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The Connected Components of the Auslander–Reiten Quiver of a Tilted Algebra

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INTRODUCTION

Let A be an artin algebra and T a splitting tilting module in mod A, the category of fintely generated left A-modules. If we let $B = \mathrm{End}_A(T)$ and denote by mod B the category of finitely generated left B-modules, then T determines a splitting torsion theory $(\mathscr{Y}(T), \mathscr{X}(T))$ in mod B. We know that all but finitely many connected components of the Auslander–Reiten quiver Γ_B of B either completely lie in $\mathscr{X}(T)$ or completely lie in $\mathscr{Y}(T)$ [10]. In this paper we show that all connected components of Γ_B lying in $\mathscr{X}(T)$ are preinjective components, quasi-serial components, or isomorphic to valued translation quivers obtained from quasi-serial translation quivers by coray insertions (see Section 1 for definition).

In general we are not able to get a similar result about those components of Γ_B lying in $\mathscr{Y}(T)$. However, if A is an hereditary algebra of type Δ, then we are able to give a complete description of the possible shapes of the connected components of Γ_B. We show that a connected component of Γ_B is a preprojective component, a preinjective component, the connecting component determined by T, a quasi-serial component, or isomorphic to a valued translation quiver obtained from a quasi-serial translation quiver by ray insertions or by coray insertions. The connecting component of Γ_B determined by T can be embedded in $\mathbb{Z}\Delta^*$, where Δ^* is the opposite quiver of Δ [17]. The shapes of preprojective and preinjective components of an arbitrary Auslander–Reiten quiver are well known.

The problem of describing the shapes of the connected components of the Auslander–Reiten quiver Γ_B of a tilted algebra B of infinite representa-

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tion type goes back to the work of Happel and Ringel [7] where they gave a complete description of the shape of Γ_B when B is a tilted algera of tame type. Later Ringel proved in general that any regular component of Γ_B which is not a connecting component is quasi-serial [18]. A description of the shapes of the non-regular components of Γ_B when B is a tilted algebra of wild type might be concluded from results in [11, 12] which were obtained from a detailed study of the homomorphisms between regular modules of a wild hereditary algebra.

1. Prelmiminaries

Throughout this paper A denotes a connected artin algebra. We denote by mod A the category of finitely generated left A-modules, by Γ_A the Auslander–Reiten quiver of A, and by τ the Auslander–Reiten translation $D\,\mathrm{Tr}$. We will use freely the standard notion and terminology as well as some basic results of Auslander–Reiten theory, which may be found in [3, 4].

When no possible confusion can occur, we do not distinguish between an indecomposable module X in mod A and the corresponding vertex $[X]$ in Γ_A.

Let Γ be a valued translation quiver and x a vertex of Γ. Let x_1, $x_2, ..., x_s$ be the immediate predecessors of x in Γ and (b_i, b_i') the valuation of the arrow $x_i \to x$ for $1 \leqslant i \leqslant s$. Then we say that x has b' *left neighbours* where $b' = \sum_1^s b_i'$. Let $y_1, y_2, ..., y_r$ be the immediate successors of x in Γ and (c_j, c_j') the valuation of the arrow $x \to y_j$ for $1 \leqslant j \leqslant r$. Then we say that x has c *right neighbours* where $c = \sum_1^r c_j$. We say that an arrow of Γ has *trivial valuation* if its valuation is $(1, 1)$, and we say that Γ has trivial valuation when each arrow of Γ has trivial valuation.

We recall that Γ_A is a valued translation quiver and the valuation (b_{XY}, b'_{XY}) of an arrow $X \to Y$ is defined so that Y occurs b_{XY} times in the codomain of the source map for X while X occurs b'_{XY} times in the domain of the sink map for Y. Thus if X and $Y_1, Y_2, ..., Y_t$ are modules in Γ_A, then $Y_1, Y_2, ..., Y_t$ are the left neighbours of X in Γ_A if and only if $\bigoplus_1^t Y_i$ is the domain of the sink map for X, and $Y_1, Y_2, ..., Y_t$ are the right neighbours of X in Γ_A if and only if $\bigoplus_1^t Y_i$ is the codomain of the source map for X.

In order to describe the shapes of translation quivers which we are interested in, we quote from [17] the definition of coray insertions and ray insertions in an appropriate translation quiver.

Let Γ be a translation quiver. A vertex x of Γ is called a *coray vertex* if there is an infinite sectional path

$$\cdots \to x_n \to x_{n-1} \to \cdots \to x_2 \to x_1 = x$$

in Γ, which is called a *coray* ending at x, such that for each integer $n>0$, the path $x_n \to x_{n-1} \to \cdots x_2 \to x_1 = x$ is the only sectional path of length n in Γ which ends at x.

Let x be a coray vertex of Γ with a coray as above. For a positive integer n, we define a translation quiver $\Gamma[x, n]$ as follows.

The vertices of $\Gamma[x, n]$ are those of Γ together with the pairs (i, j) with $1 \leqslant j \leqslant n$, $i \geqslant 1$.

The arrows of $\Gamma[x, n]$ are those of Γ, excluding those ending at x_i other than $x_{i+1} \to x_i$ with $i \geqslant 1$, together with the following arrows: $(i+1, j) \to (i, j)$ for $1 \leqslant j \leqslant n$, $i \geqslant 1$; $(i, j+1) \to (i+1, j)$ for $1 \leqslant j < n$, $i \geqslant 1$; $(n+i-1, 1) \to x_i$ for all $i \geqslant 1$; and $y \to (i, n)$ if $y \to x_i$ is an arrow of Γ other than $x_{i+1} \to x_i$.

Let ρ be the translation of Γ. The translation ρ' of $\Gamma[x, n]$ is defined so that if z belongs to Γ but $z \neq x_i$ for all $i \geqslant 1$ and ρz is defined, then $\rho' z = \rho z$, whereas $\rho' x_i = (n+i, 1)$ for all $i \geqslant 1$; $\rho'(i, j) = (i, j+1)$ for $1 \leqslant j < n$; and $\rho'(i, n) = \rho x_i$ if ρx_i is defined.

We define by induction a translation quiver $\Gamma[x_0, n_0][x_1, n_1] \cdots [x_s, n_s]$ where the n_i are positive integers, x_0 is a coray vertex of Γ, while x_i is a coray vertex of $\Gamma[x_0, n_0] \cdots [x_{i-1}, n_{i-1}]$. Where there is no need to emphasize the coray vertices, we say that a translation quiver Γ' is of form $\Gamma[n_0, n_1, ..., n_s]$ if $\Gamma' \cong \Gamma[x_0, n_0][x_1, n_1] \cdots [x_s, n_s]$ with some appropriate coray vertiees x_i, and call Γ' a translation quiver obtained from Γ by *coray insertions*. There is a dual concept of a *ray vertex* and a dual construction of *ray insertions* at ray vertices.

Recall that a *quasi-serial* translation quiver is by definition either a stable tube or is of form $\mathbf{ZA}_\infty$ [18]. We say that a valued translation quiver is *obtained from a quasi-serial translation quiver by coray insertions* (or *by ray insertions*) if it has trivial valuation, and its underlying translation quiver is obtained from a quasi-serial translation quiver by coray insertions (or by ray insertions). In particular, a valued translation quiver is said to be a *coray tube* (or *ray tube*) if it is obtained from a stable tube by coray insertions (or by ray insertions) [14].

Let us now quote the following results from [13, 14]. Recall that an indecomposable module X in mod A is said to be *stable* if $\tau^n X \neq 0$ for all integers n.

1.1. LEMMA [13]. *Let X be a stable module in Γ_A which is not τ-periodic. If X has at least three stable left neighbours in Γ_A, then we have* $\lim_{n\to\infty} l(\tau^{-n}X) = \infty$ *and* $\lim_{n\to\infty} l(\tau^n X) = \infty$, *where $l(Y)$ is the composition length of Y.*

1.2. LEMMA [14]. *Let Γ be a connected component of Γ_A. Then we have the following*:

(1) *If Γ contains no projective module but does contain an oriented cycle, then Γ is either a stable tube or a coray tube.*

(2) *If Γ contains no injective module but does contain an oriented cycle, then Γ is either a stable tube or a ray tube.*

Finally we collect some basic facts of Auslander–Reiten theory in subcategories which can be found in [4, 17, 5]. In this paper a subcategory of mod A is always full and closed under isomorphisms, direct sums, direct summands, and extensions. Thus a subcategory of mod A is a Krull–Schmidt category itself.

