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Journal of Combinatorial Theory, Series A 113 (2006) 1565–1571

Journal of Combinatorial Theory Series A

www.elsevier.com/locate/jcta

Note

Generalized quadrangles of order (p, t) admitting a 2-transitive regulus, p a prime

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Received 22 December 2004

Available online 20 March 2006

Abstract

We classify generalized quadrangles of order (p, t) admitting a 2-transitive regulus, p a prime.

Keywords: Generalized quadrangle; 2-Transitive group; Regulus; Classification

1. 2-Transitive reguli in generalized quadrangles

In [2] J. De Kaey and H. Van Maldeghem classified the finite generalized quadrangles ("GQs") of order s, $1 < s$, having an automorphism group that induces the natural action of degree $s + 1$ of $\mathbf{PSL}_2(s)$ on a regulus $\mathfrak{R}$ while fixing the opposite regulus elementwise, and also inducing that action on the opposite regulus. (They call this action the "natural action of $\mathbf{PSL}_2(s) \times \mathbf{PSL}_2(s)$" on the subGQ of order $(s, 1)$ defined by $\mathfrak{R}$.)

Theorem 1.1. (J. De Kaey and H. Van Maldeghem [2].) *Suppose $\mathcal{S}$ is a GQ of order s, $\infty > s > 1$, admitting the natural action of $\mathbf{PSL}_2(s) \times \mathbf{PSL}_2(s)$ on a subGQ of order $(s, 1)$. Then $\mathcal{S}$ is isomorphic to the GQ $\mathcal{Q}(4, s)$ formed by the points and lines on a nonsingular parabolic quadric in $\mathbf{PG}(4, s)$.*

In [14] Theorem 1.1 was then generalized for arbitrary parameters:

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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2005.12.009

Theorem 1.2. (K. Thas [14].) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $\infty > s,\ t > 1$, admitting the natural action of $\mathbf{PSL}_2(s) \times \mathbf{PSL}_2(s)$ on a subGQ of order $(s,1)$. Then $t \in \{s, s^2\}$ and $\mathcal{S}$ is isomorphic to a GQ $\mathcal{Q}(d,s)$ formed by the points and lines on a nonsingular quadric of Witt index 2 in $\mathbf{PG}(d,s)$, where $d \in \{4,5\}$.*

In this paper, we consider general 2-transitive groups acting "naturally" on a subGQ of order $(s,1)$ of a GQ of order (s,t), $s,t > 1$. Perhaps this aim is not plausible without further restrictions (as the proofs of the results in [7,12,13,15] seem to indicate). In this paper, we will do the classification for s a prime. However, we only demand a 2-transitive action on one regulus of the subGQ which fixes the opposite regulus elementwise.

Besides Theorems 1.1 and 1.2, our result is motivated by attempting to construct new generalized quadrangles. To do so, it is always useful to have assumptions on the automorphism group of the quadrangle, especially when it admits a nice action on some interesting subgeometry. Our principal result seems to indicate that no unknown (thick) quadrangle of order (s,t) with a subquadrangle of order $(s,1)$ has a "large" automorphism group stabilizing the subquadrangle.

We do not use the classification of finite simple groups; we rely only on rather elementary group theory to obtain sufficient geometrical information in order to obtain the main results.

2. Statement of the main result

In this paper we only work with finite GQs—see the monograph [10] for a comprehensive introduction to these structures.

Recall that, for a subset A of the point set P of a GQ, $A^{\perp} = \bigcap_{a \in A} a^{\perp}$, and the elements of $A^{\perp}$ are called *centers* (of A). Also, $A^{\perp\perp} = (A^{\perp})^{\perp}$, and the same notations are used for lines. If A is a set of three two by two noncollinear points, A is called a *triad*. Let U, V be distinct nonconcurrent lines of the GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$. If $|\{U,V\}^{\perp\perp}| = s+1$, then $\{U,V\}^{\perp\perp}$ is called a *regulus*. Note that if $\{U,V\}^{\perp\perp}$ is a regulus, $\{U,V\}^{\perp}$ also is one.

