On the C-Numerical Range of a Matrix

Jean Dazord*

Laboratoire de Probabilités et Statistique
I.A.S.B.S.E.
Université Claude Bernard-Lyon 1
43, boulevard du 11 Novembre 1918
69622 Villeurbanne Cedex, France

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ABSTRACT

Given two $n \times n$ complex matrices C and T, we prove that if the differentiable mapping $q : U(n, \mathbf{C}) \to \mathbf{R}^2$ defined by $q(U) = \operatorname{tr}(CU^*TU)$ is of rank at most 1 on a nonempty open set, then the C-numerical range $W(C, T)$ of T is a line segment. The same conclusion holds whenever the interior of $W(C, T)$ is empty.

1. INTRODUCTION

Let us denote by $M_n(\mathbf{C})$ the algebra of complex $n \times n$ matrices and by $U(n, \mathbf{C})$, the group of unitary $n \times n$ matrices. Given two matrices $C, T \in M_n(\mathbf{C})$ the C-numerical range of T is the subset $W(C, T)$ of the complex plane defined by $W(C, T) = \{\operatorname{tr}(CU^*TU); U \in U(n, \mathbf{C})\}$, where $\operatorname{tr}(X)$ denotes the trace of the matrix X. When C is an orthogonal rank-one projection, the C-numerical range of T is the usual numerical range $W(T)$ of T.

Throughout the paper $\mathbf{C}$ is identified with $\mathbf{R}^2$ and the C-numerical range $W(C, T)$ of T is considered as a subset of $\mathbf{R}^2$.

In [3, Theorem 3], Marcus and Sandy gave a necessary and sufficient condition for the C-numerical range of T to be real. A necessary and sufficient condition for the C-numerical range of T to be either a point or a nondegenerate line segment was given by Chi-Kwong Li in [2, Theorems 2.5, 2.7].

*E-mail: DAZORD@algebre.univ-lyon1.fr.

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

In this paper we obtain another necessary and sufficient condition for the set $W(C,T)$ to be either a point or a line segment and an extension of these results.

2. THE RANK OF THE MAPPING q

Our study requires notions of differential geometry included in most textbooks. The reader may also consult References [8] and [4].

Consider the mapping $q : U(n, \mathbf{C}) \to \mathbf{R}^2$ defined by $q(U) = \mathrm{tr}(CU^*TU)$. The C-numerical range of T is the set $W(C,T) = q(U(n, \mathbf{C}))$. Both $U(n, \mathbf{C})$ and $\mathbf{R}^2$ are real C^∞ manifolds, and q is a C^∞ mapping. Let $u(n, \mathbf{C})$ be the $\mathbf{R}$-algebra of skew-Hermitian matrices. Let us denote by $T_U q : Uu(n, \mathbf{C}) \to \mathbf{R}^2$ the mapping tangent to q at $U \in U(n, \mathbf{C})$.

In our first result, the brackets stand for the commutator, $[X,Y] = XY - YX$; $\mathrm{lin}(E)$ and $\dim(\mathrm{lin}(E))$ denote respectively the linear subspace of $\mathbf{R}^2$ generated by a subset E of $\mathbf{R}^2$ and the dimension of $\mathrm{lin}(E)$; and i denotes a square root of -1.

THEOREM 2.1. *The mapping $T_U q$ tangent to q at $U \in U(n, \mathbf{C})$ is given by $T_U q(UA) = tr([C, U^*TU]A), A \in u(n, \mathbf{C})$. In particular, the rank of the mapping q at U is $rk_U(q) = \dim(\mathrm{lin}(W([C, U^*TU])))$.*

Proof. The proof rests on an easy computation. Consider $A \in u(n, \mathbf{C})$, and consider the one-parameter unitary group $U(t) = \exp(tA)$, $t \in \mathbf{R}$, where $\exp(X)$ denotes the exponential of the matrix $X \in M_n(\mathbf{C})$. We have

$$\begin{aligned} T_{U(t)}q(U(t)A) &= \{\mathrm{tr}(C\exp(-tA)T\exp(tA))\}' \\ &= \mathrm{tr}(C(-A)\exp(-tA)T\exp(tA) + C\exp(-tA)TA\exp(tA)) \\ &= \mathrm{tr}([A,C]U(t)^*TU(t)) = \mathrm{tr}([C, U(t)^*TU(t)]A). \end{aligned}$$

