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A two-dimensional matrix Padé-type approximation in the inner product space ☆

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ABSTRACT

By introducing a bivariate matrix-valued linear functional on the scalar polynomial space, a general two-dimensional (2-D) matrix Padé-type approximant (*BMPTA*) in the inner product space is defined in this paper. The coefficients of its denominator polynomials are determined by taking the direct inner product of matrices. The remainder formula is developed and an algorithm for the numerator polynomials is presented when the generating polynomials are given in advance. By means of the *Hankel-like* coefficient matrix, a determinantal expression of *BMPTA* is presented. Moreover, to avoid the computation of the determinants, two efficient recursive algorithms are proposed. At the end the method of *BMPTA* is applied to partial realization problems of 2-D linear systems.

1. Introduction

Let $f(x, y)$ be a matrix-valued formal power series in x and y

$$f(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}x^i y^j, \quad c_{ij} \in \mathbb{C}^{q\times q}.$$

The problem of 2-D matrix-valued Padé approximants to $f(x, y)$ has been studied for a long time and has wide applications to many fields, such as scattering physics, multiport network synthesis, design of multi-input multi-output digital filters, ARMA model, reduction of a high degree multivariate systems, signal processing and systems with the number of inputs different from the number of outputs [1–3,20]. Bose and Basu [4] introduced the classical 2-D matrix-valued Padé approximant and discussed the existence, uniqueness and recursive computation of the approximant. By denoting

$$P(x, y) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \zeta_{i,j}x^i y^j, \qquad Q(x, y) = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} \eta_{i,j}x^i y^j, \quad \zeta_{i,j},\ \eta_{i,j} \in \mathbb{C}^{q\times q},$$

the classical 2-D matrix-valued Padé approximant is defined as $P(x, y)(Q(x, y))^{-1}$ such that

$$f(x, y)Q(x, y) - P(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varepsilon_{i,j}x^i y^j$$

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with $\varepsilon_{i,j} = 0$ for $i = 0, 1, \ldots, n_1,\ j = 0, 1, \ldots, n_2$ and $i = n_1 + 1, \ldots, n_1 + m_1 + 1,\ j = n_2 + 1, \ldots, n_2 + m_2 + 1$; excluding the 2-tuple $(i, j) = (n_1 + m_1 + 1, n_2 + m_2 + 1)$.

Zheng and Li [5] presented a 2-D matrix-valued Padé approximant with matrix-valued generating polynomial $V(x, y) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \beta_{ij}x^iy^j$, $\beta_{ij} \in \mathbb{R}^{q\times q}$, $\beta_{n_1n_2} = I$. By giving a matrix linear functional γ:

$$\gamma(Bu^iv^j) = c_{ij}, \quad B \in \mathbb{R}^{q\times q},\ i, j = 0, 1, \ldots,$$

the bivariate associated matrix polynomial W of V is defined by

$$W(x, y) = \gamma\left(\frac{V(x, y) + V(u, v) - V(x, v) - V(u, y)}{(u - x)(v - y)}\right),$$

and the bivariate matrix polynomials $\overline{V}$ and $\overline{W}$ are defined respectively as

$$\overline{V}(x, y) = x^{n_1}y^{n_2}V(x^{-1}, y^{-1}), \qquad \overline{W}(x, y) = x^{n_1-1}y^{n_2-1}W(x^{-1}, y^{-1}).$$

Thus the approximant in [5] is defined by $\overline{W}(x, y)(\overline{V}(x, y))^{-1}$.

By means of the direct inner product of two matrices and the generalized inverse of a matrix, Gu [6] defined a practical 2-D matrix-valued Padé approximant with branched Thiele-type continued fraction with scalar denominator polynomials. However, the denominators of the approximants in [4,5] are matrix-valued polynomials, thus left-handed and right-handed aproximants are defined respectively in the construction process. On the other hand, the rational approximant in [6] has its restriction, that is, the degree of the denominator polynomial is always even.

The problem of scalar Padé-type approximants was introduced in [7] and has been studied for a long time by many authors, for example, Brezinski [8,9], Sablonniere [10], Arioka [11] and Draux [12]. Following the Padé-type approximants [7], Gu [13] gave a matrix-valued linear functional on the scalar polynomial space and defined the 1-D matrix Padé-type approximants. Motivated by the idea of [7] and [13], in this paper we define a bivariate matrix-valued linear functional on the bivariate scalar polynomial space. By giving a scalar generating polynomial, we construct a general 2-D matrix Padé-type approximant (*BMPTA*) in the inner product space. As compared to the existing 2-D matrix-valued Padé approximants [4,5], the *BMPTA* in this paper doesn't need multiplication of matrices in the construction process, and hence, we don't have to define left-handed and right-handed approximants. Moreover, its denominator polynomials can have general degrees.

This paper is organized as follows: Section 2 reviews the definition of 1-D matrix-valued Padé-type approximation in [13]. Section 3 presents the definition of *BMPTA* and a recursive algorithm for *BMPTA* with given generating polynomials. Section 4 gives three choices of the index set in the remainder formula and the determinants expressions for *BMPTA*. In order to avoid the computation of the determinants, Section 5 proposes two efficient recursive algorithms: Sylvester type algorithm derived from Sylvester theorem [14] and E-algorithm extended from Brezinski's algorithm [15]. Section 6 applies the method of *BMPTA* to partial realization problems of 2-D linear systems.

