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Anomaly cancellation and conformality in quiver gauge theories

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Abstract

Abelian quiver gauge theories provide non-supersymmetric candidates for the conformality approach to physics beyond the standard model. Written as $\mathcal{N} = 0$, $U(N)^n$ gauge theories, however, they have mixed $U(1)_p U(1)_q^2$ and $U(1)_p SU(N)_q^2$ triangle anomalies. It is shown how to construct explicitly a compensatory term $\Delta \mathcal{L}_{\text{comp}}$ which restores gauge invariance of $\mathcal{L}_{\text{eff}} = \mathcal{L} + \Delta \mathcal{L}_{\text{comp}}$ under $U(N)^n$. It can lead to a negative contribution to the $U(1)$ β-function and hence to one-loop conformality at high energy for all dimensionless couplings.
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1. Introduction

One alternative to supersymmetry and grand unification is to postulate conformality, four-dimensional conformal invariance at high energy, for the non-gravitational extension of the standard model. Although much less vigorously studied than supersymmetry, the conformality approach suggested [1] in 1998 has made considerable progress. Models which contain the standard model fields have been constructed [2] and a model which grand unifies at about 4 TeV [3] has been examined.

Such models are inspired by the AdS/CFT correspondence [4,5] specifically based on compactification of the IIB superstring on the Abelian orbifold $AdS_5 \times S^5/Z_n$ with N coalescing parallel D3 branes. A model is specified by N and by the embedding $Z_n \subset SU(4)$ which is characterized by integers A_m ($m = 1, 2, 3, 4$) which specify how the **4** of $SU(4)$ transforms under Z_n. Only three of the A_m are independent because of the $SU(4)$ requirement that $\sum_m A_m = 0 \pmod{n}$. The number of vanishing A_m is the number $\mathcal{N}$ of surviving supersymmetries. Here we focus on the non-supersymmetric $\mathcal{N} = 0$ case.

In [6], the original speculation [1] that such models may be conformal has been refined to exclude models which contain scalar fields transforming as adjoint representations because only if all scalars are in bifundamentals are there chiral fermions and, also only if all scalars are in bifundamentals, the one-loop quadratic divergences cancel in the scalar propagator. We regard it as encouraging that these two desirable properties select the same subset of models.

Another phenomenological encouragement stems from the observation [7] that the standard model representations for the chiral fermions can all be accommodated in bifundamentals of $SU(3)^3$ and can appear naturally in the conformality approach.

In the present Letter we address the issue of triangle anomalies. Although the purely non-Abelian anomalies involving $SU(N)^3$ subgroups of the $U(N)^n$ gauge group are cancelled, there do survive triangle anomalies of the types $U(1)_p U(1)_q^2$ and $U(1)_p SU(N)_q^2$. Since the original superstring is anomaly free, one expects such anomalies to be cancelled. This cancellation is well understood [8] in terms of the closed string axions coupling to $F\tilde{F}$. Here we shall construct a compensatory term $\Delta \mathcal{L}_{\text{comp}}$

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which is non-polynomial in the bifundamental scalars and which when added to the gauge Lagrangian $\mathcal{L}$ gives rise to an effective Lagrangian $\mathcal{L}_{\text{eff}} = \mathcal{L} + \Delta\mathcal{L}_{\text{comp}}$ which is $U(N)^n$ gauge invariant.

In the next section, we shall discuss the anomaly cancellation by a compensatory term. The following section explains the explicit construction. There is then a treatment of the evolution of the $U(1)$ couplings and finally there is some discussion.