Let $\mathscr{C}$ be a subcategory of mod A. A non-zero map $f: Y \to Z$ in $\mathscr{C}$ is said to be *$\mathscr{C}$-irreducible* if (i) f is neither a split epimorphism nor a split monomorphism and (ii) if $f = gh$ with g and h in $\mathscr{C}$, then either g is a split monomorphism or h is a split epimorphism.

Let $\mathscr{C}$ be a subcategory of mod A. A map $g: Y \to Z$ in $\mathscr{C}$ is said to be a *$\mathscr{C}$-sink map* for Z if (1) g is not a split epimorphism; (2) if $g = hg$, then h is an automorphism of Y; and (3) if $h: Y' \to Z$ in $\mathscr{C}$ is not a split epimorphism, then there exists $h': Y' \to Y$ such that $h = h'g$. A $\mathscr{C}$-source map for Z is defined dually. If there is a $\mathscr{C}$-sink map (respectively, a $\mathscr{C}$-source map) for Z then it is unique up to isomorphism, and moreover, Z is indecomposable. Sink maps and source maps are called minimal right almost split maps and minimal left almost split maps in [4].

1.3. LEMMA [4]. *Let $\mathscr{C}$ be a subcategory of* mod A. *If $0 \neq g: Y \to Z$ is a $\mathscr{C}$-sink map for Z, then a morphism $h: W \to Z$ with $W \neq 0$ in $\mathscr{C}$ is $\mathscr{C}$-irreducible if and only if there is a split monomorphism $s: W \to Y$ such that $h = sg$. Dually if $0 \neq f: X \to Y$ is a $\mathscr{C}$-source map for X, then a morphism $h: X \to W$ with $W \neq 0$ in $\mathscr{C}$ is $\mathscr{C}$-irreducible if and only if there is a split epimorphism $r: Y \to W$ such that $h = fr$.*

Let $\mathscr{C}$ be a subcategory of mod A. If X is an indecomposable module in mod A, then $k_X = \mathrm{End}(X)/\mathrm{rad}(\mathrm{End}(X))$ is a division ring. Let X and Y be indecomposable modules in $\mathscr{C}$. We define $\mathrm{Irr}_{\mathscr{C}}(X, Y) = \mathrm{rad}(X, Y)/\mathrm{rad}^2_{\mathscr{C}}(X, Y)$ where $\mathrm{rad}^2_{\mathscr{C}}(X, Y)$ is the set of maps $f: X \to Y$ which are of form $f = f_1 f_2$ with $f_1 \in \mathrm{rad}(X, Z)$, $f_2 \in \mathrm{rad}(Z, Y)$ for some module Z in $\mathscr{C}$. Note that $\mathrm{Irr}_{\mathscr{C}}(X, Y)$ is a left k_X vector space and a right k_Y vector space. Define $d_{XY} = \dim_{k_Y} \mathrm{Irr}_{\mathscr{C}}(X, Y)$ and $d'_{XY} = \dim_{k_X} \mathrm{Irr}_{\mathscr{C}}(X, Y)$. Using an argument similar to that used in the proof of the Lemma in [18, Sect. 2.5], we have the following:

1.4. LEMMA. *Let $\mathscr{C}$ be a subcategory of* mod A *and X, Y indecomposable modules in $\mathscr{C}$. If $f: X \to W$ is a $\mathscr{C}$-source map for X, then d_{XY} is the multiplicity of Y appearing in W. If $g: V \to Y$ is a $\mathscr{C}$-sink map for Y, then d'_{XY} is the multiplicity of X appearing in V.*

A short exact sequence $0 \to X \xrightarrow{f} E \xrightarrow{g} Z \to 0$ in mod A, with X and Z in $\mathscr{C}$, is said to be a *$\mathscr{C}$-almost split sequence* if f is a $\mathscr{C}$-source map for X and g is a $\mathscr{C}$-sink map for Z. In this case both X and Z are indecomposable.

It follows from the definition that two $\mathscr{C}$-almost split sequences are isomorphic if either the start-terms are isomorphic or the end-terms are isomorphic.

1.5. LEMMA. *Let $\mathscr{C}$ be a subcategory of* mod A. *If there is a $\mathscr{C}$-almost split sequence $0 \to X \to E \to Z \to 0$, then $k_X \cong k_Z$.*

Proof. The lemma follows easily from the definition of a $\mathscr{C}$-sink map and a $\mathscr{C}$-source map.

The proof of the following lemma is similar to that of the corresponding result of Aulander–Reiten theory in mod A.

1.6. LEMMA. *Let $\mathscr{C}$ be a subcategory of* mod A. *If there is a $\mathscr{C}$-almost split sequence $0 \to X \to E \to Z \to 0$ then, for each indecomposable summand Y of E, we have that $d_{XY} = d'_{YZ}$ and $d'_{XY} = d_{YZ}$.*

Proof. Assume that $0 \to X \to E \to Z \to 0$ is a $\mathscr{C}$-almost split sequence and Y an indecomposable summand of E. The equality $d_{XY} = d'_{YZ}$ follows directly from Lemma 1.3. Let C be the center of A. Then $k = C/\mathrm{rad}(C)$ is a field since A is connected by assumption. We have $d'_{XY} \dim_k k_X = \dim_k \mathrm{Irr}_{\mathscr{C}}(X, Y) = d_{XY} \dim_k k_Y = d'_{YZ} \dim_k k_Y = \dim_k \mathrm{Irr}_{\mathscr{C}}(Y, Z) = d_{YZ} \dim_k k_Z$. Thus we have $d'_{XY} = d_{YZ}$ since $\dim_k k_X = \dim_k k_Z$ by Lemma 1.5.

Let $\mathscr{C}$ be a subcategory of mod A. A module Z in $\mathscr{C}$ is said to be *$\mathscr{C}$-projective* if each short exact sequence $0 \to X \to Y \to Z \to 0$ in mod A with X in $\mathscr{C}$ splits. Dually a module X in $\mathscr{C}$ is said to be *$\mathscr{C}$-injective* if each short exact sequence $0 \to X \to Y \to Z \to 0$ in mod A with Z in $\mathscr{C}$ splits.

A subcategory $\mathscr{C}$ of mod A is said to have *relative almost split sequences* if, for each indecomposable module X in $\mathscr{C}$, (1) there exist both a $\mathscr{C}$-sink map and a $\mathscr{C}$-source map for X; (2) if X is not $\mathscr{C}$-projective, then there exists a $\mathscr{C}$-almost split sequence ending at X; and (3) if X is not $\mathscr{C}$-injective, then there exists a $\mathscr{C}$-almost split sequence starting at X.

Let $\mathscr{C}$ be an subcategory of mod A having relative almost split sequences. *The Auslander–Reiten quiver* $\Gamma(\mathscr{C})$ of $\mathscr{C}$ is a valued translation quiver defined as follows: Its vertices are the isoclasses $[X]$ of indecomposable modules X in $\mathscr{C}$. There is an arrow $[X] \to [Y]$ between two vertices $[X]$ and $[Y]$ if there is a $\mathscr{C}$-irreducible map from X to Y. Each arrow $[X] \to [Y]$ is endowed with a valuation (d_{XY}, d'_{XY}) where d_{XY} and d'_{XY} are defined as before. Finally, the translation $\tau_{\mathscr{C}}$ is defined so that if $0 \to X \xrightarrow{f} E \xrightarrow{g} Z \to 0$ is a $\mathscr{C}$-almost split sequence, then $[X] = \tau_{\mathscr{C}}[Z]$ and $[Z] = \tau_{\mathscr{C}}^{-}[X]$.

