (N(\hat{u}, S(\hat{x})), \eta(y, \hat{x})) \geq 0, \quad \forall y \in K.$$

(2) If $F(x, y) = \psi(y) - \psi(x)$ and $N(x, y) = 0$ for all $x, y \in E$, where $\psi : K \to R$ is a real valued function, then problem (2.2) reduces to the following minimization problem subject to implicit constraints:

Find $\hat{x} \in K$ such that

$$\psi(\hat{x}) \leq \psi(y), \quad y \in K,$$

which shows that problem (2.1) has potential and useful applications in nonlinear analysis and mathematical economics.

(3) If $F(x, y) = \sup_{\xi \in M(y)} (\xi, \eta(y, g(x)))$ with $M : K \to 2^E$ a maximal $\eta$-monotone mapping (see [10]), then problem (2.1) becomes the basic case of implicit variational-like inclusions as follows:

Find $\hat{x} \in K, \hat{u} \in T(\hat{x})$ such that

$$0 \in N(\hat{u}, S(\hat{x})) + M(g(\hat{x})),

which is studied by Agarwal et al. [1] when $N(u, S(x)) = N(x)$ for all $u \in T(x)$ and $g = I$. Moreover, if $F(x, y) = \psi(y) - \psi(x)$, then problem (2.1) reduces to find $\hat{x} \in K, \hat{u} \in T(\hat{x})$ such that

$$(N(\hat{u}, S(\hat{x})), \eta(y, g(\hat{x}))) + \psi(y) - \psi(g(\hat{x})) \geq 0, \quad \forall y \in K,$$

which can be written as the following implicit complementarity problem if $\psi = 0$, $K$ is a closed convex cone and $\eta(y, x) = y - x$ for all $x, y \in E$:

Find $\hat{x} \in K, \hat{u} \in T(\hat{x})$ such that

$$g(\hat{x}) \in K, \quad N(\hat{u}, S(\hat{x})) \in K^*, \quad (N(\hat{u}, A(\hat{x})), g(\hat{x})) = 0,$$

where $K^* = \{x \in E | (x, y) \geq 0, \forall y \in K\}$ is the polar cone to $K$.

(4) If $N(x, y) = y$ for all $x \in E$ and $\eta(y, x) = y - x$ for all $x, y \in E$, then problem (2.1) is equivalent to finding $\hat{x} \in K$ such that

$$F(g(\hat{x}), y) + (S(\hat{x}), y - g(\hat{x})) \geq 0, \quad \forall y \in K.$$

(2.3)
Problem (2.3) is said to be mixed equilibrium problem which is studied by Moudafi [13] which includes variational inequalities as well as complementarity problems, convex optimization, saddle point problems, problems of finding a zero of a maximal monotone operator, and Nash equilibria problems as special case.

**Remark 2.1.** For a suitable choice of $F, g, N, T, S, \eta$, a number of classes of variational inequality, optimization problem, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems can be obtained as special cases of the nonlinear generalized mixed implicit equilibrium problem (2.1).

In the sequel, we need the following concepts and lemmas.

**Definition 2.1 (Blum and Oettli [3]).** A real valued bifunction $F : K \times K \rightarrow \mathbb{R}$ is said to be

(i) monotone if

$$F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in K;$$

(ii) strictly monotone if

$$F(x, y) + F(y, x) < 0, \quad \forall x, y \in K \text{ with } x \neq y;$$

(iii) upper-hemicontinuous if

$$\limsup_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y), \quad \forall x, y, z \in K.$$

**Definition 2.2.** A function $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be lower semi-continuous at $x_0$ if for all $f(x_0) < \infty$, there exists a constant $\beta > 0$ such that

$$f(y) \geq f(x_0) - \beta, \quad \forall y \in B(x_0, \beta),$$

where $B(x_0, \beta)$ denotes the ball with the center $x_0$ and the radius $\beta$, i.e.,

$$B(x_0, \beta) = \{y | \|y - x_0\| \leq \beta\}.$$  

$f$ is said to be lower semi-continuous if it is lower semi-continuous at every point of $E$.

**Lemma 2.1 (Blum and Oettli [3]).** If the following conditions hold:

1. $F$ is monotone and upper hemicontinuous;
2. $F(x, \cdot)$ is convex and lower semi-continuous for each $x \in K$;
3. there exists a compact subset $X$ of $E$ and there exists $y_0 \in X \cap K$ such that $F(x, y_0) < 0$ for each $x \in K \setminus X$;

then the set of solutions to the following problem:

$$\text{Find } \hat{x} \in K \text{ such that } F(\hat{x}, y) \geq 0, \quad \forall y \in K,$$

is nonempty convex and compact.