2. Anomaly cancellation by a compensatory term

The Lagrangian for the non-supersymmetric Z_n theory can be written in a convenient notation which accommodates simultaneously both adjoint and bifundamental scalars as

$$
\begin{aligned}
\mathcal{L} = {} & -\frac{1}{4} F^{ab}_{\mu\nu;r,r} F^{ba}_{\mu\nu;r,r} + i\bar{\lambda}^{ab}_{r+A_4,r} \gamma^\mu D_\mu \lambda^{ba}_{r,r+A_4} + 2D_\mu \Phi^{ab\dagger}_{r+a_i,r} D_\mu \Phi^{ba}_{r,r+a_i} + i\bar{\Psi}^{ab}_{r+A_m,r} \gamma^\mu D_\mu \Psi^{ba}_{r,r+A_m} \\
& - 2ig\big[\bar{\Psi}^{ab}_{r,r+A_i} P_L \lambda^{bc}_{r+A_i,r+A_i+A_4} \Phi^{\dagger ca}_{r+A_i+A_4,r} - \bar{\Psi}^{ab}_{r,r+A_i} P_L \Phi^{\dagger bc}_{r+A_i,r-A_4} \lambda^{ca}_{r-A_4,r}\big] \\
& - \sqrt{2} ig\epsilon_{ijk}\big[\bar{\Psi}^{ab}_{r,r+A_i} P_L \Psi^{bc}_{r+A_i,r+A_i+A_j} \Phi^{ca}_{r-A_k-A_4,r} - \bar{\Psi}^{ab}_{r,r+A_i} P_L \Phi^{bc}_{r+A_i,r+A_i+A_k+A_4} \Psi^{ca}_{r-A_j,r}\big] \\
& - g^2\big(\Phi^{ab}_{r,r+a_i} \Phi^{\dagger bc}_{r+a_i,r} - \Phi^{\dagger ab}_{r,r-a_i} \Phi^{bc}_{r-a_i,r}\big)\big(\Phi^{cd}_{r,r+a_j} \Phi^{\dagger da}_{r+a_j,r} - \Phi^{\dagger cd}_{r,r-a_j} \Phi^{da}_{r-a_j,r}\big) \\
& + 4g^2\big(\Phi^{ab}_{r,r+a_i} \Phi^{\dagger bc}_{r+a_i,r+a_i+a_j} \Phi^{cd}_{r+a_i+a_j,r+a_j} \Phi^{\dagger da}_{r+a_j,r} - \Phi^{ab}_{r,r+a_i} \Phi^{bc}_{r+a_i,r+a_i+a_j} \Phi^{\dagger cd}_{r+a_i+a_j,r+a_i} \Phi^{\dagger da}_{r+a_i,r}\big),
\end{aligned}
\tag{1}
$$

where $\mu, \nu = 0, 1, 2, 3$ are Lorentz indices; $a, b, c, d = 1$ to N are $U(N)^n$ group labels; $r = 1$ to n labels the node of the quiver diagram (when the two node subscripts are equal it is an adjoint plus singlet and the two superscripts are in the same $U(N)$: when the two node subscripts are unequal it is a bifundamental and the two superscript labels transform under different $U(N)$ groups); a_i ($i = \{1, 2, 3\}$) label the first three of the **6** of $SU(4)$; A_m ($m = \{1, 2, 3, 4\}) = (A_i, A_4)$ label the **4** of $SU(4)$. By definition A_4 denotes an arbitrarily-chosen fermion (λ) associated with the gauge boson, similarly to the notation in the $\mathcal{N} = 1$ supersymmetric case. Recall that $\sum_{m=1}^{m=4} A_m = 0 \pmod{n}$.

As mentioned above we shall restrict attention to models where all scalars are in bifundamentals which requires all a_i to be non-zero. Recall that $a_1 = A_2 + A_3$, $a_2 = A_3 + A_1$, $a_3 = A_1 + A_2$.

The Lagrangian in Eq. (1) is classically $U(N)^p$ gauge invariant. There are, however, triangle anomalies of the $U(1)_p U(1)_q^2$ and $U(1)_p SU(N)_q^2$ types. Making gauge transformations under the $U(1)_r$ ($r = 1, 2, \ldots, n$) with gauge parameters Λ_r leads to a variation

$$
\delta\mathcal{L} = -\frac{g^2}{4\pi^2} \sum_{p=1}^{p=n} A_{pq} F^{(p)}_{\mu\nu} \tilde{F}^{(p)\mu\nu} \Lambda_q
\tag{2}
$$

which defines an $n \times n$ matrix A_{pq} which is given by

$$
A_{pq} = \text{Tr}\big(Q_p Q_q^2\big),
\tag{3}
$$

where the trace is over all chiral fermion links and Q_r is the charge of the bifundamental under $U(1)_r$. We shall adopt the sign convention that **N** has $Q = +1$ and **N*** has $Q = -1$.

