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# Large-Time Behavior of Solutions of Abstract Wave Equations

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## 1. INTRODUCTION

Consider the telegrapher's equation

$$u_{tt} + bu_t = u_{xx}, \tag{1}$$

$b$ a positive constant, with initial conditions $u(0, x) = \phi(x)$, $u_t(0, x) = 0$. The change of dependent variables $\hat{t} = \epsilon t$, $\hat{x} = \epsilon^{1/2}x$ reduces this problem to

$$\epsilon\hat{u}_{\hat{t}\hat{t}} + b\hat{u}_{\hat{t}} = \hat{u}_{\hat{x}\hat{x}}, \qquad \hat{u}\mid_{\hat{t}=0} = \hat{\phi}, \qquad \hat{u}_{\hat{t}}\mid_{\hat{t}=0} = 0,$$

where $\hat{u} = \hat{u}_\epsilon(\hat{t}, \hat{x}) = u(t, x)$, $\hat{\phi} = \hat{\phi}_\epsilon(\hat{x}) = \phi(x)$. It is well known that the solution of this problem tends, as $\epsilon \to 0+$, to the solution of

$$b\hat{U}_{\hat{t}} = \hat{U}_{\hat{x}\hat{x}}, \qquad \hat{U}\mid_{\hat{t}=0} = \hat{\phi};$$

indeed, it is known [1, among many others] that

$$\lim_{\epsilon\to 0} \sup_{-\infty<\hat{x}<\infty} \mid \hat{u}_\epsilon(\hat{t}, \hat{x}) - \hat{U}(\hat{t}, \hat{x})\mid = 0$$

for each fixed $\hat{t} > 0$. Returning to the original variables and keeping $\hat{t}$ fixed, we thus have that

$$\lim_{t\to\infty} \sup_{-\infty<x<\infty} \mid u(t, x) - U(t, x)\mid = 0,$$

where $U(t, x) \equiv \hat{U}(\epsilon t, \epsilon^{1/2}x)$ satisfies

$$bU_t = U_{xx}, \qquad U\mid_{t=0} = \phi. \tag{2}$$

Thus we have shown that, as $t \to \infty$, the solution of the telegrapher's equation (1) with the given initial data tends to the solution of the heat equation (2).

The foregoing argument depends on the special form of the initial conditions and on the special circumstance that the right-hand side of (1) is a homogeneous



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differential operator. It also depends on the particular norm chosen; for, in the $L_1$-norm it is known [1] that

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} | \hat{u}_\epsilon(\hat{t}, \hat{x}) - \hat{U}(\hat{t}, \hat{x})| \, d\hat{x} = 0,$$

which becomes, for $\hat{t}$ fixed,

$$\lim_{t \to \infty} \left(\frac{\hat{t}}{t}\right)^{1/2} \int_{-\infty}^{\infty} | u(t, x) - U(t, x)| \, dx = 0,$$

yielding no information about $\| u(t, x) - U(t, x)\|_{L_1}$ as $t \to \infty$.

The object here is to generalize the first result above so that it applies to an equation with reasonably general initial conditions and right-hand side, to an arbitrary norm, and to the mixed initial boundary value problem. Indeed, we shall examine the large-time behavior of the solution of the problem

$$u'' + bu' = A^2 u + f(t), \qquad u(0) = u_0 , \qquad u'(0) = u_1 ,$$

where $u$ takes values in a Banach space $\mathscr{B}$ and the (possibly unbounded) operator $A$ generates a $c_0$-group $T(t)$ in $\mathscr{B}$; this formulation includes the generalizations just mentioned. As a consequence of our development and known results for parabolic equations, we shall also obtain conditions guaranteeing that $u$ tends to zero as $t \to \infty$.

## 2. The Homogeneous Case

Let the closed operator $A$ generate the $c_0$-contraction group $T(t)$; we shall show that the solution of the abstract wave equation

$$u'' + bu' = A^2 u, \qquad u(0) = u_0 , \qquad u'(0) = u_1 \tag{3}$$

tends in the norm of $\mathscr{B}$ to the solution of

$$bU' = A^2 U, \qquad U(0) = u_0 + \frac{1}{b} u_1 \tag{4}$$

as $t$ increases. The appearance of the term $(1/b)\, u_1$ in the initial condition for the heat equation is perhaps to be expected on physical grounds; for if $u_0 = 0$, $u_1 \neq 0$ there is energy present in the system described by (3) which must still be accounted for by (4).