Let Γ be a proper subGQ of order $(s,1)$, $s > 1$, of a GQ $\mathcal{S}$ of order (s,t). Let $\mathfrak{R}$ be a regulus of Γ. We say that $\mathfrak{R}$ is a 2-*transitive regulus* if $\mathcal{S}$ has an automorphism group that fixes $\mathfrak{R}^{\perp}$ elementwise and acts 2-transitively on $\mathfrak{R}$.

The main objective of this note is to obtain the following result.

Theorem 2.1. *Let $\mathcal{S}$ be a GQ of order (p,t), p a prime and $t > 1$, admitting a 2-transitive regulus, with associated 2-transitive group H. Then one of the following occurs*:

(1) $\mathcal{S} \cong \mathcal{Q}(4,p)$;
(2) $\mathcal{S} \cong \mathcal{Q}(5,p)$;
(3) *H/N acts sharply 2-transitively, where N is the kernel of the action on $\mathfrak{R}$, and $7 \leqslant p+2 \leqslant t \leqslant p^2$. Here $p+1$ is a power of 2.*

A *spread* of a GQ of order (s,t), $s,t > 1$, is a set of $st+1$ mutually nonconcurrent lines (this means that each point is on exactly one line of the spread). A spread is a *spread of symmetry* if there is an automorphism group of the GQ that fixes the spread line by line, and acts regularly on any (and then each) of these lines (we call this group the "associated group").

A *generalized broken grid* with *carriers* L and M in a GQ of order (s,t), $s,t > 1$, consists of two disjoint sets of lines $G_1 \subseteq L^{\perp}$ and $G_2 \subseteq M^{\perp}$, where $L \nsim M$, $L \in G_1$ and $M \in G_2$, such that each point of $L \cup M$ (the lines seen as point sets) is incident with a constant number c of lines of

$G_1 \cup G_2$, $1 < c < t+1$, and such that the point set of $G_1 \setminus \{L\}$ equals that of $G_2 \setminus \{M\}$. If $c = 2$, we just speak of a *broken grid*.

Analyzing the situation in the third part of Theorem 2.1, we will obtain the following extra information:

Theorem 2.2. *Suppose we are in* (3) *of Theorem* 2.1*, and suppose that H is minimal w.r.t. its action, i.e., H has no subgroup which also acts* 2*-transitively on $\mathfrak{R}$. Suppose first that $|N| = 1$. If $\mathcal{S}$ has no broken grids with carriers in $\mathfrak{R}^{\perp}$, then $p+1$ divides $t-1$ implies that $\mathcal{S}$ has a spread of symmetry, with associated elementary abelian* 2*-group.*

Suppose that $|N| > 1$. Then $|N|$ is even.

In general, if $\mathcal{S}$ has a spread of symmetry, $p+1$ divides $t-1$.

Moreover, we will show that there is a class of examples in the third case with $t = p+2$ (and $|N| = 1$), each example being associated to a Mersenne prime.

3. Proof of Theorems 2.1 and 2.2

Suppose $L \in \mathfrak{R}$ and let N be the kernel of the action of H on $\mathfrak{R}$. Let Γ be the subGQ of order $(p, 1)$ defined by $\mathfrak{R}$. By D. Passman [9, Theorem 7.3, p. 53], H_L/N either is a 2-transitive group, or $(H_L/N, \mathfrak{R} \setminus \{L\})$ can be identified with the permutation group $(H', \mathbf{GF}(p))$, where

$$H' \subseteq \{cx + d \parallel c \neq 0;\ c, d \in \mathbf{GF}(p)\}.$$

If we are in the latter case, $|H'|$ divides $p(p-1)$, so H_L/N has exactly one (Sylow) p-subgroup. So either H/N acts 3-transitively on $\mathfrak{R}$, or H/N has a *split BN-pair of rank* 1, meaning that

- for each line $R \in \mathfrak{R}$, $(H/N)_R$ contains a normal subgroup N_R acting sharply transitively on $\mathfrak{R} \setminus \{R\}$;
- the groups N_R generate H/N, and are mutually conjugate.

As the GQs of order $(3, t)$, $t > 1$, are known (see [10, Chapter 6]), we suppose $p > 3$ throughout.