It is straightforward to write A_{pq} in terms of Kronecker deltas because the content of chiral fermions is

$$
\sum_{m=1}^{m=4} \sum_{r=1}^{r=n} (\mathbf{N}_r, \mathbf{N}^*_{r+A_m}).
\tag{4}
$$

This implies that the antisymmetric matrix A_{pq} is explicitly

$$
A_{pq} = -A_{qp} = \sum_{m=1}^{m=4} (\delta_{p,q-A_m} - \delta_{p,q+A_m}).
\tag{5}
$$

Now we are ready to construct $\mathcal{L}^{(1)}_{\text{comp}}$, the compensatory term. Under the $U(1)_r$ gauge tansformations with gauge parameters Λ_r we require that

$$
\delta\mathcal{L}^{(1)}_{\text{comp}} = -\delta\mathcal{L} = +\frac{g^2}{4\pi^2} \sum_{p=1}^{p=n} A_{pq} F^{(p)}_{\mu\nu} \tilde{F}^{(p)\mu\nu} \Lambda_q.
\tag{6}
$$

To accomplish this property, we construct a compensatory term in the form[2]

$$\mathcal{L}_{\text{comp}}^{(1)} = \frac{g^2}{4\pi} \sum_{p=1}^{p=n} \sum_{k} B_{pk} \, \text{Im} \, \text{Tr} \ln\left(\frac{\Phi_k}{v}\right) F_{\mu\nu}^{(p)} \tilde{F}^{(p)\mu\nu}, \tag{7}$$

where $\sum_k$ runs over scalar links. We believe this *form* for the compensatory term to be unique[3] because $\mathcal{L}_{\text{comp}}^{(1)}$ must be invariant under $SU(N)^n$. To see that $\mathcal{L}_{\text{comp}}^{(1)}$ of Eq. (7) has such invariance rewrite $\text{Tr} \ln \equiv \exp \det$ and note that the $SU(N)$ matrices have unit determinant. It is inconceivable that any other non-trivial function of the bifundamental, other than a closed loop of links in the quiver diagram, has the full $SU(N)^n$ invariance but a closed loop, unlike Eq. (7), is $U(N)^n$ invariant.

We note en passant that one cannot take the $v \to 0$ limit in Eq. (7); the chiral anomaly enforces a breaking of conformal invariance.

Explicit construction of the matrix B_{pk} will the subject of the subsequent section. But first we investigate the transformation properties of the "$\text{Im} \, \text{Tr} \ln(\Phi/v)$" term in Eq. (7). Define a matrix C_{kq} by

$$\delta\left(\sum_{p=1}^{p=n} \sum_{k} \text{Im} \, \text{Tr} \ln\left(\frac{\Phi_k}{v}\right)\right) = \sum_{q=1}^{q=n} C_{kq} \Lambda_q \tag{8}$$

where upon Eq. (6) will be satisfied if the matrix B_{pk} satisfies $A = BC$. The inversion $B = AC^{-1}$ is non-trivial because C is singular but C_{kq} can be written in terms of Kronecker deltas by noting that the content of complex scalar fields in the model is

$$\sum_{i=1}^{i=3} \sum_{r=1}^{r=n} (\mathbf{N}_r, \mathbf{N}_{r\pm a_i}^*) \tag{9}$$