It follows from Lemmas 1.3 and 1.6 and the uniqueness of $\mathscr{C}$-almost split sequences that the quiver $\Gamma(\mathscr{C})$ defined above is a valued translation quiver. By definition, if $[X]$ and $[Y_1]$, $[Y_2]$, ..., $[Y_t]$ are vertices in $\Gamma(\mathscr{C})$, then $[Y_1]$, $[Y_2]$, ..., $[Y_t]$ are the left neighbours of $[X]$ in $\Gamma(\mathscr{C})$ if and only if the module $\bigoplus_1^t Y_i$ is the domain of the $\mathscr{C}$-sink map for the module X and $[Y_1]$, $[Y_2]$, ..., $[Y_t]$ are the right neighbours of $[X]$ in $\Gamma(\mathscr{C})$ if and only if the module $\bigoplus_1^t Y_i$ is the codomain of the $\mathscr{C}$-source map for the module X.

Let $\mathscr{C}$ be a subcategory of mod A having relative almost split sequences. When no confusion can arise, we do not distinguish between an indecomposable module X in $\mathscr{C}$ and the corresponding vertex $[X]$ of $\Gamma(\mathscr{C})$. From this point of view, the translation $\tau_{\mathscr{C}}$ of the quiver $\Gamma(\mathscr{C})$ can be regarded as an operation on the indecomposable modules in $\mathscr{C}$ which are not $\mathscr{C}$-projective while $\tau_{\mathscr{C}}^-$ can be regarded as an operation on the indecomposable modules in $\mathscr{C}$ which are not $\mathscr{C}$-injective. We extend these operations to the whole subcategory $\mathscr{C}$ by defining $\tau_{\mathscr{C}} X = 0$ for an indecomposable $\mathscr{C}$-projective module X and $\tau_{\mathscr{C}}(\bigoplus X_i) = \bigoplus \tau_{\mathscr{C}} X_i$; similarly $\tau_{\mathscr{C}}^- Y = 0$ for an indecomposable $\mathscr{C}$-injective module Y and $\tau_{\mathscr{C}}^-(\bigoplus Y_i) = \bigoplus \tau_{\mathscr{C}}^- Y_i$. We call $\tau_{\mathscr{C}}$ the *relative Auslander–Reiten translation* of the subcategory $\mathscr{C}$.

2. A Subcategory $\mathscr{C}$ with $\tau_{\mathscr{C}}$ Preserving Monomorphisms

In this section we describe the shapes of connected components of the Auslander–Reiten quiver of a subcategory whose relative Auslander–Reiten translation either preserves monomorphisms or preserves epimorphisms.

Throughout the rest of this paper, if $\mathscr{C}$ is a subcategory of mod A having relative almost split sequences, then we denote by $\Gamma(\mathscr{C})$ its Auslander–Reiten quiver and by $\tau_{\mathscr{C}}$ its relative Auslander–Reiten translation.

Let $\mathscr{C}$ be a subcategory of mod A having relative almost split sequences and let Γ be a connected component of $\Gamma(\mathscr{C})$. By saying that $\tau_{\mathscr{C}}$ *preserves monomorphisms in* Γ we mean that if there is a monomorphism $f: X \to Y$ in $\mathscr{C}$, where all indecomposable summands of X and Y are modules in Γ, none of which is $\mathscr{C}$-projective, then there is a monomorphism $f': \tau_{\mathscr{C}} X \to \tau_{\mathscr{C}} Y$. Saying that $\tau_{\mathscr{C}}$ *preserves* $\mathscr{C}$-*irreducible monomorphisms in* Γ means that if there is a $\mathscr{C}$-irreducible monomorphism $f: X \to Y$ with X and Y in Γ and not $\mathscr{C}$-projective, then there is a $\mathscr{C}$-irreducible monomorphism $f': \tau_{\mathscr{C}} X \to \tau_{\mathscr{C}} Y$. Under the dual conditions we say that $\tau_{\mathscr{C}}^-$ *preserves epimorphisms or* $\mathscr{C}$-*irreducible epimorphisms in* Γ.

For convenience, we give the following definition.

2.1. Definition. A subcategory $\mathscr{C}$ of mod A having relative almost split sequences is said to be an *Auslander–Smalø subcategory* if $\mathscr{C}$ is either closed under submodules or closed under factor modules, and there is only a finite number of isoclasses of indecomposable $\mathscr{C}$-projective modules and a finite number of isoclasses of indecomposable $\mathscr{C}$-injective modules in $\mathscr{C}$.

2.2. LEMMA. *Let $\mathscr{C}$ be an Auslander–Smalø subcategory of* mod A. *Then each $\mathscr{C}$-irreducible map in $\mathscr{C}$ is either an epimorphism or a monomorphism.*

Proof. It follows from the fact that $\mathscr{C}$ is either closed under submodules or closed under factor modules.

Let $\mathscr{C}$ be an Auslander–Smalø subcategory of mod A and Γ a connected component of $\Gamma(\mathscr{C})$. It follows from the above lemma that τ_c preserves $\mathscr{C}$-irreducible monomorphisms in Γ if and only if τ_c^- preserves $\mathscr{C}$-irreducible epimorphisms in Γ.

We will make frequent use of the following easy lemma.

2.3. LEMMA. *If*

$$0 \to X \xrightarrow{(f,f')} Y \oplus Y' \xrightarrow{\binom{g}{g'}} Z \to 0$$

is a short exact sequence in mod A, *then f is a monomorphism if and only if g' is a monomorphism and f is an epimorphism if and only if g' is an epimorphism.*

The following two lemmas are generalisations of similar results in [2].

2.4. LEMMA. *Let $\mathscr{C}$ be an Auslander–Smalø subcategory of* mod A. *Assume that*

$$\tau_c X \xrightarrow{f} Y \xrightarrow{f'} X \xrightarrow{g} \tau_c^- Y \xrightarrow{g'} \tau_c^- X$$

is a chain of $\mathscr{C}$-irreducible maps between indecomposable modules with f a monomorphism and g' an epimorphism. Then the module X has at most two left neighbours in $\Gamma(\mathscr{C})$ which are not $\mathscr{C}$-injective and at most two right neighbours in $\Gamma(\mathscr{C})$ which are not $\mathscr{C}$-projective.

Proof. It suffices to prove the first part of the statement. Assume that the module X has at least three left neighbours, say $Y_1 = Y$, Y_2, and Y_3, in $\Gamma(\mathscr{C})$ which are not $\mathscr{C}$-injective. Then $\tau_c^- Y_1$, $\tau_2^- Y_2$, and $\tau_c^- Y_3$ are right neighbours of X in $\Gamma(\mathscr{C})$ while X is a right neighbour of Y_i, $i = 1, 2, 3$, in $\Gamma(\mathscr{C})$ by Lemma 1.6. Thus we have

$$l(\tau_c X) + l(X) \geqslant l(Y_1) + l(Y_2) + l(Y_3);$$

$$l(Y_i) + l(\tau_c^- Y_i) \geqslant l(X),\ i = 1, 2, 3;$$

$$l(\tau_c^- X) + l(X) \geqslant l(\tau_c^- Y_1) + l(\tau_c^- Y_2) + l(\tau_c^- Y_3).$$

Hence we have $l(\tau_c X)+l(\tau_c^- X)\geqslant l(\tau_c^- Y)+l(Y)$. However, we know that $l(\tau_c X)<l(Y)$ since f is a proper monomorphism, and $l(\tau_c^- X)<l(\tau_c^- Y)$ since g' is a proper epimorphism. This gives rise to a contradiction. The proof is completed.