**Remark 2.2.** If $F$ is strictly monotone, then the solution of (2.4) is unique.

**Definition 2.3.** Let $\rho$ be a positive number. For a given bifunction $F$ the associated Yosida approximation $F_\rho$ over $K$ and the corresponding regularized operator $A^F_\rho$ are defined as follows:

$$F_\rho(x, y) = \left\langle \frac{1}{\rho}(x - J^F_\rho(x)), \eta(y, x) \right\rangle$$

and

$$A^F_\rho(x) := \frac{1}{\rho}(x - J^F_\rho(x)).$$
in which \( J^F_\rho (x) \in K \) is the unique solution of
\[
\rho F(J^F_\rho (x), y) + \langle J^F_\rho (x) - x, \eta (y, J^F_\rho (x)) \rangle \geq 0, \quad \forall y \in K.
\] (2.5)

**Remark 2.3.** It is easy to see that Definition 2.3 is the extension of the Yosida approximation notion introduced in [14]. The existence and uniqueness of the solution of (2.5) follow by invoking Lemma 2.1 and Remark 2.1.

**Example 2.1.** Let \( F(x, y) = \sup_{\xi \in M(x)} \langle \xi, \eta (y, g(x)) \rangle \) with \( M \) a maximal \( \eta \)-monotone mapping and \( K = E \). Then it directly yields
\[
J^F_\rho (x) = (I + \rho M)^{-1}(x) \quad \text{and} \quad A^F_\rho (x) = M_\rho (x),
\]
where \( M_\rho := (1/\rho)(I - (I + \rho M)^{-1}) \) is the Yosida approximation of \( M \), and we recover the classical concepts.

**Definition 2.4.** A mapping \( \eta : E \times E \to E \) is said to be

(i) \( \delta \)-strongly monotone if there exists a constant \( \delta > 0 \) such that
\[
\langle x - y, \eta (x, y) \rangle \geq \delta \| x - y \|^2, \quad \forall x, y \in E,
\]
(ii) \( \tau \)-Lipschitz continuous if there exists a constant \( \tau > 0 \) such that
\[
\| \eta (x, y) \| \leq \tau \| x - y \|, \quad \forall x, y \in E.
\]

**Lemma 2.2.** If \( F : K \times K \to R \) is monotone and \( \eta : E \times E \to E \) is a \( \delta \)-strongly monotone and \( \tau \)-Lipschitz continuous mapping with \( \eta (x, y) + \eta (y, x) = 0 \) for all \( x, y \in E \), then the operator \( J^F_\rho \) is Lipschitz continuous with constant \( \tau / \delta \), i.e.,
\[
\| J^F_\rho (x) - J^F_\rho (y) \| \leq \frac{\tau}{\delta} \| x - y \|, \quad \forall x, y \in E.
\]

**Proof.** From (2.5), for all \( x, y \in E \) we can obtain
\[
\rho F(J^F_\rho (x), J^F_\rho (y)) + \langle J^F_\rho (x) - x, \eta (J^F_\rho (y), J^F_\rho (x)) \rangle \geq 0
\]
and
\[
\rho F(J^F_\rho (y), J^F_\rho (x)) + \langle J^F_\rho (y) - y, \eta (J^F_\rho (x), J^F_\rho (y)) \rangle \geq 0.
\]
Since \( \eta \) is \( \delta \)-strongly monotone and \( \tau \)-Lipschitz continuous and \( \eta (x, y) + \eta (y, x) = 0 \) for all \( x, y \in E \), by adding the last two inequalities and using the monotonicity of \( F \), we have
\[
\langle x - y - (J^F_\rho (x) - J^F_\rho (y)), \eta (J^F_\rho (x), J^F_\rho (y)) \rangle \geq 0,
\]
and so
\[
\delta \| J^F_\rho (x) - J^F_\rho (y) \|^2 \leq \langle J^F_\rho (x) - J^F_\rho (y), \eta (J^F_\rho (x), J^F_\rho (y)) \rangle
\leq \langle x - y, \eta (J^F_\rho (x), J^F_\rho (y)) \rangle
\leq \| x - y \| \| \eta (J^F_\rho (x), J^F_\rho (y)) \|
\leq \tau \| x - y \| \| J^F_\rho (x) - J^F_\rho (y) \|.
\]
This completes the proof. \( \square \)

Now, in relation to problem (2.1), we consider the following equation:

Find \( z \in E \) such that
\[
\forall u \in T(x), \quad N(u, S(x)) + A^F_\rho (z) = 0 \quad \text{and} \quad g(x) = J^F_\rho (z).
\] (2.6)
Lemma 2.3. It is easy to see that for given \( x \in E \) and \( u \in T(x) \), \((x, u)\) is a solution of problem (2.1) if and only if the generalized Wiener–Hopf Eq. (2.6) has solution \( z \), where
\[
g(x) = J^F_\rho(z) \quad z = g(x) - \rho N(u, S(x)),
\]
i.e.,
\[
g(x) = J^F_\rho(g(x) - \rho N(u, S(x))).
\]

Proof. The proof directly follows from the definition of \( J^F_\rho \). \( \square \)

Lemma 2.4 (Nadler [15]). Let \((E, d)\) be a complete metric space. Suppose that \( Q : E \to CB(E) \) satisfies
\[
H(Q(x), Q(y)) \leq \nu d(x, y), \quad \forall x, y \in E,
\]
where \( \nu \in (0, 1) \) is a constant. Then the mapping \( Q \) has a fixed point in \( E \).

3. Existence of solutions

In the forthcoming study, we consider the parametric version of problems (2.1) and (2.6). To formulate the problems, let \((A, d_1)\) and \((\Omega, d_2)\) be two metric spaces in which \(| \cdot |\) the norm generated by its scalar product and the parameters \( \lambda \) and \( \mu \) take values, respectively. Let \( g : E \times A \to E, F : E \times E \times A \to R, T : E \times \Omega \to 2^E, S : E \times \Omega \to E, N : E \times E \times \Omega \to E, \eta : E \times E \times \Omega \to E \) be nonlinear mappings. The parametric set-valued nonlinear generalized mixed implicit equilibrium problem is to find \( \hat{x} \in K, \hat{u} \in T(\hat{x}, \mu) \) such that
\[
F(g(\hat{x}, \lambda), y, \lambda) + \langle N(\hat{u}, S(\hat{x}, \mu), \mu), \eta(y, g(\hat{x}, \lambda), \mu) \rangle \geq 0, \quad \forall y \in K. \tag{3.1}
\]
For any \((\hat{x}, \hat{u}, \lambda, \mu) \in E \times E \times A \times \Omega\), the associated Wiener–Hopf equation is:
Find \( \hat{z} \in E \) such that
\[
\forall \hat{u} \in T(\hat{x}, \mu), \quad N(\hat{u}, S(\hat{x}, \mu), \mu) + A^F_{\rho(\cdot, \cdot, \cdot, \cdot)}(\hat{z}) = 0 \quad \text{and} \quad g(\hat{x}, \lambda) = J^F_{\rho(\cdot, \cdot, \cdot, \cdot)}(\hat{z}). \tag{3.2}
\]

Now we first give the existence theorems of solutions for problems (3.1) and (3.2).

Definition 3.1. Let \( S : E \times \Gamma \to E \) and \( N : E \times E \times \Gamma \to E \) be single-valued mappings, where \((\Gamma, d)\) are metric spaces. Then \( N \) is said to be
(i) \( \alpha \)-strongly monotone with respect to \( S \) in the second argument if, for each given \( \mu \in \Gamma \), there exists a constant \( \alpha > 0 \) such that
\[
\langle N(\cdot, S(x, \mu), \mu) - N(\cdot, S(y, \mu), \mu), x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in E,
\]
(ii) \( N \) is said to be \( \beta \)-Lipschitz continuous with respect to the second argument, for each given \( \mu \in \Gamma \), there exists a constant \( \beta > 0 \) such that
\[
\|N(\cdot, \cdot, \mu) - N(\cdot, y, \mu)\| \leq \beta \|x - y\|, \quad \forall x, y \in E.
\]
Similarly, we can define the Lipschitzian continuity with respect to the first argument and the third argument of \( N(\cdot, \cdot, \cdot, \cdot) \).

Definition 3.2. Let \( g : E \times \Gamma \to E \) be a nonlinear mapping, where \((\Gamma, d)\) is a metric space. Then \( g \) is said to be
(i) \( \xi \)-strongly monotone with respect to the first argument if there exists a constant \( \xi > 0 \) such that
\[
\langle g(x, \cdot) - g(y, \cdot), x - y \rangle \geq \xi \|x - y\|^2, \quad \forall x, y \in E,
\]
(ii) \(\varepsilon\)-Lipschitz continuous with respect to the first argument if there exists a constant \(\varepsilon > 0\) such that
\[
\|g(x, \cdot) - g(y, \cdot)\| \leq \varepsilon\|x - y\|, \quad \forall x, y \in E,
\]
(iii) continuous with respect to the second argument if \(g(x, \cdot) : \Gamma \to E\) is continuous for all \(x \in E\).