which implies that the matrix C_{kq} must be of the form

$$C_{kq} = 3\delta_{kq} - \sum_{i} \delta_{k+a_i,q}. \tag{10}$$

The $U(1)_p SU(N)_q^2$ triangle anomalies necessitate the addition of a second compensatory term $\mathcal{L}_{\text{comp}}^{(2)}$. The derivation of $\mathcal{L}_{\text{comp}}^{(2)}$ is similar to, but algebraically simpler than, that for $\mathcal{L}_{\text{comp}}^{(1)}$. Under $U(1)_r$ with gauge parameter Λ_r and $SU(N)_s$ gauge transformations the variation in $\mathcal{L}$ of Eq. (1) is

$$\delta\mathcal{L} = -\frac{g^2}{4\pi^2} \sum_{p=1}^{p=n} A_{pq} i' F_{\mu\nu\alpha_p}^{(p)\beta_p} \tilde{F}_{\beta_p}^{(p)\mu\nu\alpha_p} \Lambda_q \tag{11}$$

which defines an $n \times n$ matrix A'_{pq} which is given by

$$A'_{pq} = \text{Tr}(Q_p n_q) \tag{12}$$

where the trace is over all chiral fermion links, Q_r is the charge of the bifundamental under $U(1)_r$ and n_q is the number of fundamentals and antifundamentals of $SU(N)_q$ corresponding to all fermionic links between nodes p and q. As before, we adopt the sign convention that $\mathbf{N}$ has $Q = +1$ and $\mathbf{N}^*$ has $Q = -1$.

It is straightforward to write A'_{pq} in terms of Kronecker deltas as

$$A'_{pq} = -A'_{qp} = \sum_{m=1}^{m=4} (-\delta_{p,q-A_m} + \delta_{p,q+A_m}) \tag{13}$$

$\mathcal{L}_{\text{comp}}^{(2)}$, the compensatory term for the $U(1)_p SU(N)_q^2$ triangle anomalies is

$$\mathcal{L}_{\text{comp}}^{(2)} = \frac{g^2}{4\pi} \sum_{p=1}^{p=n} \sum_{k} B'_{pk} \, \text{Im} \, \text{Tr} \ln\left(\frac{\Phi_k}{v}\right) F_{\mu\nu\alpha_p}^{(p)\beta_p} \tilde{F}_{\beta_p}^{(p)\mu\nu\alpha_p} \tag{14}$$

where $\sum_k$ runs over scalar links.

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[2] For a related construction in a different context, see [9].
[3] Although the general form is unique, there can be a technical ambiguity in the matrix B to be discussed below.

Explicit construction of the matrix B'_{pk} in $\mathcal{L}^{(2)}_{\text{comp}}$ is more straightforward than B_{pk} in $\mathcal{L}^{(1)}$ because when we define a matrix C'_{kq} by the variation under mixed Abelian–non-Abelian gauge transformations

$$\delta\left(\sum_{p=1}^{p=n}\sum_{k} \operatorname{Im} \operatorname{Tr} \ln\left(\frac{\Phi_k}{v}\right)\right) = \sum_{q=1}^{q=n} C'_{kq} \Lambda_q \tag{15}$$

we find $C'_{kq} = 3\delta_{pk}$ so B'_{pk} in Eq. (14) is $B'_{pq} = \frac{1}{3}A'_{pq}$ with A'_{pq} defined by Eq. (13).

3. Explicit construction of the matrix B in $\mathcal{L}^{(1)}_{\text{comp}}$

Construction of the anomaly compensatory term $\mathcal{L}^{(1)}_{\text{comp}}$ of Eq. (7) has been reduced to the explicit construction of the matrix B_{pk}. Although $B = AC^{-1}$ is inadequate because $\text{Rank}(C) < n$, a necessary and sufficient condition for the existence of B is $\text{Rank}(A) \leqslant \text{Rank}(C)$. Proving this in general would be one approach but the large number of special cases will make the proof lengthy. Of course, we strongly suspect that the matrix B must exist from indirect string theory arguments [8] but we shall convince the reader directly by explicit construction of B in two extremes which we call the totally degenerate and the totally non-degenerate cases respectively.