Lemma 3.1. *H/N cannot act* 4*-transitively.*

Proof. Suppose it does act as such. Then the size of H is $(p+1)p(p-1)(p-2)r|N|$, where r is natural. Suppose some nontrivial element α of H fixes some point of $\mathcal{S}$ not in Γ. Then combining [10, 2.4.1 and 2.2.2], it follows that α fixes a subGQ of order p pointwise, and so $\alpha \in N$. Then one notes that $(p+1)p(p-1)(p-2)$ divides $(p+1)p(t-1)$, so that $(p-1)(p-2)$ divides $t-1$. If there is no such element, H acts semiregularly on the points of $\mathcal{S} \setminus \Gamma$, so we have the same conclusion. By Higman's Inequality [10, 1.2.3], it follows that either $p \leqslant 5$ or that $t - 1 = (p-1)(p-2)$. If $p = 5$, and the latter equality is not satisfied, $t = 25$. A 4-transitive group on 6 letters contains $\mathbf{A}_6$ (see [1]), so H/N also contains $\mathbf{PSL}_2(5)$. Now it follows from [14] that $\mathcal{S} \cong \mathcal{Q}(5,5)$, but this GQ does not admit $\mathbf{A}_6$ in this action. Suppose $t = (p-1)(p-2)+1$; then the "standard divisibility condition for GQs" [10, 1.2.2] leads to a contradiction. □

We now proceed in a number of steps to obtain the proofs of the main results.

3.1. 3-Transitive case

Let $L \in \mathfrak{R}$, and suppose $np+1$ is the number of Sylow p-subgroups in H_L, and denote these groups with $H_0, H_1, \ldots, H_{np}$.[2] Let $M \in \mathfrak{R}^{\perp}$. Suppose n' is the number of fixed lines through $L \cap M$ of H_0 which are not in Γ, and note that this number is the same for each of these Sylow groups. *For now, suppose that no two of these groups can fix the same line incident with $L \cap M$ not in Γ*. Then $(np+1)n' \leqslant t-1$. One also notes that $t-1-n'$ is divisible by p. We now consider several cases.

(a) $t-1=n'$. Then $n=0$, and H_0 fixes each line concurrent with L, implying, by definition, that L is an axis of symmetry. So each line of $\mathfrak{R}$ is an axis of symmetry, so that $\mathcal{S}$ is, again by definition, a span-symmetric generalized quadrangle. But then [7,12,13] (see also [15, Chapter 7]) imply that $H/N \cong \mathbf{PSL}_2(p)$ in its natural permutation representation of degree $p+1$, contradicting the 3-transitive action.

(b) Let $p=t-1-n'$. Then $(np+1)(t-1-p) \leqslant t-1 \Rightarrow t \leqslant p+1=s+1$, implying that $s=t$ by the standard divisibility condition of GQs ($p \leqslant t$ as $\mathcal{S}$ has a subGQ of order $(p,1)$), contradiction.

(c) Let $mp=t-1-n'$, $m \geqslant 2$. We suppose that $t-1=a(p-1)$ for some nonzero integer a (we will show later that this is no restriction)—it follows that $a \leqslant p+1$. As $p(a-m)=n'+a$, the assumption that $a-m \geqslant 2$ would assert that $n' \geqslant p-1$. So $n=1$ and $t=p^2$. Suppose $N_{(H/N)_L}(H_0)$ is the normalizer of $H_0 \cong H_0N/N$ in $(H/N)_L$; then clearly $p(p-1)/2$ divides $|N_{(H/N)_L}(H_0)|$. Applying the result of D. Passman quoted in the beginning of this section, it is easy to see that $[(H/N)_L]' = [(H/N)_L, (H/N)_L]$ either is a 2-transitive group (so that $(H/N)'$ is again 3-transitive), or H/N has a split BN-pair of rank 1. Suppose for now that the latter is not the case, so that $(H/N)_L$ may be assumed to be unsolvable. Then by P.M. Neumann [8], Lemma 3.1 implies that $|N_{(H/N)_L}(H_0)|$ equals $p(p-1)/2$. By [8], we then have that $(H/N)_{X,Y,Z}$, where X, Y, Z are three distinct lines in $\mathfrak{R}$, has at most two orbits in $\mathfrak{R} \setminus \{X,Y,Z\}$. So by Lemma 3.1 precisely two orbits. As $|(H/N)_{X,Y,Z}| = (p+1)/2$, it easily follows that $p=7$ or $p=11$. The group $\mathbf{S}_8$ does not contain 3-transitive subgroups (on 8 letters) of order $8.7.6.4=1344$ by [1], so $p=7$ does not occur. By [1], the only 3-transitive subgroup (on 12 letters) of $\mathbf{S}_{12}$ of order $12.11.10.6=7920$ is the Mathieu group $\mathbf{M}_{11}$. This permutation group contains $\mathbf{PSL}_2(11)$ in its natural action (on 12 letters), so that the main result of [14] implies that $\mathcal{S} \cong \mathcal{Q}(5,11)$. But this quadrangle does not allow $\mathbf{M}_{11}$ to act in this way.