Since our concern is with the asymptotics of solutions, we shall assume that

(3) and (4) are well posed; conditions on $u_0$, $u_1$ sufficient for this may be found in [2–5]. As may be checked by tedious calculation, the solution of (3) is

$$\begin{aligned} u(t) &= \frac{1}{2} e^{-bt/2}[T(t) + T(-t)]u_0 \\ &\quad + \frac{1}{2} e^{-bt/2} \int_{-t}^{t} J_0(ib(t^2 - s^2)^{1/2}/2)\, T(s)\Big[u_1 + \frac{b}{2} u_0\Big]\, ds \\ &\quad - \frac{1}{4} ibe^{-bt/2} \int_{-t}^{t} J_1(ib(t^2 - s^2)^{1/2}/2)\, t(t^2 - s^2)^{-1/2}\, T(s)\, u_0 ds \\ &= I_1 + I_2 + I_3 ; \end{aligned}$$

here $J_0$, $J_1$ are Bessel functions of the first kind. This representation is that of Hersh [6]. Since $A$ generates the $c_0$-group $T(\cdot)$, $A^2$ generates the analytic semigroup $S(\cdot)$ given by

$$S(t)a = \frac{1}{2(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/4t}\, T(s)a\, ds$$

[2, p. 92]. It is then immediate that the solution of (4) can be written as

$$U(t) = \frac{b^{1/2}}{2(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-bs^2/4t}\, T(s)u_0\, ds.$$

Since $\| T(t)\| \leqslant 1$ and $b > 0$ by assumption, it is evident that $\| I_1 \| \to 0$ as $t \to \infty$. We thus have to show that $\| I_2 + I_3 - U \| \to 0$ as $t \to \infty$; we shall first show that

$$I_2 \to K_2 \equiv \frac{1}{2} (\pi bt)^{-1/2} \int_{-\infty}^{\infty} e^{-bs^2/4t}\, T(s) \left[u_1 + \frac{b}{2} u_0\right] ds. \tag{5}$$

Using $k$ as a generic constant, we have

$$\begin{aligned} \| I_2 - K_2 \| &\leqslant kt^{-1/2} \int_{-\infty}^{-t} e^{-bs^2/4t}\, ds + kt^{-1/2} \int_{t}^{\infty} e^{-bs^2/4t}\, ds \\ &\quad + k \int_{-t}^{t} | e^{-bt/2} J_0(ib(t^2 - s^2)^{1/2}/2) - (\pi bt)^{-1/2} e^{-bs^2/4t} |\, ds. \end{aligned}$$

The first two of these integrals may be written in terms of Erfc, whence it is obvious that they tend to zero as $t \to \infty$. We bound the third integral by a constant times the sum of three integrals:

$$\int_{-t}^{-t^{3/4}} | \quad | \, ds + \int_{-t^{3/4}}^{t^{3/4}} | \quad | \, ds + \int_{t^{3/4}}^{t} | \quad | \, ds = I_{21} + I_{22} + I_{23} .$$

For $I_{23}$ we have

$$I_{23} \leqslant \int_{t^{3/4}}^{t} | e^{-bt/2} J_0(ib(t^2 - s^2)^{1/2}/2)| \, ds + (\pi bt)^{-1/2} \int_{t^{3/4}}^{\infty} e^{-bs^2/4t} \, ds; \qquad (6)$$

the second of these can be written as

$$2\pi^{-1/2} b^{-1} \int_{b^{1/2}t^{1/4}/2}^{\infty} e^{-u^2} \, du$$

which tends to zero as $t \to \infty$.

For large positive $x$ we have [7, p. 373] that

$$J_0(ix) = (2\pi x)^{-1/2} e^x [1 + O(1/x)], \qquad (7)$$

where $| O(1/x)| \leqslant k/x$; since $J_0(0) = 1$, it follows that for some value of $k$ the estimate

$$J_0(ix) \leqslant ke^x$$

holds for all $x \geqslant 0$. Since in general $(a^2 - b^2)^{1/2} \leqslant a - \frac{1}{2}(b^2/a)$, we have the estimate

$$\begin{aligned} | e^{-bt/2} J_0(ib(t^2 - s^2)^{1/2}/2)| &\leqslant ke^{-bt/2} e^{b(t^2-s^2)^{1/2}/2} \\ &\leqslant ke^{-bs^2/4t} \qquad (8) \end{aligned}$$

for $t^2 \geqslant s^2$. Using this, we get the following bound on the first integral of the right-hand side of (6):

$$\begin{aligned} k \int_{t^{3/4}}^{\infty} e^{-bs^2/4t} \, ds &\leqslant kt^{1/2} \int_{b^{1/2}t^{1/4}/2}^{\infty} e^{-u^2} \, du \\ &\leqslant kt^{1/2} \operatorname{Erfc}(b^{1/2}t^{1/4}/2); \end{aligned}$$

this tends to zero as $t \to \infty$ since $\operatorname{Erfc}(x) \leqslant ke^{-x^2}$. Thus we have shown that $I_{23} \to 0$ as $t \to \infty$, and an entirely similar argument shows that $I_{21} \to 0$.