Given the form of A_{pq} in Eq. (5) and of C_{kq} in Eq. (10), it is irresistible to make a corresponding ansatz for B_{pk}

$$B_{pk} = \sum_{\eta} C_\eta \delta_{p,k+\eta} \tag{16}$$

and this ansatz works by setting up recursion relations for the C_η and allows explicit solution for the matrix B in *any* special case. Writing a general formula for B will now be demonstrated in two extreme cases.

3.1. Totally degenerate case

We assume $A_m = (A, A, A, -3A) \ (\text{mod}\, n)$. In this case, from Eq. (5),

$$A_{pq} = 3\delta_{p,q-A} - 3\delta_{p,q+A} + \delta_{p,q+3A} - \delta_{p,q-3A} \tag{17}$$

and from Eq. (10)

$$C_{kq} = 3(\delta_{kq} - \delta_{k+2A,q}). \tag{18}$$

Using Eq. (16) and comparing coefficients gives the series of recursion relations

$$3C_{-A} - 3C_A = 3, \tag{19}$$
$$3C_A - 3C_{3A} = -3, \tag{20}$$
$$3C_{-3A} - 3C_{-A} = -1, \tag{21}$$
$$3C_{3A} - 3C_{5A} = +1, \tag{22}$$
$$3C_{-5A} - 3C_{-3A} = 0, \tag{23}$$
$$3C_{5A} - 3C_{7A} = 0, \tag{24}$$

and so on, with solution $C_A = -2/3$, $C_{-A} = C_{3A} = 1/3$ and all other $C_A = 0$. The explicit B matrix is thus

$$B_{pk} = \frac{1}{3}(-2\delta_{p,k+A} + \delta_{p,k+3A} + \delta_{p,k-A}). \tag{25}$$

From Eqs. (17), (25), (18) one confirms $A = BC$.

3.2. Totally non-degenerate case

At an opposite extreme we may assume that

$$\pm A_m \quad \text{and} \quad (a_i \pm A_m) \text{ are all non-degenerate integers } (\text{mod}\, n). \tag{26}$$

Assumption (26) requires $n \gg 1$ and so is not a physical case. In this limit the recursion relations become

$$3C_{-A_m} - \sum_i C_{a_i - A_m} = +1, \tag{27}$$

$$3C_{A_m} - \sum_i C_{a_i + A_m} = -1. \tag{28}$$

Because the A_m enter symmetrically in the model, one can put $C_{A_m} = x$ and $C_{-A_m} = y$ both independent of m. This yields $C_{a_i + A_m} = (x + \frac{1}{3})$ and $C_{a_i - A_m} = (y - \frac{1}{3})$. Comparing coefficients of Kronecker deltas gives $x = -2/9$ and $y = +1/9$ and hence an explicit form for B_{pk} by substitution in Eq. (16).

3.3. Intermediate cases

When there are some degeneracies which violate assumption (26), there are too many special cases to permit any succinct general formula. Nevertheless, one can find fairly easily the explicit B matrix for *any* specific model, as we have done for dozens of cases, using MATHEMATICA software.

3.4. Technical non-uniqueness

As mentioned in an earlier footnote, although the form of $\mathcal{L}^{(1)}_{\text{comp}}$ in Eq. (7) is unique the matrix B can have technical non-uniqueness which is best illustrated by a specific example.

Take $n = 6$ and the totally degenerate example $A_m = (1, 1, 1, 3)$. Following the analysis given earlier one finds for C_{kq}

$$
C = \begin{pmatrix}
3 & 0 & -3 & 0 & 0 & 0 \\
0 & 3 & 0 & -3 & 0 & 0 \\
0 & 0 & 3 & 0 & -3 & 0 \\
0 & 0 & 0 & 3 & 0 & -3 \\
-3 & 0 & 0 & 0 & 3 & 0 \\
0 & -3 & 0 & 0 & 0 & 3
\end{pmatrix},
\tag{29}
$$

while for the matrix B_{pk}

$$
3B = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & -2 \\
-2 & 0 & 1 & 0 & 1 & 0 \\
0 & -2 & 0 & 1 & 0 & 1 \\
1 & 0 & -2 & 0 & 1 & 0 \\
0 & 1 & 0 & -2 & 0 & 1 \\
1 & 0 & 1 & 0 & -2 & 0
\end{pmatrix}.
\tag{30}
$$