Now suppose that $a-m=1$ ($a-m>0$!). We know that $|H/N|$ is divisible by $(p+1) \times p(p-1)(np+1)/r$, where r divides $(n+1, p-1)$. First suppose that H acts semiregularly on the point set of $\mathcal{S} \setminus \Gamma$; then $(p+1)p(t-1)$ is divisible by $|N|(p+1)p(p-1)(np+1)/r$, implying that $|N|(p-1)(np+1)/r$ divides $t-1$. If $r \leqslant (n+1)/2$, Higman's Inequality is violated unless $n=0$ or $n=1$. The case $n=1$ was already essentially handled, and $n=0$ implies H/N to have a split BN-pair of rank 1. We may assume $r=n+1$, so that $n+1$ divides $p-1$. Then $(p-1)(np+1)/(n+1)$ divides $t-1$ implies that $(p-1)(p+(1-p)/(n+1))$ divides $t-1$. As $t \leqslant p^2$, a contradiction easily follows if $n \geqslant 2$. So $n=0$ or $n=1$, and we proceed as before. If the action of H on the point set of $\mathcal{S} \setminus \Gamma$ is not semiregular, an element θ that fixes a point not in Γ is contained in N, so that again $(p-1)(np+1)/r$ divides $t-1$, and we are done.

[2] The number of Sylow p-subgroups in $(H/N)_L$ is at most $np+1$, but by abuse of notation, we will assume that this number is also $np+1$—the intention is to prove that the number of Sylow p-subgroups in $(H/N)_L$ is 1.

Suppose that either $p-1 \nmid t-1$, or that there are two distinct Sylow p-subgroups H_i and H_j in H_L that fix the same line M not in Γ (meeting L). In the first case, the stabilizer in H of two distinct lines of $\mathfrak{R}$, having a size divisible by $p-1$, has some nontrivial element that fixes some line meeting Γ but not contained in it. So [10, 2.4.1 and 2.2.2] imply that this element fixes some subGQ of order p pointwise, and $t=p^2$. But then $p-1 \mid t-1$, contradiction. If we are in the second case, by a similar argument, $\langle H_i, H_j\rangle$ has such an element fixing a subGQ of order p pointwise. Moreover, if $N' \in \mathfrak{R} \setminus \{L\}$, then each element of $\langle H_i, H_j\rangle_{N'}$ fixes a subGQ of order p pointwise, so that $|\langle H_i, H_j\rangle_{N'}| = 2$, contradicting $p > 2$.

We have shown that H/N always has a split BN-pair of rank 1.

3.2. H/N has a split BN-pair of rank 1

If a group G has a split BN-pair of rank 1 in its (faithful) action on the set X, where $|X| = s+1 \in \mathbb{N}$, then G acts as a sharply 2-transitive group on X, or is isomorphic, as a permutation group, to one of the following: (a) $\mathbf{PSL}_2(s)$, (b) the Ree group $\mathbf{R}(\sqrt[3]{s})$, (c) the Suzuki group $\mathbf{Sz}(\sqrt{s})$, (d) the unitary group $\mathbf{PSU}_3(\sqrt[3]{s^2})$, each with its natural action of degree $s+1$ [6,11]. Clearly, only the sharply 2-transitive groups and $\mathbf{PSL}_2(s)$ turn up here, with $s=p$. If we are in the latter case, Theorem 3.1 of K. Thas [14] implies that either $\mathcal{S} \cong \mathcal{Q}(4,p)$ or $\mathcal{Q}(5,p)$.