**Definition 3.3.** Let \(T : E \times \Gamma \to CB(E)\) be a nonlinear mapping, where \((\Gamma, d)\) is a metric space. Then \(T\) is said to be \(\kappa\)-\(H\)-Lipschitz continuous with respect to the first argument if there exists a constant \(\kappa > 0\) such that
\[
H(T(x, \cdot), T(y, \cdot)) \leq \kappa\|x - y\|, \quad \forall x, y \in E,
\]
where \(H(\cdot, \cdot)\) is the Hausdorff metric on \(CB(E)\).

Following the ideas of Dafermos [4], we consider the mapping \(G : E \times A \times \Omega \to 2^E\) defined as
\[
G(z, \lambda, \mu) = J_{\rho}^{F(\cdot, \lambda, \mu)}(z) - \rho N(T(x, \mu), S(x, \mu), \mu) = g(x, \lambda) - \rho N(T(x, \mu), S(x, \mu), \mu), \quad (3.3)
\]
where \(g(x, \lambda) = J_{\rho}^{F(\cdot, \lambda, \mu)}(z)\).

We have to show that the map \(z \mapsto h \in G(z, \lambda, \mu)\) has a fixed point, which is also a solution of problem (3.2). First of all, we prove that the mapping is contraction with respect to \(z\).

**Theorem 3.1.** Let \(g : K \times A \to K\) be \(\alpha\)-strongly monotone and \(\beta\)-Lipschitz continuous with respect to the first argument, \(F(x, y, \cdot)\) be a monotone function, \(T : E \times \Omega \to CB(E)\) be \(\gamma\)-\(H\)-Lipschitz continuous with respect to the first argument, \(S : E \times \Omega \to E\) be \(\sigma\)-Lipschitz continuous with respect to the first argument and \(\eta(x, y, \cdot)\) be \(\delta\)-strongly monotone and \(\tau\)-Lipschitz continuous with \(\eta(x, y, \cdot) + \eta(y, x, \cdot) = 0\) for all \(x, y \in E\). Suppose that \(N : E \times E \times \Omega \to E\) is \(\xi\)-strongly monotone and \(\kappa\)-Lipschitz continuous with respect to \(S\) in the second argument, and \(\varepsilon\)-Lipschitz continuous with respect to the second argument and \(\kappa\)-Lipschitz continuous with respect to the first argument. Then
\[
H(G(z, \lambda, \mu), G(z', \lambda, \mu)) \leq L\|z - z'\|, \quad (3.4)
\]
for all \((z, z', \lambda, \mu) \in E \times E \times A \times \Omega\), where
\[
L = \sqrt{\frac{k + \rho\kappa\eta + \sqrt{1 - 2\rho\xi + \rho^2\tau^2\sigma^2}}{\delta(1 - k)}} < 1
\]
for
\[
\begin{align*}
k &= \sqrt{1 - 2\alpha + \beta^2} < \frac{\delta}{\delta + \tau}, \quad \alpha > \alpha\gamma, \quad \rho\kappa\gamma < \delta\tau^{-1}(1 - k), \\
\xi &= \kappa\gamma[\delta\tau^{-1}(1 - k) - k] + \sqrt{(\kappa^2\alpha^2 - \kappa^2\gamma^2)(1 - [\delta\tau^{-1}(1 - k) - k]^2)}, \\
\rho &= \frac{\kappa\gamma[\delta\tau^{-1}(1 - k) - k]}{\kappa^2\alpha^2 - \kappa^2\gamma^2} < \frac{\sqrt{[\xi - \kappa\gamma[\delta\tau^{-1}(1 - k) - k]]^2 - (\kappa^2\alpha^2 - \kappa^2\gamma^2)(1 - [\delta\tau^{-1}(1 - k) - k]^2)}}{\kappa^2\alpha^2 - \kappa^2\gamma^2}. \quad (3.5)
\end{align*}
\]
Furthermore, for each fixed \((\lambda, \mu) \in A \times \Omega\), the problem (3.2) has a solution.

**Proof.** For any given \(z, z' \in E\), \((\lambda, \mu) \in A \times \Omega\), \(\varepsilon > 0\) and \(a \in G(z, \lambda, \mu)\), there exist \(u \in T(x, \mu)\) such that \(a = g(x, \lambda) - \rho N(u, S(x, \mu), \mu)\). Since \(T(x, \mu) \in CB(X)\), it follows from (3.3) and Nadler [15] that there exist \(u' \in T(x', \mu)\) such that
\[
\|u - u'\| \leq (1 + \varepsilon)H(T(x, \mu), T(x', \mu)).
\]
Let \(b = g(x', \lambda) - \rho N(u', S(x', \mu), \mu)\). Then \(b \in G(z', \lambda, \mu)\). Thus, for all \((x, x', \lambda, \mu) \in E \times E \times A \times \Omega\), we obtain
\[
\|a - b\| = \|g(x, \lambda) - \rho N(u, S(x, \mu), \mu) - [g(x', \lambda) - \rho N(u', S(x', \mu), \mu)]\|
\leq \|x - x' - (g(x, \lambda) - g(x', \lambda))\| + \rho\|N(u, S(x, \mu), \mu) - N(u', S(x', \mu), \mu)\|
+ \|x - x' - \rho(N(u, S(x, \mu), \mu) - N(u, S(x', \mu), \mu))\|. \quad (3.6)
\]
Since $g(x, \cdot)$ is $\alpha$-strongly monotone and $\beta$-Lipschitz continuous, it follows that:

$$
\|x - x' - (g(x, \lambda) - g(x', \lambda))\|^2 \\
= \|x - x'\|^2 - 2\langle g(x, \lambda) - g(x', \lambda), x - x'\rangle + \|g(x, \lambda) - g(x', \lambda)\|^2 \\
\leq (1 - 2\alpha + \beta^2)\|x - x'\|^2,
$$

(3.7)

for all $(x, x', \lambda) \in \mathbb{R} \times \mathbb{R} \times A$. Further, since $T(x, \cdot)$ is $\gamma$-$H$-Lipschitz continuous, $S(x, \cdot)$ is $\sigma$-Lipschitz continuous and $N(x, y, \cdot)$ be $\zeta$-strongly monotone with respect to $S$ in the second argument, and $\varepsilon$-Lipschitz continuous with respect to the second argument and $\mu$-Lipschitz continuous with respect to the first argument, we have

$$
\|N(u, S(x', \mu), \mu) - N(u', S(x', \mu), \mu)\| \\
\leq \kappa\|u - u'\| \\
\leq \kappa(1 + \varepsilon)H(T(x, \mu), T(x', \mu)) \\
\leq \kappa\gamma(1 + \varepsilon)\|x - x'\|
$$

(3.8)

and

$$
\|x - x' - \rho(N(u, S(x, \mu), \mu) - N(u, S(x', \mu), \mu))\|^2 \\
= \|x - x'\|^2 - 2\rho\langle N(u, S(x, \mu), \mu) - N(u, S(x', \mu), \mu), x - x'\rangle \\
+ \rho^2\|N(u, S(x, \mu), \mu) - N(u, S(x', \mu), \mu)\|^2 \\
\leq (1 - 2\rho\zeta + \rho^2\varepsilon^2\sigma^2)\|x - x'\|^2
$$

(3.9)

for all $(x, x', \mu) \in \mathbb{R} \times \mathbb{R} \times \Omega$. From (3.6)–(3.9), we get

$$
\|a - b\| \leq \left[\sqrt{1 - 2\alpha + \beta^2} + \rho\kappa\gamma(1 + \varepsilon) + \sqrt{1 - 2\rho\zeta + \rho^2\varepsilon^2\sigma^2}\right]\|x - x'\| \\
= [k + \rho\kappa\gamma(1 + \varepsilon) + \sqrt{1 - 2\rho\zeta + \rho^2\varepsilon^2\sigma^2}]\|x - x'\|,
$$

(3.10)

for all $(x, x', \lambda, \mu) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega$, where $k = \sqrt{1 - 2\alpha + \beta^2}$.

According to (3.2) and using Lemma 2.2, we can write

$$
\|x - x'\| \leq \|x - x' - (g(x, \lambda) - g(x', \lambda)) + J_{\rho}^F(\cdot; \lambda)(z) - J_{\rho}^F(\cdot; \lambda)(z')\| \leq k\|x - x'\| + \frac{\tau}{\delta}\|z - z'\|.
$$

Thus,

$$
\|x - x'\| \leq \frac{\tau}{\delta(1 - k)}\|z - z'\|.
$$

(3.11)

Substituting (3.11) into (3.10), we obtain

$$
\|a - b\| \leq \left[k + \rho\kappa\gamma(1 + \varepsilon) + \sqrt{1 - 2\rho\zeta + \rho^2\varepsilon^2\sigma^2}\right] \frac{\tau}{\delta(1 - k)}\|z - z'\| \\
= L(\varepsilon)\|z - z'\|,
$$

(3.12)

for all $(z, z', \lambda, \mu) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega$, where

$$
L(\varepsilon) = \frac{k + \rho\kappa\gamma(1 + \varepsilon) + \sqrt{1 - 2\rho\zeta + \rho^2\varepsilon^2\sigma^2}}{\delta(1 - k)}.
$$

From (3.12), we know that

$$
\sup_{a \in G(z, \lambda, \mu)} d(a, G(z', \lambda, \mu)) \leq L(\varepsilon)\|z - z'\|, \quad \forall (z, z', \lambda, \mu) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega.
$$