Multiplication of BC gives the required

$$
A = \begin{pmatrix}
0 & 3 & 0 & 0 & 0 & -3 \\
-3 & 0 & 3 & 0 & 0 & 0 \\
0 & -3 & 0 & 3 & 0 & 0 \\
0 & 0 & -3 & 0 & 3 & 0 \\
0 & 0 & 0 & -3 & 0 & 3 \\
3 & 0 & 0 & 0 & -3 & 0
\end{pmatrix}.
\tag{31}
$$

There is, however, the following matrix $\hat{B}$

$$
\hat{B} = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\tag{32}
$$

which has the property that $\hat{B}C = 0$. This means there is a one parameter family $B' = B + \alpha \hat{B}$ with α a continuous parameter which can be used in $\mathcal{L}_{\text{comp}}$. However, this also means that the term

$$
\hat{\mathcal{L}} = \frac{g^2}{4\pi} \sum_{p=1}^{p=n} \sum_k \hat{B}_{pk} \operatorname{Im} \operatorname{Tr} \ln\left(\frac{\Phi_k}{v}\right) F^{(p)}_{\mu\nu} \tilde{F}^{(p)\mu\nu}
\tag{33}
$$

is $U(N)^n$ invariant and therefore need not be added for purposes of anomaly cancellation.

The non-uniqueness of the matrix B under $B \to B + \alpha \hat{B}$ has no immediate physical interpretation but it may suggest an undiscovered residual symmetry.

4. String theory

In [10] the anomaly A_{pq} is written in a factorized form $A = TU$ following from the closed string axion exchange using the Green–Schwarz mechanism so here we compare the two factorized expressions $A = BC$ and $A = TU$ to become convinced that there is no connection.

The expression from [10] is

$$A_{pq} = \sum_{l=1}^{l=n} T_{pl} U_{lq} V_l \tag{34}$$

where

$$T_{pl} = \exp\left(\frac{2\pi i pl}{n}\right), \qquad U_{lq} = \exp\left(\frac{-2\pi i lq}{n}\right) \tag{35}$$

and

$$V_l = \prod_{i=1}^{i=3} \sin\left(\frac{\pi l a_i}{n}\right). \tag{36}$$

Let us take the example of $n = 6$ and $A_m = (1, 1, 1, 3)$ for which the matrices A, B and C are given above. In Eq. (34) there is an ambiguity in whether V is accommodated in T or U so let us look at three possibilities: (i) $T' = TV$, (ii) $U' = VU$ and (iii) $T'' = T\sqrt{V}$, $U'' = \sqrt{V}U$. The factorizing matrices are then, up to overall normalization which has no effect on the matrix textures, as follows. We define $\alpha = \exp(i\pi/3)$.

$$A = T'U = \begin{pmatrix} \alpha & \alpha^2 & 0 & \alpha & \alpha^2 & 0 \\ \alpha^2 & -\alpha & 0 & -\alpha^2 & \alpha & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ -\alpha & \alpha^2 & 0 & \alpha & -\alpha^2 & 0 \\ -\alpha^2 & -\alpha & 0 & -\alpha^2 & -\alpha & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -\alpha^2 & -\alpha & -1 & \alpha^2 & \alpha & 1 \\ -\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ \alpha^2 & -\alpha & 1 & \alpha^2 & -\alpha & 1 \\ \alpha & \alpha^2 & -1 & -\alpha & -\alpha^2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{37}$$

$$A = TU' = \begin{pmatrix} \alpha & \alpha^2 & -1 & -\alpha & -\alpha^2 & 1 \\ \alpha^2 & -\alpha & 1 & \alpha^2 & -\alpha & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ -\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\ -\alpha^2 & -\alpha & -1 & \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\alpha^2 & -\alpha & -1 & \alpha^2 & \alpha & 1 \\ -\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha^2 & \alpha & -1 & -\alpha^2 & \alpha & -1 \\ -\alpha & -\alpha^2 & 1 & \alpha & \alpha^2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{38}$$