It remains to show that $I_{22} \to 0$ as $t \to \infty$. Since for large $t$, $t^2 \gg t^{3/2} \geqslant s^2$, we have $(t^2 - s^2)^{1/2} \gg 1$ and so can use the asymptotic expansion (7) for $J_0$. Now $2/[b(t^2 - s^2)^{1/2}] = o(1)$ as $t \to \infty$, uniformly for $s \in (-t^{3/4}, t^{3/4})$, where $o(1)$ stands for a function which tends to zero as $t \to \infty$. Thus

$$J_0(ib(t^2 - s^2)^{1/2}/2) = \frac{e^{b(t^2-s^2)^{1/2}/2}}{(b\pi(t^2 - s^2)^{1/2})^{1/2}} [1 + o(1)].$$

Since $t^2 \gg s^2$ for $s \in (-t^{3/4}, t^{3/4})$, we can expand the square root as

$$t\left(1 - \frac{s^2}{t^2}\right)^{1/2} = t - \frac{s^2}{2t}\left(1 + \frac{s^2}{2t^2} + \cdots\right) = t - \frac{s^2}{2t}(1 + O(t^{-1/2})).$$

Hence

$$e^{-bt/2}J_0(ib(t^2 - s^2)^{1/2}/2) = (\pi bt)^{-1/2}e^{-bs^2/4t}(1 + o(1)).$$

It follows that

$$\| I_{22} \| < o(1) \int_{-\infty}^{\infty} (\pi t)^{-1/2}e^{-bs^2/4t}\, ds = o(1).$$

This establishes (5).

Since $-iJ_1(ix)$ is asymptotic to $J_0(ix)$ [7, p. 372–373], $I_3$ can be treated much as $I_2$ to show that

$$I_3 \to K_3 \equiv \frac{b^{1/2}}{4(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-bs^2/4t}\, T(s)u_0\, ds. \tag{9}$$

Indeed, the only nonobvious alteration needed is in the estimation of

$$\int_{t^{3/4}}^{t} | e^{-bt/2}J_1(ib(t^2 - s^2)^{1/2}/2)\, t(t^2 - s^2)^{-1/2} |\, ds,$$

occasioned by the singularity of the integrand at $s = t$. But by using (8) we see that this integral is bounded by

$$\int_{t^{3/4}}^{t} te^{-bs^2/4t}(t^2 - s^2)^{-1/2}\, ds < te^{-bt^{1/2}/4} \int_0^t (t^2 - s^2)^{-1/2}\, ds$$
$$= \frac{\pi}{2}\, te^{-bt^{1/2}/4},$$

which tends to zero as $t \to \infty$.

From (5) and (9) follows at once the desired result that the solution of (3) tends to the solution of (4) as $t \to \infty$.

## 3. The Homogeneous Case

Here we shall show that the solution of the nonhomogeneous wave equation

$$u'' + bu' = A^2u + f(t), \qquad u(0) = u_0\,, \qquad u'(0) = u_1 \tag{10}$$

tends to the solution of

$$bU' = A^2U + f(t), \qquad U(0) = u_0 + \frac{1}{b}\, u_1 \tag{11}$$

as $t \to \infty$, provided $f$ satisfies certain restrictions. The presence of the driving term $f$ in (10) or (11) means that energy is being pumped into the system. If no restriction on the growth of $f$ is made, the energy in the system is unbounded and hence the solutions $u$ and $U$ cannot be expected to remain bounded. For unbounded $U$ one anticipates a result of the form $\lim_{t\to\infty} \| u(t) - U(t)\|/\| U(t)\| = 0$, rather than a stronger result of the form $\lim_{t\to\infty} \| u(t) - U(t)\| = 0$. We therefore assume that $f$, besides being a continuous function from $[0, \infty)$ to $\mathscr{B}$, satisfies the growth requirement

$$\|f(t)\| \leqslant g(t), \qquad \text{where} \quad \int_0^\infty g(t)\, dt < \infty \qquad \text{and} \quad \sup_{t\geqslant 0} g(t) < \infty. \tag{12}$$

We may use Duhamel's principle in the standard way to solve for $u$, $U$:

$$u(t) = v(t) + \frac{1}{2}\int_0^t \int_{-(t-\mu)}^{(t-\mu)} e^{-b(t-\mu)/2} J_0(ib[(t-\mu)^2 - s^2]^{1/2}/2) \cdot T(s) f(\mu)\, ds\, d\mu, \tag{13}$$

$$U(t) = V(t) + \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} [\pi b(t-\mu)]^{-1/2} e^{-bs^2/[4(t-\mu)]} T(s) f(\mu)\, ds\, d\mu, \tag{14}$$

where $v$, $V$ solve the homogeneous problems (3), (4), respectively. Since we showed earlier that $v(t) \to V(t)$ as $t \to \infty$, it is enough to show that the integral of (13) approaches the integral of (14) as $t \to \infty$.