Let $\mathscr{C}$ be an Auslander–Smalø subcategory of mod A. A module X in $\Gamma(\mathscr{C})$ is said to be *left $\mathscr{C}$-stable* if $\tau_c^n X\neq 0$ for all positive integers n, *right $\mathscr{C}$-stable* if $\tau_c^n X\neq 0$ for all negative integers n, and *$\mathscr{C}$-stable* if $\tau_c^n X\neq 0$ for all integers n. An arrow $X\to Y$ in $\Gamma(\mathscr{C})$ is said to be *monic* if there is a $\mathscr{C}$-irreducible monomorphism from X to Y, and to be *epic* if there is a $\mathscr{C}$-irreducible epimorphism from X to Y. By Lemma 2.2, each arrow in $\Gamma(\mathscr{C})$ is either monic or epic.

2.5. Lemma. *Let $\mathscr{C}$ be an Auslander–Smalø subcategory of* mod A *and let Γ be a connected component of $\Gamma(\mathscr{C})$, containing no $\mathscr{C}$-injective module. Assume that τ_c preserves $\mathscr{C}$-irreducible monomorphisms in Γ. Then we have the following*:

(1) *If $X\to Y$ is an arrow in Γ, then either $X\to Y$ or $Y\to\tau_c^- X$ is monic, and furthermore, if X and Y are both $\mathscr{C}$-stable, then one of the arrows $X\to Y$ and $Y\to\tau_c^- X$ is monic and the other is epic.*

(2) *Let $X\to Y$ be a monic arrow in Γ. If X and Y are $\mathscr{C}$-stable, then for all integers $n\geqslant 0$, the arrow $\tau_c^{-n}X\to\tau_c^{-n}Y$ is monic.*

(3) *Let X be a module in Γ which has exactly two right neighbours Y and Y' in Γ. If Y, Y', and X are all $\mathscr{C}$-stable, then $Y\neq Y'$ and one of the arrows $X\to Y$ and $X\to Y'$ is monic and the other is epic.*

(4) *If $Y\to Z$ is an epic arrow in Γ with Z not $\mathscr{C}$-projective, then $\tau_c Z$ has at most two right neighbours in Γ.*

Proof. (1) First note that τ_c^- preserves $\mathscr{C}$-irreducible epimorphisms in Γ since τ_c preserves $\mathscr{C}$-irreducible monomorphisms in Γ by assumption. Assume that both $X\to Y$ and $Y\to\tau_c^- X$ are epic. Then for an arbitrary integer $n\geqslant 0$, there is a chain of $\mathscr{C}$-irreducible epimorphisms

$$X\to Y\to\tau_c^- X\to\tau_c^- Y\to\cdots\to\tau_c^{-n}X\to\tau_c^{-n}Y,$$

which is impossible. So at least one of $X\to Y$ and $Y\to\tau_c^- X$ is monic. Further assume that X and Y are $\mathscr{C}$-stable and that both $X\to Y$ and $Y\to\tau_c^- X$ are monic then, for an arbitrary integer $m\geqslant 0$, there is a chain of $\mathscr{C}$-irreducible monomorphisms

$$\tau_c^m X\to\tau_c^m Y\to\cdots\to\tau_c X\to\tau_c Y\to X\to Y,$$

which is again impossible.

(2) Assume that X and Y are $\mathscr{C}$-stable modules. By (1), the arrow $Y \to \tau_{\mathscr{C}}^{-} X$ is epic. Thus for any $n \geq 0$, the arrow $\tau_{\mathscr{C}}^{-n} Y \to \tau_{\mathscr{C}}^{-n-1} X$ is epic since $\tau_{\mathscr{C}}^{-}$ preserves $\mathscr{C}$-irreducible epimorphisms. Using (1) again, we see that the arrow $\tau_{\mathscr{C}}^{-n} X \to \tau_{\mathscr{C}}^{-n} Y$ is monic.

(3) By assumption, $Y \oplus Y'$ is the middle term of a $\mathscr{C}$-almost split sequence starting at X. Assume that Y, Y', and X are all $\mathscr{C}$-stable. If the arrow $X \to Y$ is monic, then so is $Y' \to \tau_{\mathscr{C}}^{-} X$ by Lemma 2.3. Hence $X \to Y'$ is epic by (1). Similarly if $X \to Y$ is epic, then $X \to Y'$ is monic.

(4) Assume that Z is not $\mathscr{C}$-projective and that $Y \to Z$ is an epic arrow in Γ. Then $\tau_{\mathscr{C}} Z \to Y$ is monic by (1), and $\tau_{\mathscr{C}}^{-} Y \to \tau_{\mathscr{C}}^{-} Z$ is epic since $\tau_{\mathscr{C}}^{-}$ preserves $\mathscr{C}$-irreducible epimorphisms in Γ. Thus there is a chain of $\mathscr{C}$-irreducible maps

$$\tau_{\mathscr{C}} Z \xrightarrow{f} Y \xrightarrow{f'} Z \xrightarrow{g} \tau_{\mathscr{C}}^{-} Y \xrightarrow{g'} \tau_{\mathscr{C}}^{-} Z$$

with f a monomorphism and g' an epimorphism. Since Γ contains no $\mathscr{C}$-injective module, it follows from Lemma 2.4 that Z has at most two left neighbours in Γ, that is, $\tau_{\mathscr{C}} Z$ has at most two right neighbours in Γ.

The basic idea in the proof of the following lemma comes from [16].

2.6. LEMMA. *Let $\mathscr{C}$ be an Auslander–Smalø subcategory of* mod A *and let Γ be a connected component of $\Gamma(\mathscr{C})$, containing no $\mathscr{C}$-injective module. Assume that $\tau_{\mathscr{C}}$ preserves monomorphisms in Γ. If a $\mathscr{C}$-stable module X in Γ has at least three right neighbours in Γ, then it has at least three $\mathscr{C}$-stable right neighbours in Γ, and for each* $n > 0$, *there is a monomorphism from $\tau_{\mathscr{C}}^{n} X$ to $\bigoplus_s X$, s copies of X, where $s = l(X)$.*

Proof. By assumption $\tau_{\mathscr{C}}$ preserves $\mathscr{C}$-irreducible monomorphisms in Γ and so $\tau_{\mathscr{C}}^{-}$ preserves $\mathscr{C}$-irreducible epimorphisms in Γ.

Assume that X is a $\mathscr{C}$-stable module in Γ. Since there is no $\mathscr{C}$-injective module in Γ, there is an arrow $X \to Y$ in Γ with Y a $\mathscr{C}$-stable module. Assume that X has at least three right neighbours in Γ, then so does $\tau_{\mathscr{C}}^{-n+1} X$ for each $n > 0$. Thus by Lemma 2.5(4), each arrow $W \to \tau_{\mathscr{C}}^{-n} X$ in Γ is monic. Hence each arrow $W' \to \tau_{\mathscr{C}}^{n-1} X$ in Γ is monic. In particular, the arrow $\tau_{\mathscr{C}}^{n} Y \to \tau_{\mathscr{C}}^{n-1} X$ is monic. So $\tau_{\mathscr{C}}^{n} X$ has at least two right neighbours in Γ. If $\tau_{\mathscr{C}}^{n} X$ has exactly two right neighbours, say $\tau_{\mathscr{C}}^{n} Y$ and Y', in Γ, then by Lemma 2.3, the arrow $\tau_{\mathscr{C}}^{n} X \to \tau_{\mathscr{C}}^{n} Y$ is monic since $Y' \to \tau_{\mathscr{C}}^{n-1} X$ is monic. Therefore $\tau_{\mathscr{C}}^{n} X \to \tau_{\mathscr{C}}^{n} Y$ and $\tau_{\mathscr{C}}^{n} Y \to \tau_{\mathscr{C}}^{n-1} X$ are both monic, which contradicts Lemma 2.5(1). Thus $\tau_{\mathscr{C}}^{n} X$ has at least three right neighbours in Γ. So X has at least three $\mathscr{C}$-stable right neighbours in Γ since there is no $\mathscr{C}$-injective modules in Γ.