The formula $T_U q(UA) = \mathrm{tr}([C, U^*TU]A), A \in u(n, \mathbf{C})$, is now clear. On the other hand, for any matrix $X \in M_n(\mathbf{C})$, we have

$$\mathrm{lin}(W(X)) = \mathrm{tr}(X \cdot iu(n, \mathbf{C})) = i\ \mathrm{tr}(Xu(n, \mathbf{C}))$$

and hence the range of the mapping $T_U q$ tangent to q at U is given by $T_U q(Uu(n, \mathbf{C})) = \mathrm{tr}([C, U^*TU]u(n, \mathbf{C})) = i\ \mathrm{lin}(W([C, U^*TU]))$. In particular the rank of q at U is, as indicated in the theorem, $\mathrm{rk}_U(q) = \dim(\mathrm{lin}(W([C, U^*TU])))$. ■

3. STUDY OF THE CASE WHERE $W(C,T)$ IS A POINT

The following lemma is well known [5, Corollary 0.14, p. 8]:

LEMMA 3.1. *Consider the matrices* $A \in M_r(\mathbf{C}), B \in M_s(\mathbf{C})$, $T \in M_n(\mathbf{C})$, *with* $r+s=n$. *If the matrix* T *commutes with the block diagonal matrix* $\operatorname{diag}(A,B)$ *and if the spectra of the matrices* A *and* B *are disjoint, then* T *is a block diagonal matrix* $\operatorname{diag}(C,D)$ *where* $C \in M_r(\mathbf{C})$ *commutes with* A, *and* $D \in M_s(\mathbf{C})$ *commutes with* B.

The unitary orbit of a matrix X is the subset of $M_n(\mathbf{C})$ defined by $\{U^*XU; U \in U(n,\mathbf{C})\}$. We say that the unitary orbits of X and Y commute if $[U^*XU, V^*YV] = 0$ for any $(U,V) \in U(n,\mathbf{C}) \times U(n,\mathbf{C})$. Since we have

$$[U^*TU, V^*TV] = U^*[C, (VU^*)^*T(VU^*)]U = V^*[(UV^*)^*C(UV^*), T]V, \tag{3.1}$$

it is clear that the unitary orbits of C and T commute if and only if $[C, V^*TV] = 0$ for any $V \in U(n,\mathbf{C})$, if and only if $[U^*CU, T] = 0$ for any $U \in U(n,\mathbf{C})$.

The equivalence between assertions (i) and (iii) in the following theorem is just Theorem 2.5 in [2].

THEOREM 3.2. *For two matrices* $C, T \in M_n(\mathbf{C})$ *the following assertions are equivalent:*

(i) *the C-numerical range $W(C,T)$ of T is a point;*
(ii) *the unitary orbits of C and T commute;*
(iii) *C or T is a scalar matrix.*

Proof. (i)⇒(ii): Since the mapping q is constant, it is of rank zero, i.e., we have $\operatorname{tr}([C, U^*TU]A) = 0$ for any $A \in u(n,\mathbf{C})$ and any $U \in U(n,\mathbf{C})$. By linearity of the trace, we get $\operatorname{tr}([C, U^*TU]X) = 0$ for any $X \in M_n(\mathbf{C})$, i.e., $[C, U^*TU] = 0$.

(ii)⇒(iii): Assume first that C is a nonscalar normal matrix, and let a be an eigenvalue of C; we may assume that C is the block diagonal matrix $C = \operatorname{diag}(A,B)$, where the spectrum of A is reduced to $\{a\}$, a not being an eigenvalue of B. We have $A \in M_r(\mathbf{C})$, with $1 \le r \le n-1$. Let P be the orthogonal projection on the eigenspace of C corresponding to the eigenvalue a. We have $[C, U^*TU] = 0$ for any $U \in U(n,\mathbf{C})$; hence, by Lemma 3.1, $[P, U^*TU] = 0$ for any $U \in U(n,\mathbf{C})$, i.e., $[U^*PU, T] = 0$ for any $U \in U(n, \mathbf{C})$. Since the projection P is of rank r, clearly the matrix T commutes with any orthogonal projection of rank r.