3.3. The sharply 2-transitive groups

Put $\mathfrak{R} = \{L_0, L_1, \ldots, L_p\}$, and let H_i be a Sylow p-subgroup in H_{L_i}.

First suppose that $t=p$. Then clearly H_i fixes $L_i^{\perp}$ elementwise, so that L_i is an axis of symmetry for each i, and $\mathcal{S}$ a span-symmetric generalized quadrangle. But by [13,15], the groups H_i only generate a sharply 2-transitive group acting on $\mathfrak{R}$ if $p \in \{2,3\}$, leading us to the classical case. As $p+1=t$ is not allowed by the standard divisibility condition for GQs, we have $p+2 \leqslant t \leqslant p^2$ by Higman's Inequality.

For now, suppose that $|N|=1$ and that $\mathcal{S}$ has no broken grids with carriers in $\mathfrak{R}^{\perp}$. As H acts sharply 2-transitively on $\mathfrak{R}$, H has a normal regular elementary abelian 2-subgroup A. So $H = H_j A$ for any j.

Suppose that

$$\frac{t-1}{p+1} = a \in \mathbb{N}.$$

Then for a fixed but arbitrary i, H_i fixes at least a lines not in Γ through each point of L_i. So H_i acts regularly on the points not in Γ of these lines. Fix $l\,\mathrm{I}\,L_i$, and suppose $M\,\mathrm{I}\,l$ is fixed by H_i, $M \notin \Gamma$. Then the H-orbit M^H has $(p+1)p$ points not in Γ, and $p+1$ lines. Let this point set be denoted by Ω. Let $l \neq l'\,\mathrm{I}\,L_i$, and take an arbitrary line R incident with l' and meeting Ω in a point. Then we show that no nontrivial element of H_i can fix R. If such an element would exist, H_i fixes R. It now follows that M^H and R^H are the line sets of a broken grid with carriers the lines of $\mathfrak{R}^{\perp}$ containing l and l', respectively.

So $R^{H_i} \cap \Omega$ consists of a set of p mutually noncollinear points. Note that l' was essentially arbitrary, and that this observation is independent of i (it works for all $i \in \{0,1,\ldots,p\}$). As $H = (\bigcup_i H_i) \cup A$, it follows easily that all elements of A map points of Ω onto collinear points. Let $o \in \Omega$ be arbitrary, as well as $\alpha, \beta \in A \setminus \{\mathbf{1}\}$, $\alpha \neq \beta$. Then $o \sim o^{\alpha} \sim o^{\alpha\beta}$, but also $o \sim o^{\alpha\beta}$, so $o, o^{\alpha}, o^{\beta}, o^{\alpha\beta}$ are on the same line. It now easily follows that the points of $\Omega \cup l^H$ together with the lines of $\mathcal{S}$ meeting this point set in at least 2 distinct points, form a subGQ of order $(s,1)$, of which M^H is a regulus. Moreover, A fixes the opposite regulus $[M^H]^{\perp} =: \mathbf{S}(M)$ linewise.

Now let M vary through l (obtaining at least a lines), and also $l\,I\,L_i$. Then we obtain $a(p+1)$ different reguli which are disjoint outside Γ, and fixed linewise by A. So we obtain $a(p+1)p = (t-1)p$ mutually nonconcurrent lines which are all fixed by A. Together with the lines of $\mathfrak{R}^{\perp}$, they form a spread of symmetry with associated group A.

Suppose $|N| > 1$. If H_i is not a unique Sylow p-subgroup in H_{L_i}, then, using 1.4.1 of [10], one easily shows that $|N| = p+1$ is even. So we suppose now H_i is unique as such. As $N \cap H_i = \{\mathbf{1}\}$ and N and H_i normalize each other, they commute, so N is in the center $Z(H)$ of H. Suppose A/N is the regular normal elementary abelian 2-subgroup of H/N as before. As $N \leqslant Z(A)$, $A/Z(A)$ is nilpotent, so A is nilpotent. Hence A has a unique Sylow 2-subgroup U. As UN/N is abelian,

$$(UN/N)' = \{\mathbf{1}\} = U'N/N,$$

so $U' \leqslant N$. Suppose that U is abelian. As H_i acts naturally on U and $(p,|U|) = 1$, Maschke's Theorem [5] implies that $U = C_U(H_i) \times [U, H_i]$. So $[U, H_i]$ is an H_i-invariant subgroup of U of size $p+1$, which is hence normal in H. So $[U, H_i]H_i$ is a subgroup of H which acts sharply 2-transitively on $\mathfrak{R}$ without a kernel. We are in a previous situation. If $U' \neq \{\mathbf{1}\}$, then it follows that $|N|$ is even.