Now, for each integer $n > 0$, let

$$0 \to \tau_{\mathscr{C}}^{-n+1} X \xrightarrow{(f_i)} \bigoplus_{i=1}^{s_n} Y_i \xrightarrow{(f_i')} \tau_{\mathscr{C}}^{-n} X \to 0$$

be a $\mathscr{C}$-almost split sequence. Then each $\mathscr{C}$-irreducible map f_i' is a monomorphism as we proved above. So there is a monomorphism from $\tau_{\mathscr{C}}^{-n+1} X$ to $\bigoplus_{s_n} \tau_{\mathscr{C}}^{-n} X$, that is, the module $\tau_{\mathscr{C}}^{-n+1} X$ is cogenerated by $\tau_{\mathscr{C}}^{-n} X$. Then, by induction, the module X is cogenerated by $\tau_{\mathscr{C}}^{-n} X$. Hence there is a monomorphism $h: X \to \bigoplus_s \tau_{\mathscr{C}}^{-n} X$ with $s = l(X)$. So there is a monomorphism $h': \tau_{\mathscr{C}}^{n} X \to \bigoplus_s X$ since $\tau_{\mathscr{C}}$ preserves monomorphisms in Γ. The proof is completed.

Note that if $\mathscr{C}$ is not closed under submodules, there may be a $\mathscr{C}$-irreducible epimorphism with codomain an indecomposable $\mathscr{C}$-projective module, and if $\mathscr{C}$ is not closed under factor modules, there may be a $\mathscr{C}$-irreducible monomorphism with domain an indecomposable $\mathscr{C}$-injective module.

2.7. THEOREM. *Let $\mathscr{C}$ be an Auslander–Smalø subcategory of* mod A *and let Γ be a connected component of $\Gamma(\mathscr{C})$, containing neither $\mathscr{C}$-injective modules nor oriented cycles. Assume that $\tau_{\mathscr{C}}$ preserves $\mathscr{C}$-irreducible monomorphisms in Γ. If Γ contains $\mathscr{C}$-stable modules, and each $\mathscr{C}$-stable module in Γ has at most two right neighbours in Γ, then either Γ is isomorphic to $\mathbf{Z}\mathbf{A}_\infty$ or Γ is obtained from $\mathbf{Z}\mathbf{A}_\infty$ by ray insertions.*

Proof. Since $\tau_{\mathscr{C}}$ preserves $\mathscr{C}$-irreducible monomorphisms in Γ by assumption, $\tau_{\mathscr{C}}^{-}$ preserves $\mathscr{C}$-irreducible epimorphisms in Γ. Assume now that each $\mathscr{C}$-stable module in Γ has at most two right neighbours. We divide the proof into several steps.

(1) *Let $X \to Y$ be an arrow in Γ between $\mathscr{C}$-stable modules X and Y. Then the arrow $X \to Y$ has trivial valuation.*

Since X is $\mathscr{C}$-stable, it has at most two right neighbours in Γ by assumption. If Y appears more than once as a right neighbour of X, then X has exactly two right neighbours Y, Y. This contradicts Lemma 2.5(3) since Y is $\mathscr{C}$-stable. So Y appears just once as a right neighbour of X. It follows similarly that $\tau_{\mathscr{C}}^{-} X$ appears just once as a right neighbour of Y. Thus the arrow $X \to Y$ has trivial valuation.

(2) *Let $X_1 \to X_2 \to \cdots \to X_{s-1} \to X_s$ be a sectional path in Γ with a monic arrow $X_{s-1} \to X_s$ between $\mathscr{C}$-stable modules X_{s-1} and X_s. For each i, $1 \leqslant i \leqslant s-1$, the following hold:*

(a) *The arrow $X_i \to X_{i+1}$ is monic while the arrow $X_{i+1} \to \tau_{\mathscr{C}}^{-} X_i$ is epic;*

(b) *The module X_i has at most two right neighbours;*

(c) *The arrow $X_i \to X_{i+1}$ has trivial valuation.*

We use induction to prove (2). Since X_{s-1} and X_s are both $\mathscr{C}$-stable, X_{s-1} has at most two right neighbours by assumption and the arrow $X_{s-1} \to X_s$ has trivial valuation by (1). Since $X_{s-1} \to X_s$ is monic, the arrow $X_s \to \tau_c^- X_{s-1}$ is epic by Lemma 2.5(1). So the statements (a), (b), and (c) hold for $i = s-1$. Assume that they are true for i, $1 < i \leqslant s-1$. Then the module X_i has exactly two right neighbours X_{i+1} and $\tau_c^- X_{i-1}$ in Γ by (b). Since $X_{i+1} \to \tau_c^- X_i$ is epic, so is the arrow $X_i \to \tau_c^- X_{i-1}$ by Lemma 2.3. Hence the arrow $X_{i-1} \to X_i$ is monic by Lemma 2.5(1) and X_{i-1} has at most two right neighbours in Γ by Lemma 2.5(4). Thus it follows that X_i appears just once as a right neighbour of X_{i-1}. Moreover, we have shown that $\tau_c^- X_{i-1}$ appears just once as a right neighbour of X_i. So the arrow $X_{i-1} \to X_i$ has trivial valuation.

(3) *Let $X \to Y$ be a monic arrow in Γ between $\mathscr{C}$-stable modules X and Y. Then there is a sectional path $X \to Y \to Z$ in Γ.*

By (1), the arrow $X \to Y$ has trivial valuation. Since the arrow $X \to Y$ is monic, the module Y has at least two left neighbours, say X and Y', which are different since $X \to Y$ has trivial valuation. So $X \to Y \to \tau_c^- Y'$ is a sectional path in Γ since Γ contains no $\mathscr{C}$-injective module.

(4) *Let $X \to Y \to Z$ be a sectional path in Γ with a monic arrow $X \to Y$ between $\mathscr{C}$-stable modules X and Y. Then Z is $\mathscr{C}$-stable and the arrow $Y \to Z$ is monic.*

Since Y is $\mathscr{C}$-stable, it has exactly two right neighbours $\tau_c^- X$ and Z in Γ. For each integer $n \geqslant 0$, the arrow $\tau_c^n X \to \tau_c^n Y$ is monic since $X \to Y$ is monic. Therefore, there is a sectional path $\tau_c^n X \to \tau_c^n Y \to X_n$ in Γ by (3). So $X \to Y \to \tau_c^{-n} X_n$ is also a sectional path in Γ. Thus $\tau_c^{-n} X_n = Z$. This implies that Z is $\mathscr{C}$-stable. By Lemma 2.5(1) the arrow $Y \to \tau_c^- X$ is epic, and so is the arrow $Z \to \tau_c^- Y$. Hence the arrow $Y \to Z$ is monic by Lemma 2.5(1) again.

(5) *Let $X \to Y$ be a monic arrow in Γ between $\mathscr{C}$-stable modules X and Y. Then the arrow $X \to Y$ is contained in a unique infinite sectional path*

$$\sigma_0 \colon X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to \cdots$$

in Γ which satisfies the following conditions:

(a) *The module X_1 has no immediate predecessor other than $\tau_c X_2$ in Γ, and has at most two immediate successors in Γ;*

(b) *For all integers $i > 1$, the module X_i has exactly two immediate successors X_{i+1} and $\tau_c^- X_{i-1}$;*

(c) *For all integers $i \geqslant 1$, the arrow $X_i \to X_{i+1}$ is monic and has trivial valuation;*

(d) *The modules X_i belong to pairwise different $\tau_{\mathscr{C}}$-orbits;*

(e) *There is some integer $m_0 \geqslant 0$ such that X_i is $\mathscr{C}$-stable if and only if $i > m_0$;*

(f) *If p is the number of $\mathscr{C}$-projective modules in Γ, then $0 \leqslant m_0 \leqslant p$. If $m_0 > 0$ and $X_i = \tau^{-n_i} P_i$ where P_i is $\mathscr{C}$-projective, then $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{m_0}$.*

From (2) there is a bound on the lengths of the sectional paths in Γ ending with the arrow $X \to Y$. Let

$$X_1 \to X_2 \to \cdots \to X_{s-1} \to X_s$$

be a sectional path in Γ of maximal length with $X_{s-1} = X$ and $X_s = Y$. Then X_1 has no immediate predecessor other than $\tau_{\mathscr{C}} X_2$ in Γ. Moreover, by (2), the module X_1 has at most two immediate successors in Γ while for $1 < i < s$, the module X_i has exactly two immediate successors X_{i+1} and $\tau_{\mathscr{C}}^{-} X_{i-1}$, and for all i, $1 \leqslant i < s$, the arrow $X_i \to X_{i+1}$ is monic and has trivial valuation.