Consider now two rank-one orthogonal projections P_1 and P_2 such that $P_1 P_2 = 0$, and let Q be a projection of rank r such that $P_1 \leq Q, P_2 \leq I - Q$. From the equality $(I - Q)TQ = 0$ we get $P_2(I - Q)TQP_1 = 0$, i.e. $P_2 T P_1 = 0$. The proof shows that for any rank-one orthogonal projection P_1 we have $(I - P_1)TP_1 = 0$, i.e., T is a scalar matrix.

Suppose now that the matrix C is not normal and satisfies the equality $[C, U^*TU] = 0$ for any unitary matrix U. In particular we have $\mathrm{tr}([C, U^*TU]C^*) = 0$, i.e., $\mathrm{tr}([C^*, C]U^*TU]) = 0$ for any unitary matrix U. So we get $W([C^*, C], T) = \{0\}$. Since the matrix C is not normal, the hermitian matrix $[C^*, C]$ is not scalar, and the proof of the implication (ii)⇒(iii), whenever the matrix C is assumed to be normal, shows that T is a scalar matrix.

(iii)⇒(i): This implication is easy, since whenever C or T is a scalar matrix, the C-numerical range of T is $W(C, T) = \{(1/n)\mathrm{tr}(C)\mathrm{tr}(T)\}$. ■

4. STUDY OF THE CASE WHERE $W(C, T)$ IS A LINE SEGMENT

For a matrix $X \in M_n(\mathbf{C})$, we use the notation $X_{i,j}$, $1 \leq i,\ j \leq n$, to denote then n^2 coefficients of X. For completeness, we will indicate the proof of the following lemma, which appears in [1, Lemma 5]:

LEMMA 4.1. *For a matrix $X \in M_n(\mathbf{C})$ the following assertions are equivalent:*

(i) *the numerical range $W(T)$ of T is a line segment;*
(ii) *T satisfies the equalities $|(U^*TU)_{1,2}| = |(U^*TU)_{2,1}|$ for any unitary matrix $U \in U(n, \mathbf{C})$.*

Proof. (i)⇒(ii): The numerical range $W(T)$ of T being a line segment, T is given by $T = \alpha I + \beta A$ with $\alpha, \beta \in \mathbf{C}$, $A \in u(n, \mathbf{C})$; this is an easy consequence of the fact that a matrix is Hermitian if its numerical range is real. So we get $(U^*TU)_{1,2} = \beta(U^*AU)_{1,2} = -\beta(U^*AU)_{2,1} = -(U^*TU)_{2,1}$.

(ii)⇒(i): The hypothesis implies the relations $|(U^*TU)_{i,j}| = |(U^*TU)_{j,i}|$ for any unitary matrix $U \in U(n, \mathbf{C})$, and for any pair $\{i, j\}$ with $1 \leq i \leq j \leq n$. Let U be a unitary matrix such that U^*TU is an upper triangular matrix. Clearly U^*TU is a diagonal matrix, and consequently T is a normal matrix. Now a computation shows that any three eigenvalues of T necessarily belong to the same straight line. ■

We need a local version of Lemma 4.1:

LEMMA 4.2. *Let T be a complex $n \times n$ matrix. If there exists in $U(n, \mathbf{C})$ a nonempty open set Ω such that: $|(U^*TU)_{1,2}| = |(U^*TU)_{2,1}|$ for any unitary matrix $U \in \Omega$, then these equalities are valid for any $U \in U(n, \mathbf{C})$, and hence the numerical range of T is a line segment.*

Proof. Let $U_0 \in \Omega$. We may suppose that U_0 is the unit matrix I, since the matrices T and $U_0^* T U_0$ both have the same numerical range. Consider $U \in U(n, \mathbf{C})$, and consider $A \in u(n, \mathbf{C})$ such that $U = \exp(A)$. Let $U(t) = \exp(tA)$ be the one-parameter unitary group with generator A. Let $\eta > 0$ be a real such that $U(t) \in \Omega$ for any $t \in]-\eta, \eta[$. The real-analytic mapping

$$t \to |(U(t)^* T U(t))_{1,2}|^2 - |(U(t)^* T U(t))_{2,1}|^2,$$

being null in $]-\eta, \eta[$, is null in $\mathbf{R}$. The first assertion is now clear, and the second one follows from Lemma 4.1. ∎

In the sequel we may assume that neither C nor T is a scalar matrix, since otherwise $W(C, T)$ is a point.