Following [1], we introduce the notations

$$K_1(t, \mu, s) \equiv e^{-b(t-\mu)/2} J_0(ib[(t-\mu)^2 - s^2]^{1/2}/2),$$

$$K_2(t, \mu, s) \equiv [\pi b(t-\mu)]^{-1/2} e^{-bs^2/[4(t-\mu)]}.$$

To show that the integral of (13) approaches that of (14), it will be enough to show that

$$I \equiv \left\| \int_0^t \left\{ \int_0^{(t-\mu)} K_1 T(s) f(\mu)\, ds - \int_0^\infty K_2 T(s) f(\mu)\, ds \right\} d\mu \right\|$$

tends to zero as $t \to \infty$. Since for some $k$, $\operatorname{Erfc}(x) \leqslant ke^{-x^2}$ and $\| T(\cdot)\| \leqslant 1$, we can estimate $\int_0^t \int_{(t-\mu)}^\infty \| K_2 T(s) f(\mu)\|\, ds\, d\mu$ as follows.

$$\begin{aligned}
&\int_0^t \int_{(t-\mu)}^\infty \| K_2 T(s) f(\mu)\|\, ds\, d\mu \\
&\quad \leqslant k \int_0^t \int_{t-\mu}^\infty (t-\mu)^{-1/2} e^{-bs^2/[4(t-\mu)]} g(\mu)\, ds\, d\mu \\
&\quad = k \int_0^t \operatorname{Erfc}\left(\frac{1}{2}[b(t-\mu)]^{1/2}\right) g(\mu)\, d\mu \\
&\quad \leqslant k \int_0^{t-N} e^{-b(t-\mu)/4}\, d\mu + k \int_{t-N}^t g(\mu)\, d\mu \\
&\quad = k[e^{-bN/4} - e^{-bt/4}] + k \int_{t-N}^t g(\mu)\, d\mu,
\end{aligned}$$

which can be made arbitrarily small by choosing $N$ large and then $t$ $N$ sufficiently large. Thus to show that I goes to zero it is enough to show that

$$\int_0^t \int_0^{t-\mu} | K_1 - K_2 | g(\mu) \, ds \, d\mu \to 0.$$

Now

$$\int_0^t \int_0^{t-\mu} | K_1 - K_2 | g(\mu) \, ds \, d\mu = \int_0^{t-N} \int_0^{t-\mu-N} | K_1 - K_2 | g(\mu) \, ds \, d\mu$$

$$+ \int_{t-N}^t \int_0^{t-\mu} | K_1 - K_2 | g(\mu) \, ds \, d\mu + \int_0^{t-N} \int_{t-\mu-N}^{t-\mu} | K_1 - K_2 | g(\mu) ds \, d\mu$$

$$\equiv I_1 + I_2 + I_3 .$$

We estimate $I_2$ as follows.

$$I_2 \leqslant \int_{t-N}^t \int_0^{t-\mu} | K_1 | g(\mu) \, ds \, d\mu + \int_{t-N}^t \int_0^{t-\mu} K_2 g(\mu) \, ds \, d\mu; \tag{15}$$

the first of these integrals is bounded by

$$k \int_{t-N}^t \int_0^{t-\mu} e^{-bs^2/[4(t-\mu)]} g(\mu) \, ds \, d\mu$$

$$\leqslant k \int_{t-N}^t (t-\mu)^{1/2} \operatorname{Erf}(\tfrac{1}{2}[b(t-\mu)]^{1/2}) \, g(\mu) \, d\mu$$

$$\leqslant kN^{1/2} \operatorname{Erf}(\tfrac{1}{2}[bN]^{1/2}) \int_{t-N}^t g(\mu) \, d\mu,$$

which tends to zero as $t \to \infty$ for any fixed $N$. Here we have used (8) with $t$ replaced by $t - \mu$ to estimate $K_1$. The second integral of (15) can be estimated in a similar fashion.