Since $X_{s-1} \to X_s$ is monic and X_{s-1} and X_s are $\mathscr{C}$-stable, it follows from (3) and (4) that there is a sectional path $X_{s-1} \to X_s \to X_{s+1}$ in Γ such that the module X_{s+1} is $\mathscr{C}$-stable and the arrow $X_s \to X_{s+1}$ is monic. Hence, by induction, we obtain an infinite sectional path

$$X_{s-1} \to X_s \to X_{s+1} \to \cdots \to X_i \to X_{i+1} \to \cdots$$

in Γ such that for all $i \geqslant s-1$, the module X_i is $\mathscr{C}$-stable and the arrow $X_i \to X_{i+1}$ is monic. Moreover, for all $i \geqslant s-1$, the arrow $X_i \to X_{i+1}$ has trivial valuation by (1), and the module X_i has exactly two immediate successors X_{i+1} and $\tau_{\mathscr{C}}^{-} X_{i-1}$ from the assumption. So the infinite sectional path

$$\sigma_0\colon X_1 \to X_2 \to \cdots \to X_{s-1} \to X_s \to X_{s+1} \to \cdots \to X_i \to X_{i+1} \to \cdots$$

satisfies the conditions (a), (b), *and* (c). So σ_0 is the *unique maximal* infinite sectional path in Γ which contains the arrow $X \to Y$ and, for all $i \geqslant 1$,

$$X_i \to X_{i+1} \to \cdots \to X_{i+j} \to \cdots$$

is the unique infinite sectional path in Γ which starts with the arrow $X_i \to X_{i+1}$.

Since X_{s-1} is $\mathscr{C}$-stable, there is an integer m_0, $0 \leqslant m_0 < s-1$, such that X_{m_0+1} is $\mathscr{C}$-stable and X_i is not $\mathscr{C}$-stable if $i \leqslant m_0$. Since Γ contains no $\mathscr{C}$-injective module, there is a $\mathscr{C}$-stable module X' such that $X_{m_0+1} \to X'$ is an arrow in Γ. Assume that $X' \neq X_{m_0+2}$. Then either $m_0 > 0$ and $\tau_{\mathscr{C}}^{-} X_{m_0} = X'$, which contradicts the assumption on m_0, or $m_0 = 0$ and X_1

has an immediate predecessor $\tau_c X' \neq \tau_c X_2$, which is a contradiction to the condition (a). So $X_{m_0+2} = X'$ is $\mathscr{C}$-stable. Thus by (4), all modules X_i with $i > m_0$ are $\mathscr{C}$-stable. Hence *the sectional path* σ_0 *satisfies the condition* (e).

We show now that the modules X_i in σ_0 belong to pairwise different τ_c-orbits. We first prove that X_1 and X_j belong to different τ_c-orbits if $j > 1$. Suppose that X_1 and X_j belong to the same τ_c-orbit for some $j > 1$. If X_1 is $\mathscr{C}$-stable, then X_1 has two immediate predecessors in Γ since X_j has two $\mathscr{C}$-stable predecessors in Γ. This contradicts the condition (a). So X_1 is not $\mathscr{C}$-stable. Hence $j \leqslant m_0$. Now we may suppose that the modules X_u and X_v, $1 \leqslant u < v \leqslant m_0$, lie in the same τ_c-orbit and such that $(v-u)$ is minimal for this condition to hold. Let $X_v = \tau_c^{-r} X_u$ for some integer r. If $r \leqslant 0$, then there is an oriented cycle in Γ, contrary to assumption. So $r > 0$. Therefore it is easy to see that we obtain an infinite sectional path

$$X_u \to X_{u+1} \to \cdots \to X_{v-1} \to \tau_c^{-r} X_u \to \tau_c^{-r} X_{u+1} \to \cdots$$

in Γ containing no $\mathscr{C}$-stable module. This contradicts the uniqueness of the infinite sectional path starting with the arrow $X_u \to X_{u+1}$. So X_1 and X_j belong to different τ_c-orbits if $j > 1$.

Assume now that, for some $n > 1$ and for each i, $1 \leqslant i \leqslant n-1$, the modules X_i and X_j belong to different τ_c-orbits if $j > i$. Suppose that X_n and X_j belong to the same τ_c-orbit for some $j > n$. Then $X_j = \tau_c^{-r} X_n$ for some $r > 0$ since there is no oriented cycle in Γ. So $\tau_c^{-r-1} X_{n-1}$ is an immediate successor of X_j. Therefore either $\tau_c^{-r-1} X_{n-1} = \tau_c^{-} X_{j-1}$ or $\tau_c^{-r-1} X_{n-1} = X_{j+1}$, which contradicts the inductive assumption. Thus σ_0 *satisfies the condition* (d).

As we can see, the number m_0 is just the number of modules in σ_0 which are not $\mathscr{C}$-stable. These modules belong to pairwise different τ_c-orbits by (d). So m_0 is less than or equal to the number p of $\mathscr{C}$-projective modules in Γ. Now assume that $m_0 > 0$ and $X_i = \tau_c^{-n_i} P_i$ where P_i is $\mathscr{C}$-projective, $i = 1, \ldots, m_0$. Assume that there is some j, $1 < j \leqslant m_0$, such that $n_{j-1} > n_j$. Then $\tau_c^{n_j} X_{j-1}$ has at least one immediate predecessor, say X', in Γ since $\tau_c^{n_j+1} X_{j-1} \neq 0$. Clearly $X' \neq \tau_c^{n_j+1} X_j$ since $\tau_c^{n_j} X_j$ is $\mathscr{C}$-projective. So $\tau^{-n_j} X'$ is an immediate predecessor of X_{j-1} other than $\tau_c X_j$. So $j-1 > 1$ by (a). Hence it follows that $\tau^{-n_j} X' = X_{j-2}$. Thus the arrow $X' \to \tau_c^{n_j} X_{j-1}$ is monic and has trivial valuation. Therefore $\tau_c^{n_j} X_{j-1}$ has at least two immediate predecessors each of which is different from $\tau_c^{n_j+1} X_j$. Thus $\tau_c^{n_j} X_{j-1}$ has at least three immediate successors, so does X_{j-1}, which is a contradiction to (b). So σ_0 *satisfies the condition* (f).

We now complete the proof of the theorem. Let X be a $\mathscr{C}$-stable module in Γ. Since there is no $\mathscr{C}$-injective module in Γ, there exists an arrow $X \to Y$ with a $\mathscr{C}$-stable module Y. By Lemma 2.5(1), either $X \to Y$ or $Y \to \tau_c^{-} X$ is monic. Without loss of generality we may assume that $X \to Y$ is monic.

Then by Lemma 2.5(2), for each integer $r \geqslant 0$, the arrow $\tau_{\mathscr{C}}^{-r}X \to \tau_{\mathscr{C}}^{-r}Y$ is monic. So the arrow $\tau_{\mathscr{C}}^{-r}X \to \tau_{\mathscr{C}}^{-r}Y$ is contained in a unique maximal infinite sectional path σ_r in Γ which satisfies the conditions stated in (5). Let m_r be the number of the modules in σ_r which are not $\mathscr{C}$-stable. Then $m_r \leqslant p$, the number of $\mathscr{C}$-projective modules in Γ. Let m_{r_0} be the largest of the m_r. Without loss of generality we may assume that $r_0 = 0$. This implies that for each integer $n \geqslant 0$, the module $\tau_{\mathscr{C}}^{-n}X_1$ has only one immediate successor $\tau_{\mathscr{C}}^{-n}X_2$. So σ_0 meets each $\tau_{\mathscr{C}}$-orbit in Γ exactly once. Therefore Γ has trivial valuation. If $m_0 = 0$, then Γ is isomorphic to $\mathbf{Z}\mathbf{A}_\infty$. If $m_0 > 0$, then $X_i = \tau^{-n_i}P_i$ where P_i is $\mathscr{C}$-projective with $n_1 \leqslant \cdots \leqslant n_{m_0}$. Let $r_1, r_2, ..., r_s$ be positive integers such that $n_1 = \cdots = n_{r_1} < n_{r_1+1} = \cdots = n_{r_2} < \cdots < n_{r_{s-1}+1} = \cdots = n_{r_s} = n_{m_0}$. Then it follows now that Γ, as a translation quiver, is of form $\mathbf{Z}\mathbf{A}_\infty(r_1, r_2 - r_1, ..., r_s - r_{s-1})$. The proof is completed.