THEOREM 4.3. *If there exists a nonempty open subset Ω of $U(n, \mathbf{C})$ such that the rank of the mapping q on Ω is at most 1, then the matrices C and T are normal.*

Proof. For any $U \in \Omega$ and for any $A \in u(n, \mathbf{C})$ we have $T_U q(UA) = \operatorname{tr}([A, C] U^* T U)$. In particular,

$$\begin{aligned} T_U q(U \cdot i(C + C^*)) &= i \operatorname{tr}([C^*, C] U^* T U), \\ T_U q(U(C^* - C)) &= \operatorname{tr}([C^*, C] U^* T U). \end{aligned}$$

By hypothesis, the complex numbers $z = \operatorname{tr}([C^*, C] U^* T U)$ and iz belong to a common linear subspace of dimension 0 or 1 of $\mathbf{R}^2$. Necessarily we have $\operatorname{tr}([C^*, C] U^* T U) = 0$ for any matrix $U \in \Omega$. Now we use the same argument of analyticity as in the proof of Lemma 4.2. Consider $U_0 \in \Omega$. Replacing the matrix T by $U_0^* T U_0$, we may suppose that $I \in \Omega$. Now given any matrix $A \in u(n, \mathbf{C})$, the analytic mapping from $\mathbf{R}$ to $\mathbf{C}$ given by $t \to \operatorname{tr}([C^*, C] \exp(-tA) T \exp(tA))$ is null in some interval $]-\eta, \eta[$ with $\eta > 0$, and hence this function is null in $\mathbf{R}$. So we get $\operatorname{tr}([C^*, C] U^* T U) = 0$ for any matrix $U \in U(n, \mathbf{C})$, i.e., $W([C^*, C], T) = \{0\}$. Since T is not a scalar matrix, $[C^*, C]$ is a scalar matrix by Theorem 2.5 in [2] (cf. Theorem

3.2), i.e., $[C^*, C] = 0$. So C is a normal matrix and by the same argument, T is also a normal matrix. ■

The equivalence between assertions (i) and (iv) in the following result is Theorem 2.7 in [2].

THEOREM 4.4. *For two matrices $C, T \in M_n(\mathbf{C})$ the following assertions are equivalent:*

(i) *the C-numerical range $W(C,T)$ of T is a line segment;*
(ii) *the mapping q is of rank at most 1 on a nonempty open set Ω in $U(n, \mathbf{C})$;*
(iii) *the numerical range $W([C, U^*TU])$ is a line segment for any U in a nonempty open set Ω in $U(n, \mathbf{C})$;*
(iv) *the numerical ranges $W(C)$ and $W(T)$ are line segments.*

Proof. (i)⇒(ii): This implication is clearly true with $\Omega = U(n, \mathbf{C})$.

(ii)⇒(iii): This is clear by the equality obtained in Theorem 2.1: $\mathrm{rk}_U(q) = \dim(\mathrm{lin}(W([C, U^*TU])))$.

(iii)⇒(iv): Since condition (iii) implies condition (ii), we know by Theorem 4.3 that both matrices C and T are normal; on the other hand, we know by Lemma 4.1 that, for any unitary matrix $U \in \Omega$, the matrix $[C, U^*TU]$ satisfies the relations $|(V^*[C, U^*TU]V)_{1,2}| = |(V^*[C, U^*TU]V)_{2,1}|$ for any unitary matrix $V \in U(n, \mathbf{C})$. Let c_1 and c_2 be two distinct eigenvalues of C. Choose V such that $V^*CV = D$, where $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$ is the diagonal matrix with eigenvalues $d_1 = c_1, d_2 = c_2, d_3, \dots, d_n$. Set $UV = U_1$. Now the matrix T satisfies the relations

$$|([D, U_1^*TU_1])_{1,2}| = |([D, U_1^*TU_1])_{2,1}|, \qquad U_1 \in U\Omega.$$

So we obtain $|(d_1 - d_2)(U_1^*TU_1)_{1,2}| = |(d_2 - d_1)(U_1^*TU_1)_{2,1}|$. Hence the matrix T satisfies the relations

$$|(U_1^*TU_1)_{1,2}| = |(U_1^*TU_1)_{2,1}|, \qquad U_1 \in U\Omega.$$

By Lemma 4.2, the numerical range of T is a line segment. The same proof shows that the same conclusion holds for C.