Now suppose $\mathbf{T}$ is a spread of symmetry of $\mathcal{S}$. Then $\mathbf{T}$ is a regular spread (meaning that each two distinct lines of $\mathbf{T}$ are contained in a regulus which is part of $\mathbf{T}$), so that counting the number of reguli in $\mathbf{T}$ leads to

$$\frac{(pt+1)pt}{(p+1)p} \in \mathbb{N}.$$

So $p+1$ divides $t(t-1)$. Suppose that $t-1$ is odd; as $p+1$ is a power of 2, $p+1$ divides t, so $(p,t) = 1$ by Higman's Inequality. Now the standard divisibility condition for GQs leads to a contradiction. So t is odd, and $p+1$ divides $t-1$.

This concludes the proof of both Theorems 2.1 and 2.2.

4. A class of examples

We now construct a class of examples for the first case.

Consider a generalized quadrangle $\mathcal{S}$ of order $s > 2$, with regular point x, and point set P. Construct the following incidence structure $\mathcal{P}(\mathcal{S}, x)$ (see, e.g., [10]) from $\mathcal{S}$:

- POINTS are the points of $P \setminus x^{\perp}$;
- LINES are of two types: the lines of $\mathcal{S}$ not incident with x, and the hyperbolic lines $\{x, y\}^{\perp\perp}$, $y \not\sim x$;
- INCIDENCE is the natural one.

Then $\mathcal{P}(\mathcal{S}, x)$ is a GQ of order $(s-1, s+1)$. When $X\,I\,x$ is a regular line, an $(s+1)\times(s+1)$-grid Γ containing X clearly induces an $(s \times s)$-grid Γ' in $\mathcal{P}(\mathcal{S}, x)$. Now suppose $\mathcal{S} = W(s)$, with $s = 2^h$; then each point and each line of $W(s)$ is regular. Take a line $Y \not\sim X$. Then the symmetries about X and Y (which are collineations fixing, respectively, $X^{\perp}$ and $Y^{\perp}$ linewise), generate a group G isomorphic to $\mathbf{SL}_2(s)$ in its natural action of degree $s+1$ on $\{X, Y\}^{\perp\perp}$ [7,12]; also, G fixes $\{X, Y\}^{\perp}$ elementwise. As G acts sharply 3-transitively on $\{X, Y\}^{\perp\perp}$, G_X acts sharply

2-transitively on $\{X, Y\}^{\perp\perp} \setminus \{X\}$. Moreover, one notes that each automorphism group of $\mathcal{S}$ fixing x induces an automorphism group of $\mathcal{P}(\mathcal{S}, x)$.

So, as soon as $s - 1 = 2^h - 1$ is a prime p, we have an example of the third class of Theorem 2.1. A prime p with this property is precisely a Mersenne prime, and there are at least 43 such primes known at present (and thought of to belong to an infinite family), see [4].

In fact, for $t = p + 2$, the main result and the following theorem show that examples always essentially arise from such a construction:

Theorem 4.1. (M. De Soete and J.A. Thas [3].) *Let $\mathcal{S}$ be a GQ of order $(s, s+2)$, $s > 1$, with spread of symmetry* $\mathbf{T}$. *Then there exists a GQ $\mathcal{S}'$ of order $s+1$ with center of symmetry x, so that $\mathcal{P}(\mathcal{S}', x) \cong \mathcal{S}$ and* $\mathbf{T}$ *is the set of all lines of the second type.*

Acknowledgments

The author is grateful to Bill Kantor for various helpful remarks. The author also wants to thank the referees for several useful suggestions.

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