$$A = T''U'' = \begin{pmatrix} \alpha & \alpha^2 & 0 & -i\alpha & -i\alpha^2 & 0 \\ \alpha^2 & -\alpha & 0 & i\alpha^2 & -i\alpha & 0 \\ -1 & 1 & 0 & i & -i & 0 \\ -\alpha & \alpha^2 & 0 & -i\alpha & i\alpha^2 & 0 \\ -\alpha^2 & -\alpha & 0 & i\alpha^2 & i\alpha & 0 \\ 1 & 1 & 0 & i & i & 0 \end{pmatrix} \begin{pmatrix} -\alpha^2 & -\alpha & -1 & \alpha^2 & \alpha & 1 \\ -\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i\alpha^2 & -i\alpha & i & i\alpha^2 & -i\alpha & i \\ i\alpha & i\alpha^2 & -i & -i\alpha & -i\alpha^2 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{39}$$

By comparing the matrix textures in Eqs. (37)–(39) with those for matrices C and B in Eqs. (29), (30) we see that the factorization of the anomaly matrix is not simply related. The factorization in Eqs. (37)–(39) follows from the physical requirement of factorization at the closed string axion pole in a string tree diagram. There is no similar requirement that mandates our factorization $A = BC$ but these matrices have simple textures so the factorization is not surprising.

In particular, the field theoretical mechanism of anomaly cancellation discussed here has no connection to the string theoretical Green–Schwarz mechanism.

5. Evolution of $U(1)$ gauge couplings

In the absence of the compensatory term, the two independent $U(N)^n$ gauge couplings g_N for $SU(N)$ and g_1 for $U(1)$ are taken to be equal $g_N(\mu_0) = g_1(\mu_0)$ at a chosen scale, e.g. $\mu_0 = 4$ TeV [3,11], to enable cancellation of quadratic divergences [6]. Note that the n $SU(N)$ couplings $g_N^{(p)}$ are equal by the overall Z_n symmetry, as are the n $U(1)$ couplings $g_1^{(p)}$, $1 \leqslant p \leqslant n$.

As one evolves to higher scales $\mu > \mu_0$, the renormalization group beta function β_N for $SU(N)$ vanishes $\beta_N = 0$ at least at one-loop level so the $g_N(\mu)$ can behave independent of the scale as expected by conformality. On the other hand, the beta function β_1

for $U(1)$ is positive definite in the unadorned theory, given at one loop by, in the notation of [12]

$$b_1 = \frac{11N}{48\pi^2},\tag{40}$$

where N is the number of colors. The corresponding coupling satisfies

$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(M)} + 8\pi b_1 \ln\left(\frac{M}{\mu}\right)\tag{41}$$

so the Landau pole, putting $\alpha(\mu) = 0.1$ and $N = 3$, occurs at

$$\frac{M}{\mu} = \exp\left[\frac{20\pi}{11}\right] \simeq 302\tag{42}$$

so for $\mu = 4$ TeV, $M \sim 1200$ TeV. The coupling becomes "strong" $\alpha(\mu) = 1$ at

$$\frac{M}{\mu} = \exp\left[\frac{18\pi}{11}\right] \simeq 171\tag{43}$$

or $M \sim 680$ TeV.

We may therefore ask whether the new term $\mathcal{L}_{\text{comp}}$ in the Lagrangian, necessary for anomaly cancellation, can solve this problem for conformality?