(iv)⇒(i): Under the hypothesis we have $C = \alpha I + \beta A, T = \gamma I + \delta B$, with $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ and $A, B \in u(n, \mathbf{C})$. So we have $W(C,T) = \zeta + \beta\gamma W(A, B)$, with $\zeta \in \mathbf{C}$. Since for any $U \in U(n, \mathbf{C})$, we have

$$\overline{\mathrm{tr}(AU^*BU)} = \mathrm{tr}(U^*BUA) = \mathrm{tr}(AU^*BU),$$

we get: $W(A,B) \subset \mathbf{R}$, and this inclusion implies that $W(C,T)$ is a line segment. ■

5. STUDY OF THE CASE WHERE $W(C,T)$ IS NOWHERE DENSE

The solution of this problem is obtained by a classical reduction to the case of 2×2 matrices.

LEMMA 5.1. *Suppose that C and T are the block triangular matrices*

$$C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \quad \textit{and} \quad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where C_{11} and T_{11} are $r \times r$ matrices and C_{22} and T_{22} are $s \times s$ matrices, with $r + s = n$. The following inclusion is true: $\operatorname{tr}(C_{22}T_{22}) + W(C_{11}, T_{11}) \subset W(C,T)$.

Proof. Consider the block diagonal matrix $U \in U(n, \mathbf{C})$ given by

$$U = \begin{bmatrix} U_{11} & 0 \\ 0 & I_s \end{bmatrix} \qquad \text{with} \quad U_{11} \in U(r, \mathbf{C}),$$

where I_s denotes the unit $s \times s$ matrix. We have

$$CU^*TU = \begin{bmatrix} C_{11}U_{11}^*T_{11}U_{11} & * \\ 0 & C_{22}T_{22} \end{bmatrix}.$$

The conclusion follows from the equality

$$\operatorname{tr}(CU^*TU) = \operatorname{tr}(C_{22}T_{22}) + \operatorname{tr}(C_{11}U_{11}^*T_{11}U_{11}).$$

■

As in our study of the case where $W(C,T)$ is a line segment, a first step in our study of the case where $W(C,T)$ is nowhere dense is to show that the matrices C and T are necessarily normal:

THEOREM 5.2. *If the C-numerical range $W(C,T)$ of T is nowhere dense in $\mathbf{R}^2$, then the matrices C and T are normal.*

Proof. The matrices C and T are $n \times n$ matrices, and when $n = 2$ the result is an obvious consequences of the convexity of $W(C,T)$—a fact established by Nam-Kiu Tsing in [7, Corollary of Theorem 2]—and of Theorem 4.4. The proof when $n \geq 2$ consists in a reduction to the case $n = 2$. Given two unitary matrices $U, V \in U(n, \mathbf{C})$, we have $W(C,T) = W(U^*CU, V^*TV)$; so we may assume the matrices C and T both upper triangular. Let us consider C and T as block triangular matrices, as in Lemma 5.1, and, with the same notation as in that lemma, assume $r = 2$. Since $W(C,T)$ is nowhere dense, Lemma 5.1 shows that the set $W(C_{11}, T_{11})$ is nowhere dense; this set being convex, again by the above result of Nam-Kiu Tsing [7, Corollary of Theorem 2], it is clear that $W(C_{11}, T_{11})$ is a line segment. So the upper triangular matrices C_{11} and T_{11} are normal, hence diagonal. By repeating the same argument, we obtain the result. ∎

Now we are in a position to prove our last result:

THEOREM 5.3. *If the C-numerical range $W(C,T)$ of T is nowhere dense in $\mathbf{R}^2$, then $W(C,T)$ is a line segment.*