We have the following dual result:

2.8. THEOREM. *Let $\mathscr{C}$ be an Auslander–Smalø subcategory of* mod A *and let Γ be a connected component of $\Gamma(\mathscr{C})$, containing neither $\mathscr{C}$-projective modules nor oriented cycles. Assume that $\tau_{\mathscr{C}}^{-}$ preserves $\mathscr{C}$-irreducible epimorphisms in Γ. If Γ contains $\mathscr{C}$-stable modules, and each $\mathscr{C}$-stable module has at most two left neighbours, then either Γ is isomorphic to $\mathbf{Z}\mathbf{A}_\infty$ or Γ is obtained from $\mathbf{Z}\mathbf{A}_\infty$ by coray insertions.*

3. SPLITTING TILTING THEORY

If X is a module in mod A, then we denote by p.d.X the *projective dimension*, and i.d.X the *injective dimension* of X. A module T in mod A is called a *tilting module* if (1) p.d.$T \leqslant 1$; (2) $\mathrm{Ext}_A^1(T, T) = 0$; and (3) There is an short exact sequence $0 \to A \to T' \to T'' \to 0$ with T', T'' direct sums of direct summands of T [8].

Let T be a tilting module in mod A, and let $B = \mathrm{End}_A(T)$. Denote by mod B the category of finitely generated left B-modules. by Γ_B the Auslander–Reiten quiver of B, and by τ_B the Auslander–Reiten translation of Γ_B. Then T is a A-B-bimodule in a natural way. Hence $F = \mathrm{Hom}_A(T, -)$ and $F' = \mathrm{Ext}_A^1(T, -)$ are functors from mod A to mod B while $G = T \otimes_B -$ and $G' = \mathrm{Tor}_1^B(T, -)$ are functors from mod B to mod A. Let $\mathscr{T}(T) = \{X \mid F'(X) = 0\}$ and $\mathscr{F}(T) = \{X \mid F(X) = 0\}$, and let $\mathscr{X}(T) = \{X \mid G(X) = 0\}$ and $\mathscr{Y}(T) = \{X \mid G'(X) = 0\}$. It is shown in [8] that $(\mathscr{F}(T), \mathscr{T}(T))$ is a torsion theory in mod A with the torsion class $\mathscr{T}(T)$ containing all injective modules in mod A while $(\mathscr{Y}(T), \mathscr{X}(T))$ is a torsion theory in mod B with torsion-free class $\mathscr{Y}(T)$ containing all projective modules in mod B.

The torsion theory $(\mathscr{F}(T), \mathscr{T}(T))$ induces an *idempotent radical* $\mathbf{t}$. That is, $\mathbf{t}$ is an idempotent subfunctor of the identity functor of mod A which assigns to a module X in mod A, the submodule $\mathbf{t}(X)$ of X such that $\mathbf{t}(X)$ is in $\mathscr{T}(T)$ while $X/\mathbf{t}(X)$ is in $\mathscr{F}(T)$. We fix the notation above in the rest of this section.

3.1. BRENNER–BUTLER THEOREM [6]. *Let T be a tilting module in* mod A *and $B = \mathrm{End}_A(T)$. The functor F induces a left exact equivalence between $\mathscr{T}(T)$ and $\mathscr{Y}(T)$ while the functor F' induces a right exact equivalence between $\mathscr{F}(T)$ and $\mathscr{X}(T)$.*

A tilting module T in mod A is said to be *splitting* if each indecomposable module in mod B is either in $\mathscr{X}(T)$ or in $\mathscr{Y}(T)$.

3.2. LEMMA [8, 10]. *Let T be a splitting tilting module in* mod A *and* $B = \mathrm{End}_A(T)$.

(1) *Both $\mathscr{F}(T)$ and $\mathscr{T}(T)$ are Auslander–Smalø subcategories in* mod A.

(2) *If Q is an indecomposable injective B-module in $\mathscr{Y}(T)$, then $Q = F(I)$ with I an indecomposable injective A-module in $\mathscr{T}(T)$.*

(3) *An almost split sequence in* mod B *is either completely contained in one of $\mathscr{X}(T)$ and $\mathscr{Y}(T)$, or is of the form*

$$0 \to F(I) \to F(I/S) \oplus F'(\mathrm{rad}\, P) \to F'(P) \to 0,$$

where I is an indecomposable injective module in mod A *with socle S, while P is an indecomposable projective module in* mod A *with top S.*

3.3. LEMMA [8]. *Let T be a splitting tilting module in* mod A, *and let $B = \mathrm{End}_A(T)$. Suppose that Γ is a connected component of Γ_B. Then*

(1) *If Γ is completely contained in $\mathscr{Y}(T)$, then there is a connected component Γ' of $\Gamma(\mathscr{T}(T))$ such that the functor F induces a valued translation quiver isomorphism from Γ' to Γ.*

(2) *If Γ is completely contained in $\mathscr{X}(T)$, then there is a connected component Γ' of $\Gamma(\mathscr{F}(T))$ such that the functor F' induces a valued translation quiver isomorphism from Γ' to Γ.*

If T is a splitting tilting module in mod A, then we denote by τ_t the relative Auslander–Reiten translation of the subcategory $\mathscr{T}(T)$ and by τ_f the relative Auslander–Reiten translation of the subcategory $\mathscr{F}(T)$. We fix this notation for a splitting tilting module T in the sequel.

3.4. HOSHINO'S LEMMA [10]. *Let T be a splitting tilting module in* mod A *and* $\mathbf{t}$ *the idempotent radical determined by* T.

(1) *If X is a module in $\mathscr{T}(T)$, then $\tau_t X = \mathbf{t}(\tau X)$.*

(2) *If X is a module in $\mathscr{F}(T)$, then i.d. $X \leqslant 1$ and $\tau_f^- X = \tau^- X/\mathbf{t}(\tau^- X)$.*

COROLLARY. *Let T and* $\mathbf{t}$ *be as above.*

(1) *Assume that X and Y are modules in $\mathscr{T}(T)$. If there is a monomorphism from τX to τY, then there is a monomorphism from $\tau_t X$ to $\tau_t Y$.*

(2) *Assume that X and Y are modules in $\mathscr{F}(T)$. If there is an epimorphism from $\tau^- X$ to $\tau^- Y$, then there is an epimorphism from $\tau_f^- X$ to $\tau_f^- Y$.*

3.5. PROPOSITION. *Let T be a splitting tilting module in* mod A. *Then τ_f^- preserves epimorphisms in $\mathscr{F}(T)$.*

Proof. Note that $\mathscr{F}(T)$ is closed under submodules. Assume that X is a module in $\mathscr{F}(T)$. Since i.d.$X \leqslant 1$ by Hoshino's lemma (3.4), we have $\tau^- X = \operatorname{Tr} DX = \operatorname{Ext}^1_A(DX, A)$. (See, for example, [17, (2.4)]).