$I_3$ is handled in much the same way. Set

$$I_3 = I_{31} + I_{32} \equiv \int_0^{t-N} \int_{t-\mu-N}^{t-\mu} | K_1 | g(\mu) \, ds \, d\mu + \int_0^{t-N} \int_{t-\mu-N}^{t-\mu} K_2 g(\mu) \, ds \, d\mu;$$

we shall show only that $I_{31} \to 0$ as $t \to \infty$, as the argument for $I_{32}$ is similar. We have

$$I_{31} \leqslant k \int_0^{t-N} g(\mu) \int_{t-\mu-N}^{\infty} e^{-bs^2/[4(t-\mu)]} \, ds \, d\mu$$

$$\leqslant k \int_0^{t-N} (t-\mu)^{1/2} g(\mu) \operatorname{Erfc}\left(\tfrac{1}{2} b^{1/2} \left[(t-\mu)^{1/2} - \frac{N}{(t-\mu)^{1/2}}\right]\right) d\mu.$$

Now if $\mu$ is restricted to satisfy $\mu \leqslant t - N^2$, then $(t - \mu)^{1/2} \geqslant N$ and, since Erfc is a decreasing function of its argument and $\text{Erfc}(x) \leqslant ke^{-x^2}$.

$$\begin{aligned}
&\text{Erfc}(\tfrac{1}{2}b^{1/2}[(t - \mu)^{1/2} - N(t - \mu)^{-1/2}]) \\
&\quad \leqslant \text{Erfc}(\tfrac{1}{2}b^{1/2}[(t - \mu)^{1/2} - 1]) \\
&\quad \leqslant ke^{-b[(t-\mu)-2(t-\mu)^{1/2}]/4} \leqslant ke^{-b(t-\mu)/8}
\end{aligned}$$

provided $N \geqslant 4$. It follows that for $\mu \leqslant t - N^2$

$$(t - \mu)^{1/2}\, \text{Erfc}(\tfrac{1}{2}b^{1/2}[(t - \mu)^{1/2} - N(t - \mu)^{-1/2}]) \leqslant ke^{-b(t-\mu)/16}.$$

We can now estimate $I_{31}$ by

$$\begin{aligned}
I_{31} &\leqslant k \int_0^{t-N^2} g(\mu) e^{-b(t-\mu)/16}\, d\mu + kN \int_{t-N^2}^{t-N} g(\mu)\, d\mu \\
&\leqslant ke^{-bN^2/16} \int_0^{\infty} g(\mu)\, d\mu + kN \int_{t-N^2}^{\infty} g(\mu)\, d\mu.
\end{aligned}$$

The first term can be made small by choosing $N$ large; the second can then be made small by choosing $t$ large. Thus $I_{31} \to 0$ as $t \to \infty$.

There remains to show that $I_1 \to 0$ as $t \to \infty$. Since there $s \leqslant t - \mu - N \ll t - \mu$ for sufficiently large $N$, we may employ arguments similar to those of Section 2 to show that

$$K_1(t, \mu, s) = K_2(t, \mu, s)[1 + o(1)] \text{ as } t \to \infty$$

in the domain of integration of $I_1$. Thus

$$\begin{aligned}
I_1 &= o(1) \int_0^{t-N} \int_0^{t-\mu-N} K_2(t, \mu, s)\, g(\mu)\, ds\, d\mu \\
&\leqslant o(1) \int_0^{t-N} \left\{ \int_0^{\infty} K_2(t, \mu, s)\, ds \right\} g(\mu)\, d\mu \\
&\leqslant o(1) \int_0^{\infty} g(\mu)\, d\mu = o(1),
\end{aligned}$$

since the inner integral is a constant.

We have established the following result.

THEOREM. *Let the continuous function* $f: [0, \infty) \to \mathscr{B}$ *satisfy the growth restriction of* (12), *and let the problems* (10), (11) *be well posed. Then the solution of* (10) *tends in the norm of the Banach space* $\mathscr{B}$ *to the solution of* (11).

Under additional restrictions, the solution of the parabolic equation (11) is known to tend to a limit itself [3, p. 153]. Combining this result with the theorem, we get the following corollary.

COROLLARY. *In addition to the hypotheses of the theorem, assume that*

(i) *$f$ is uniformly Hölder continuous and* $\lim_{t\to\infty} f(t) = 0$;

(ii) *$A^2$ has an inverse.*

Then the solution of (10) tends to zero as $t \to \infty$.

Note, for example, that the corollary cannot be applied to the initial boundary value problem with boundary conditions of the form $\partial u/\partial n = 0$, where $n$ denotes the normal to the boundary, since $A^2$ is not then invertible. The theorem is, however, applicable.

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