(3.13)
Similarly, we have
\[
\sup_{b \in G(\hat{z}, \hat{\lambda}, \mu)} d(b, G(z, \lambda, \mu)) \leq L(v)\|z - z'\|, \quad \forall (z, z', \lambda, \mu) \in E \times E \times \Lambda \times \Omega.
\] (3.14)

It follows from (3.13), (3.14) and the definition of Hausdorff metric that
\[
H(G(z, \lambda, \mu), G(z', \lambda, \mu)) \leq L(v)\|z - z'\|, \quad \forall (z, z', \lambda, \mu) \in E \times E \times \Lambda \times \Omega.
\]

Let \( \varepsilon \to 0 \). Then we get
\[
H(G(z, \lambda, \mu), G(z', \lambda, \mu)) \leq L\|z - z'\|, \quad \forall (z, z', \lambda, \mu) \in E \times E \times \Lambda \times \Omega,
\] (3.15)

where
\[
L = \frac{\tau \left[ k + \rho \kappa \gamma + \sqrt{1 - 2\rho \zeta + \rho^2 \gamma^2} \right]}{\delta (1 - k)}.
\]

It follows from (3.5) that \( 0 < L < 1 \) and so by (3.15) and Lemma 2.4 that \( G(z, \lambda, \mu) \) has a fixed point in \( E \), i.e., there exists a point \( \hat{z} \in E \) such that \( \hat{z} \in G(\hat{z}, \lambda, \mu) \). This completes the proof.

From Theorem 3.1, we can obtain the following theorem.

**Theorem 3.2.** Let \( E, g \) and \( F \) be the same as in Theorem 4.1. Suppose that \( N(x, y, \mu) = y \) for all \( x \in E \) and \( \mu \in \Omega, \eta(y, x, \cdot) = y - x \) for all \( x, y \in E \), \( S(\cdot, t) \) is continuous, \( S(x, \cdot) \) is \( \xi \)-strongly monotone and \( \sigma \)-Lipschitz continuous and for each given \( (\lambda, \mu) \in \Lambda \times \Omega \), there exists a constant \( \rho > 0 \) such that

\[
k = \sqrt{1 - 2\lambda + \beta^2} < \frac{1}{2}, \quad \xi > 2\sigma \sqrt{k(1 - k)}, \quad \rho - \frac{\xi}{\sigma^2} < \frac{\sqrt{\xi^2 - 4k(1 - k)\sigma^2}}{\sigma^2}.
\] (3.16)

Then the following problem:
\[
F(g(\hat{\lambda}, \lambda), y, \lambda) + (S(\hat{\lambda}, \mu), y - g(\hat{\lambda}, \lambda)) \geq 0, \quad \forall y \in K
\] (3.17)

has a unique solution.

**4. Sensitivity analysis**

In this section, we shall study the sensitivity of problems (3.1) and (3.2) by using the alternative fixed-point formulation given in Lemma 2.3.

**Theorem 4.1.** Let \( g(\cdot, \lambda), F(\cdot, \cdot, \lambda), T(\cdot, \mu), S(\cdot, \mu), N(\cdot, \cdot, \mu) \) and \( \eta(\cdot, \cdot, \mu) \) be continuous (resp. uniformly continuous or Lipschitz continuous). Suppose that the conditions of Theorem 3.1 hold. Then the solution \( z(\lambda, \mu) \) of problem (3.2) is continuous (resp. uniformly continuous or Lipschitz continuous) from \( \Lambda \times \Omega \) into \( E \). If in addition the mapping \( \lambda \mapsto F(\cdot, \cdot; \lambda) \) is continuous (resp. uniformly continuous or Lipschitz continuous), then the solution \( x(\lambda, \mu) \) of problem (3.1) is continuous (resp. uniformly continuous or Lipschitz continuous) from \( \Lambda \times \Omega \) into \( E \).

**Proof.** For any given \( z(\lambda, \mu), \hat{z} = z(\hat{\lambda}, \hat{\mu}) \in E, (\lambda, \mu), (\hat{\lambda}, \hat{\mu}) \in \Lambda \times \Omega, \varepsilon > 0 \) and \( z(\lambda, \mu) \in G(z, \lambda, \mu) \), there exist \( u \in T(x, \mu) \) such that \( z(\lambda, \mu) = g(x, \lambda) - \lambda N(u, S(x, \mu), \mu) \). Since \( T(x, \mu) \in CB(X) \), where \( \hat{x} = x(\hat{\lambda}, \hat{\mu}) \), it follows from (3.3) and Nadler [15] that there exist \( \hat{v} \in T(\hat{x}, \mu) \) such that
\[
\|u - \hat{v}\| \leq (1 + \varepsilon)H(T(x, \mu), T(\hat{x}, \mu)).
\] (4.1)