Now suppose that $f\colon Y \to Z$ is an epimorphism in $\mathscr{F}(T)$. Then there is a short exact sequence $0 \to X \to Y \xrightarrow{f} Z \to 0$ in mod A with X in $\mathscr{F}(T)$. Hence there is a short exact sequence

$$0 \to DZ \xrightarrow{Df} DY \to DX \to 0$$

in mod A^{op}. Note that p.d.$DX \leqslant 1$, since i.d.$X \leqslant 1$. Therefore the sequence

$$\operatorname{Ext}^1_A(DX, A) \to \operatorname{Ext}^1_A(DY, A) \to \operatorname{Ext}^1_A(DZ, A) \to 0$$

induced by Df is exact, which implies that f induces an epimorphism from $\tau^- Y$ to $\tau^- Z$ in mod A. Hence by the corollary to Hoshino's lemma (3.4), f induces an epimorphism from $\tau_f^- Y$ to $\tau_f^- Z$ in $\mathscr{F}(T)$.

3.6. THEOREM. *Let T be a splitting tilting module in* mod A *and* $B = \operatorname{End}_A(T)$. *Suppose that Γ is a connected component of Γ_B which lies completely in $\mathscr{X}(T)$. If Γ is not a preinjective component, then either Γ is quasi-serial or Γ is obtained from a quasi-serial translation quiver by coray insertions.*

Proof. Since Γ is completely contained in $\mathscr{X}(T)$, then by Lemma 3.3(2) there is a connected component Γ' of $\Gamma(\mathscr{F}(T))$ such that the functor F' induces a valued translation quiver isomorphism from Γ' to Γ. Since $\mathscr{X}(T)$ does not contain projective modules in mod B, neither does Γ. We prove the theorem by considering seperately the following three cases:

(a) There is an oriented cycle in Γ. In this case by Lemma 1.2, either Γ is a stable tube or a coray tube.

(b) There is neither an oriented cycle nor a stable module in Γ. In this case each module in Γ belongs to the τ_B-orbit of an injective module in Γ_B. So Γ is a preinjective component.

(c) Γ contains stable modules but no oriented cycle. In this case Γ' contains neither $\mathscr{F}(T)$-projective modules nor oriented cycles. However, it does contain $\mathscr{F}(T)$-stable modules. In addition, by Proposition 3.5, τ_f^- preserves epimorphisms in Γ'. Assume that there is a $\mathscr{F}(T)$-stable module X in Γ' which has at least three left neighbours in Γ'. Then by the dual of Lemma 2.6, the module X has at least three $\mathscr{F}(T)$-stable left neighbours in Γ', and for each $n > 0$, there is an epimorphism from $\oplus_s X$ to $\tau_f^{-n}X$, where $s = l(X)$. Since F' induces a valued translation quiver from Γ' to Γ, the module $M = F'X$ is a stable B-module in Γ which has at least three stable left neighbours in Γ. Furthermore since F' is a right exact functor, for each $n > 0$, there is an epimorphism from $\oplus_s M$ to $\tau_B^{-n}M$, which contradicts Lemma 1.1.

So each $\mathscr{F}(T)$-stable module X in Γ' has at most two left neighbours. Hence, by Theorem 2.8, we conclude that either Γ' is isomorphic to $\mathbf{Z}\mathbf{A}_\infty$, or Γ' is obtained from $\mathbf{Z}\mathbf{A}_\infty$ by coray insertions. Therefore either Γ is isomorphic to $\mathbf{Z}\mathbf{A}_\infty$, or Γ is obtained from $\mathbf{Z}\mathbf{A}_\infty$ by coray insertions. This completes the proof.

Now we assume that A is an hereditary algebra, T is a tilting module in mod A, and $B = \operatorname{End}_A(T)$. It is well known that T is a splitting tilting module [8]. Further all modules in Γ_B of form $F(I)$, with I an indecomposable injective module in mod A, lie in a single connected component of Γ_B, called the *connecting component* determined by T. Let Γ be a connected component of Γ_B other than the connecting component. Then either Γ is completely contained in $\mathscr{Y}(T)$ and contains no injective $\mathscr{B}$-module or Γ is completely contained in $\mathscr{X}(T)$ and contains no projective B-module. Note that in this case τ_t preserves monomorphisms in $\mathscr{T}(T)$ [16] while τ_f^- preserves epimorphisms in $\mathscr{F}(T)$ (3.5).

3.7. THEOREM. *Let A be an hereditary connected artin algebra. Let T be a tilting module in* mod A, *and let* $B = \operatorname{End}_A(T)$. *Assume that Γ is a connected component of Γ_B other than the connecting component determined by T. If Γ is neither a preprojective component nor a preinjective component, then either Γ is quasi-serial or Γ is obtained from a quasi-serial translation quiver by ray insertions or by coray insertions.*

Proof. Since Γ is not the connecting component of Γ_B, either Γ is completely contained in $\mathscr{X}(T)$ or Γ is completely contained in $\mathscr{Y}(T)$. If Γ lies in $\mathscr{X}(T)$, then Theorem 3.6 shows that Γ is preinjective, quasi-serial, or obtained from a quasi-serial translation quiver by coray insertions. If Γ lies in $\mathscr{Y}(T)$, since Γ contains no injective module in mod B and τ_t preserves monomorphisms in $\mathscr{T}(T)$ then, dualising the argument used to prove Theorem 3.6, we find that Γ is preprojective, quasi-serial, or obtained from a quasi-serial translation quiver by ray insertions.

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References

1. I. Assem, "Tilting Theory—An Introduction," Rapport No. 47, Universite de Sherbrooke, September 1988.
2. M. Auslander, R. Bautista, M. I. Platzeck, I. Reiten, and S. O. Smalø, Almost split sequences whose middle term has at most two indecomposable summands, *Canad. J. Math.* **31** (1979), 942–960.
3. M. Auslander and I. Reiten, Representation theory of artin algebras. III. Almost split sequences, *Comm. Algebra* **3** (1975), 239–294.
4. M. Auslander and I. Reiten, Representation theory of artin algebras. IV. Invariants given by almost split sequences, *Comm. Algebra* **5** (1977), 443–518.
5. M. Auslander and S. O. Smalø, Almost split sequences in subcategories, *J. Algebra* **69** (1988), 426–454.
6. S. Brenner and M. C. R. Butler, Generalisations of the Bernstein–Gelfand–Ponomarev reflection functors, *in* "Lecture Notes in Math.," Vol. 832, pp. 103–169, Springer-Verlag, New York/Berlin, 1980.
7. D. Happel and C. M. Ringel, Construction of tilted algebras, *in* "Representations of Algebras," pp. 125–144, Lecture Notes in Math., Vol. 903, Springer-Verlag, New York/Berlin, 1981.
8. D. Happel and C. M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* **274** (1982), 339–443.
9. D. Happel, U. Preiser, and C. M. Ringel, Vinberg's characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules, *in* "Lecture Notes in Math.," Vol. 832, pp. 280–294, Springer-Verlag, New York/Berlin, 1980.
10. M. Hoshino, On splitting torsion theories induced by tilting modules, *Comm. Algebra* **11**, No. 4 (1983), 427–429.
11. O. Kerner, Tilting wild algebras, *J. London Math. Soc.* **39**, No. 2 (1989), 29–47.
12. O. Kerner, Stable components of wild tilted algebras, *J. Algebra* **142** (1991), 37–57.
13. S. Liu, Degrees of irreducible maps and the shapes of Auslander–Reiten quivers, *J. London Math. Soc. (2)* **45** (1992), 32–54.
14. S. Liu, Semi-stable components of an Auslander–Reiten quiver, *J. London Math. Soc. (2)* **47** (1993), 405–416.

15. C. M. RINGEL, Finite dimensional hereditary algebras of wild representation type, *Math. Z.* **161** (1978), 236–255.
16. C. M. RINGEL, Report on the Brauer–Thrall conjectures: Rojter's theorem and the theorem of Nazarova and Rojter, *in* "Lecture Notes in Math.," Vol. 831, Springer-Verlag, New York/Berlin, 1979.
17. C. M. RINGEL, Tame algebras and integral quadratic forms, *in* "Lecture Notes in Math.," Vol. 1099, Springer-Verlag, New York/Berlin, 1984.
18. C. M. RINGEL, The regular components of the Auslander–Reiten quiver of a tilted algebra, *Chinese Ann. Math. Ser. B* **9**, No. 1 (1988), 1–18.