Proof. We may assume that neither C nor T is a scalar matrix. Theorem 5.2 shows that, under the hypothesis, C and T are normal matrices. Consequently, by a result due to Nam-Kiu Tsing [6], $W(C,T)$ is a star-shaped subset of $\mathbf{R}^2$. Let z_0 be a center of the star $W(C,T)$; let $U_1 \in U(n, \mathbf{C})$ be a unitary matrix such that $U_1 \notin q^{-1}(z_0)$, i.e., the point $z_1 = q(U_1) = \mathrm{tr}(CU_1^*TU_1)$ is distinct from z_0. Suppose that the mapping $q : U(n, \mathbf{C}) \to \mathbf{R}^2$ is of rank 2 at U_1. There exists a one-parameter unitary group $(U(t))_{t \in \mathbf{R}}$ such that the range of the mapping $t \to \mathrm{tr}(CU(t)^*TU(t))$ from $\mathbf{R}$ to $\mathbf{R}^2$ is a curve containing the point z_1 and such that the tangent to this curve at z_1 is a straight line perpendicular to the straight line joining z_0 and z_1. The fact that z_0 is a center of the star-shaped set $W(C,T)$, which contains the curve just described, clearly contradicts the emptiness of the interior of $W(C,T)$. We have proved that the mapping q has rank at most 1 on the open set $\Omega = U(n, \mathbf{C}) \backslash q^{-1}(z_0)$. Since neither C nor T is a scalar matrix, this open set Ω is nonempty. Now Theorem 4.4 shows that $W(C,T)$ is a line segment. ∎

Saying that the compact set $K = W(C,T)$ is nowwhere dense, or in other words, that its interior is empty, constitutes a global assumption on $W(C,T)$. In order to replace this global assumption by a local one, similarly to what we did in Section 4 [cf. assertions (ii) and (iii) in Theorem 4.4], it is natural to ask for the shape of the C-numerical range $W(C,T)$ of T whenever there exists an open subset Ω of $\mathbf{R}^2$ satisfying $\Omega \cap K \neq \emptyset$ and $\mathrm{int}(\Omega \cap K) = \emptyset$, where $\mathrm{int}(X)$ denotes the interior of the subset X of $\mathbf{R}^2$. The existence of such an open set Ω means that the compact subset

$K = W(C,T)$ of the plane is not regular, that is, K differs from the closure of its interior. This remark leads to the following question:

PROBLEM. Whenever $W(C,T)$ is not a line segment, is $W(C,T)$ equal to the closure of its interior in $\mathbf{R}^2$?

The problem is to know if, except when it is a line segment, $W(C,T)$ is regular. Note that a nondegenerate line segment is relatively regular, i.e. regular for the topology of its supporting line.

In conclusion let us remark that Theorem 5.3 appears in a paper by W. Y. Man [*Linear and Multilinear Algebra* 32:237–247 (1992), Theorem 1.4, p. 242]. I thank Professor C.-K. Li, who brought my attention to that paper after my lecture at the meeting of the International Linear Algebra Society (ILAS) in Pensacola, Florida, in March 1993.

REFERENCES

1 G. Cassier and J. Dazord, Invariance per conjugaison unitaire pour des semi-normes de type ℓ^p sur les matrices, *J. Operator Theory*, to appear.
2 Chi-Kwong Li, The C-convex matrices, *Linear and Multilinear Algebra* 21:303–312 (1987).
3 M. Marcus and M. Sandy, Conditions for the generalized numerical range to be real, *Linear Algebra Appl.* 71:219–239 (1985).
4 R. Narasimhan, *Analysis on Real and Complex Manifolds,* Masson, Paris, 1973.
5 H. Radjavi and P. Rosenthal, *Invariant Subspaces,* Springer-Verlag, New York, 1973.
6 Nam-Kiu Tsing, On the shape of the generalized numerical ranges, *Linear and Multilinear Algebra* 10:173–182 (1981).
7 ———, The constrained bilinear form and the C-numerical range, *Linear Algebra Appl.* 56:195–206 (1984).
8 V. S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations,* Prentice-Hall, Englewood Cliffs, N.J., 1974.

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