Indeed there is the real counterpart of Eq. (7) which has the form

$$\mathcal{L}_{\text{comp}}^{(1),\text{real}} = \frac{g^2}{4\pi} \sum_{p=1}^{p=n} \sum_k B_{pk} \operatorname{Re} \operatorname{Tr} \ln\left(\frac{\Phi_k}{v}\right) F_{\mu\nu}^{(p)} F^{(p)\mu\nu}\tag{44}$$

and this contributes to the $U(1)$ gauge propagator and to the $U(1)$ β-function. Using Eq. (25) for B_{pk}, the one-loop quadratic divergence for a bifundamental scalar loop cancels because

$$\sum_k B_{pk} = 0\tag{45}$$

which confirms the cancellation found in [6].

Since the scale v breaks conformal invariance, the matter fields acquire mass, so the one-loop diagram[4] has a logarithmic divergence proportional to

$$\int \frac{d^4p}{v^2} \left[\frac{1}{(p^2 - m_k^2)} - \frac{1}{(p^2 - m_{k'}^2)}\right] \sim -\frac{\Delta m_{kk'}^2}{v^2} \ln\left(\frac{\Lambda}{v}\right)\tag{46}$$

the sign of which depends on $\delta m_{kk'}^2 = (m_k^2 - m_{k'}^2)$.

To achieve conformality of $U(1)$, a constraint must be imposed on the mass spectrum of matter bifundamentals, viz.

$$\Delta m_{kk'}^2 \propto v^2 \left(\frac{11N}{48\pi^2}\right)\tag{47}$$

with a proportionality constant of order one which depends on the choice of model, the n of Z_n and the values chosen for A_m, $m = 1, 2, 3$. This signals how conformal invariance must be broken at the TeV scale in order that it can be restored at high energy; it is interesting that such a constraint arises in connection with an anomaly cancellation mechanism which necessarily breaks conformal symmetry.

To give an explicit model, consider the case of Z_4 and $A_m = (1, 1, 1, 1)$ treated earlier for which one finds:

$$\Delta m_{kk'}^2 = \frac{3}{2} v^2 \left(\frac{11N}{48\pi^2}\right).\tag{48}$$

In a more general model, the analog of Eq. (48) involves replacement of $\frac{3}{2}$ by a generally different coefficient derivable for each case from the coefficient B_{pk} in Eq. (7).

With such a constraint, the one-loop β_1 vanishes in addition to β_N so that the couplings $\alpha_1(\mu)$ and $\alpha_N(\mu)$ can be scale invariant for $\mu \geqslant \mu_0$.

For such conformal invariance at high energy to be maintained to higher orders of perturbation theory probably requires a global symmetry, for example the explicit form of misaligned supersymmetry recently suggested in [13].

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[4] The usual one-loop β-function is of order h^2 regarded as an expansion in Planck's constant: four propagators each $\sim h$ and two vertices each $\sim h^{-1}$ (cf. Y. Nambu, Phys. Lett. B 26 (1968) 626). The diagram considered is also $\sim h^2$ since it has three propagators, one quantum vertex $\sim h$ and an additional h^{-2} associated with $\Delta m_{kk'}^2$.

6. Discussion

It has been shown how a compensatory term $\mathcal{L}_{\text{comp}} = \mathcal{L}_{\text{comp}}^{(1)} + \mathcal{L}_{\text{comp}}^{(2)}$ can be constructed respectively to cancel the $U(1)_p U(1)_q^2$ and $U(1)_p SU(N)_q^2$ triangle anomalies in the quiver gauge theories with chiral fermions. We have emphasized the uniqueness of the form of the compensatory term from the requirements of invariance under $SU(N)^n \subset U(N)^n$.

Such a term can have phenomenological consequences. We expect v to be at the TeV scale as in [3] and $\mathcal{L}_{\text{comp}}$ reveals new non-linear coupling between the bifundamental scalars and the gauge fields expected to be significant in the TeV energy regime. Such empirical consequences merit further study.

It has further been shown that the compensatory term $\mathcal{L}_{\text{comp}}$ can lead, with a suitable mass spectrum of bifundamental matter, to vanishing one-loop β-function for the $U(1)$ gauge group, this raising the possibility of one-loop scale invariance for all dimensionless couplings which may persist at all higher loops in the presence of a global symmetry.

Acknowledgements

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