For \( \hat{v} \in T(\hat{x}, \mu) \), since \( T(\hat{x}, \mu) \in CB(X) \), there exist \( \hat{u} \in T(\hat{x}, \hat{\mu}) \) such that
\[
\|\hat{u} - \hat{v}\| \leq (1 + \varepsilon)H(T(\hat{x}, \hat{\mu}), T(\hat{x}, \mu)).
\] (4.2)
Let \( \hat{z} = z(\hat{\lambda}, \hat{\mu}) = g(\hat{x}, \hat{\lambda}) - \lambda N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu}) \). Then \( z(\hat{\lambda}, \hat{\mu}) \in G(\hat{z}, \hat{\lambda}, \hat{\mu}) \). Thus, for all \((\lambda, \mu), (\hat{\lambda}, \hat{\mu}) \in \Lambda \times \Omega\), it follows from (4.1), Theorem 3.1 and (4.2) that

\[
\|z(\lambda, \mu) - z(\hat{\lambda}, \hat{\mu})\| \leq \|g(x, \lambda) - \rho N(u, S(x, \mu), \mu) - [g(\hat{x}, \hat{\lambda}) - \rho N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})]\|
\]

\[
\leq \|g(x, \lambda) - \rho N(u, S(x, \mu), \mu) - [g(\hat{x}, \hat{\lambda}) - \rho N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})]\|
\]

\[
+ \|g(\hat{x}, \hat{\lambda}) - \rho N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu}) - [g(\hat{x}, \hat{\lambda}) - \rho N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})]\|
\]

\[
\leq L(\varepsilon) \|z(\lambda, \mu) - z(\hat{\lambda}, \hat{\mu})\| + \|g(x, \lambda) - g(\hat{x}, \hat{\lambda})\|
\]

\[
+ \rho \|N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu}) - N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})\|
\]

\[
+ \rho \|N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu}) - N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})\|
\]

\[
\leq L(\varepsilon) \|z(\lambda, \mu) - z(\hat{\lambda}, \hat{\mu})\| + \|g(x, \lambda) - g(\hat{x}, \hat{\lambda})\|
\]

\[
+ \rho \|N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu}) - N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})\|
\]

\[
+ \rho \|N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu}) - N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})\|
\]

\[
+ \rho \|N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu}) - N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})\|,
\]

where

\[
L(\varepsilon) = \frac{\tau \left[ k + \rho \kappa \gamma (1 + \varepsilon) + \sqrt{1 - 2\rho \xi + \rho^2 \varepsilon^2 \sigma^2} \right]}{\delta (1 - k)},
\]

\[
k = \sqrt{1 - 2\alpha + \beta^2}.
\]

Let \( \varepsilon \to 0 \). Then from (3.5), we have

\[
L(\varepsilon) \to L = \frac{\tau \left[ k + \rho \kappa \gamma (1 + \varepsilon) + \sqrt{1 - 2\rho \xi + \rho^2 \varepsilon^2 \sigma^2} \right]}{\delta (1 - k)} < 1,
\]

and so (4.3) implies

\[
\|z(\lambda, \mu) - z(\hat{\lambda}, \hat{\mu})\|
\]

\[
\leq \frac{1}{1 - L} \left[ \|g(\hat{x}, \hat{\lambda}) - g(\hat{x}, \hat{\lambda})\| + \rho \kappa (1 + \varepsilon) H(T(\hat{x}, \mu), T(\hat{x}, \hat{\mu}))
\]

\[
+ \rho \varepsilon \|S(\hat{x}, \mu) - S(\hat{x}, \hat{\mu})\| + \rho \|N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu}) - N(\hat{u}, S(\hat{x}, \hat{\mu}), \hat{\mu})\|.
\]

(4.4)

Thus, the proof of the first part of the desired result is completed.

On the other hand, we have

\[
\|x(\lambda, \mu) - x(\hat{\lambda}, \hat{\mu})\|
\]

\[
\leq \|x(\lambda, \mu) - x(\hat{\lambda}, \hat{\mu}) - (g(x(\lambda, \mu), \lambda) - g(\hat{x}, \hat{\lambda}))\| + \|g(x(\lambda, \mu), \lambda) - g(\hat{x}, \hat{\lambda})\|
\]

\[
\leq k \|x(\lambda, \mu) - \hat{x}\| + \|g(x(\lambda, \mu), \lambda) - g(\hat{x}, \hat{\lambda})\|,
\]
Theorem 4.2. Let \( E \) be Banach spaces. Since \( g(x(\lambda, \mu), \lambda) = J^F_{\rho(\cdot, \cdot)}(z(\lambda, \mu)) \) and \( g(\hat{x}, \hat{\lambda}) = J^F_{\rho(\cdot, \cdot)}(?)(\hat{z}) \), we get

\[
\|x(\lambda, \mu) - x(\hat{\lambda}, \hat{\mu})\| \leq k\|x(\lambda, \mu) - \hat{x}\| + \|J^F_{\rho(\cdot, \cdot)}(z(\lambda, \mu)) - J^F_{\rho(\cdot, \cdot)}(\hat{z})\| \\
+ \|J^F_{\rho(\cdot, \cdot)}(\hat{z}) - J^F_{\rho(\cdot, \cdot)}(\hat{\lambda})(\hat{z})\| + \|g(\hat{x}, \hat{\lambda}) - g(\hat{x}, \hat{\lambda})\|.
\]

Since \( k < 1 \), (4.5) implies

\[
\|x(\lambda, \mu) - x(\hat{\lambda}, \hat{\mu})\| \leq \frac{\tau}{\delta(1 - k)} \|z(\lambda, \mu) - \hat{z}\| \\
+ \frac{1}{(1 - k)} \|J^F_{\rho(\cdot, \cdot)}(\hat{z}) - J^F_{\rho(\cdot, \cdot)}(\hat{\lambda})(\hat{z})\| + \|g(\hat{x}, \hat{\lambda}) - g(\hat{x}, \hat{\lambda})\|.
\]

It follows from (4.4) and (4.6) that

\[
\|x(\lambda, \mu) - x(\hat{\lambda}, \hat{\mu})\| \\
\leq \frac{\tau}{\delta(1 - k)(1 - L)} \|g(\hat{x}, \hat{\lambda}) - g(\hat{x}, \hat{\lambda})\| + \rho\kappa(1 + \varepsilon)H(T(\hat{x}, \mu), T(\hat{x}, \hat{\mu})) \\
+ \rho\varepsilon S(\hat{x}, \mu) - S(\hat{x}, \hat{\mu}) + \rho\|N(\hat{\mu}, S(\hat{x}, \hat{\mu}), \mu) - N(\hat{\mu}, S(\hat{x}, \hat{\mu}), \hat{\mu})\| \\
+ \frac{1}{(1 - k)} \|J^F_{\rho(\cdot, \cdot)}(\hat{z}) - J^F_{\rho(\cdot, \cdot)}(\hat{\lambda})(\hat{z})\| + \|g(\hat{x}, \hat{\lambda}) - g(\hat{x}, \hat{\lambda})\|
\]

\[
= \frac{1}{(1 - k)} \|J^F_{\rho(\cdot, \cdot)}(\hat{z}) - J^F_{\rho(\cdot, \cdot)}(\hat{\lambda})(\hat{z})\| + \frac{\tau + \delta(1 - L)}{\delta(1 - k)(1 - L)} \|g(\hat{x}, \hat{\lambda}) - g(\hat{x}, \hat{\lambda})\|
\]

\[
+ \frac{\rho\tau}{\delta(1 - k)(1 - L)} \kappa(1 + \varepsilon)H(T(\hat{x}, \mu), T(\hat{x}, \hat{\mu})) + \varepsilon\|S(\hat{x}, \mu) - S(\hat{x}, \hat{\mu})\|
\]

\[
+ \|N(\hat{\mu}, S(\hat{x}, \hat{\mu}), \mu) - N(\hat{\mu}, S(\hat{x}, \hat{\mu}), \hat{\mu})\|.
\]

This completes the proof. \( \square \)

From Theorem 4.1, we have the following result.

**Theorem 4.2.** Let \( E, F, g, S, N, \) and \( \eta \) be the same as in Theorem 3.2. Assume that \( g, S \) is continuous (resp. uniformly continuous or Lipschitz continuous) with respect to the second argument and the mapping \( \lambda \mapsto J^F_{\rho(\cdot, \cdot)} \) is continuous (resp. uniformly continuous or Lipschitz continuous). If conditions (3.16) hold, then the solution \( x(\lambda, \mu) \) of problem (3.17) is continuous (resp. uniformly continuous or Lipschitz continuous) from \( \Lambda \times \Omega \) into \( E \).

**Remark 4.1.** If \( \lambda = \mu \), then problem (3.17) is equivalent to problem (11) in Moudafi [13] and so we recover the main result of Moudafi [13], i.e., Theorem 4.1 can be degenerated to many known results of (generalized) variational inequalities as special cases (see, for example, [5,6,13,17,18] and the references therein).

**References**