2. 1-D matrix Padé-type approximation

Let $f(x)$ be a given power series with $p \times q$ matrix coefficients, i.e.,

$$f(x) = \sum_{i=0}^{\infty} c_ix^i, \quad c_i \in \mathbb{C}^{p\times q}.$$

Let **P** denote the set of scalar polynomials in one real variable whose coefficients belong to the complex field $\mathbb{C}$. Let $\phi : \mathbf{P} \to \mathbb{C}^{p\times q}$ be a generalized linear functional on **P**:

$$\phi(t^i) = c_i, \quad i = 0, 1, \ldots,$$

where $c_i = 0$ for $i < 0$. Let $|tx| < 1$, it follows that

$$\phi((1 - tx)^{-1}) = c(1 + tx + (tx)^2 + \cdots) = \sum_{i=0}^{\infty} c_ix^i = f(x).$$

Let $v_n(x) \in \mathbf{P}$ be a scalar polynomial of degree n

$$v_n(x) = b_0 + b_1x + \cdots + b_nx,$$

where $b_n \neq 0$. Define a matrix-valued polynomial $w_m(x)$ by

$$w_m(x) = \phi\left(\frac{t^{m-n+1}v_n(t) - x^{m-n+1}v_n(x)}{t - x}\right).$$

It is clear that $w_m(x)$ is of degree m. Set

$$\tilde{v}_n(x) = x^nv_n(x^{-1}), \qquad \tilde{w}_m(x) = x^mw_m(x^{-1}).$$

Definition 1 (*[13]*). Let $\tilde{v}_n(0) \neq 0$, then the following rational function

$$r_{m,n}(x) = \frac{\tilde{w}_m(x)}{\tilde{v}_n(x)} \tag{1}$$

is called an *MPTA* of order (m/n) with the generating polynomial $v_n(x)$ and is denoted by $(m/n)_f(x)$.

3. 2-D matrix Padé-type approximation

Consider a bivariate matrix-valued function $f(x, y)$ which has formal series in two variables x and y

$$f(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}x^i y^j, \quad c_{ij} \in \mathbb{C}^{p\times q}. \tag{2}$$

Let $\hat{\mathbf{P}}$ denote the set of scalar polynomials in two real variables whose coefficients belong to the complex field $\mathbb{C}$. Let $\hat{\phi} : \hat{\mathbf{P}} \to \mathbb{C}^{p\times q}$ be a generalized linear functional on $\hat{\mathbf{P}}$, acting on s, t, defined by

$$\hat{\phi}(s^i t^j) = c_{ij}, \quad i, j = 0, 1, \ldots, \tag{3}$$

where $c_{ij} = 0$ for $i < 0$ or $j < 0$. Assume that $|sx| < 1$, $|ty| < 1$, we have

$$\begin{aligned}(1 - sx)^{-1} &= 1 + sx + (sx)^2 + \cdots,\\ (1 - ty)^{-1} &= 1 + ty + (ty)^2 + \cdots.\end{aligned} \tag{4}$$

For the given power series (2), we obtain from (3) and (4) that

$$\begin{aligned}\hat{\phi}\left(\frac{1}{(1 - sx)(1 - ty)}\right) &= \hat{\phi}((1 + sx + (sx)^2 + \cdots)(1 + ty + (ty)^2 + \cdots))\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}x^i y^j = f(x, y).\end{aligned} \tag{5}$$

Let $V_{m_1m_2}(x, y)$ be a bivariate scalar polynomial of degree (m_1, m_2)

$$V_{m_1m_2}(x, y) = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij}x^i y^j, \tag{6}$$

and assume that the coefficient $b_{m_1m_2} \neq 0$. Define the bivariate matrix-valued polynomial $W_{n_1n_2}(x, y)$ by

$$W_{n_1n_2}(x, y) = \hat{\phi}\left(\frac{\begin{Bmatrix} x^{n_1-m_1+1}y^{n_2-m_2+1}V_{m_1m_2}(x, y) + s^{n_1-m_1+1}t^{n_2-m_2+1}V_{m_1m_2}(s, t)\\ -x^{n_1-m_1+1}t^{n_2-m_2+1}V_{m_1m_2}(x, t) - s^{n_1-m_1+1}y^{n_2-m_2+1}V_{m_1m_2}(s, y)\end{Bmatrix}}{(s - x)(t - y)}\right), \tag{7}$$

where $\hat{\phi}$ acts on s and t, and x, y are parameters.

Set

$$\widetilde{V}_{m_1m_2}(x, y) = x^{m_1}y^{m_2}V_{m_1m_2}(x^{-1}, y^{-1}) = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij}x^{m_1-i}y^{m_2-j}, \tag{8}$$

and

$$\widetilde{W}_{n_1n_2}(x, y) = x^{n_1}y^{n_2}W_{n_1n_2}(x^{-1}, y^{-1}). \tag{9}$$

Lemma 2. *Let $\widetilde{W}_{n_1n_2}(x, y)$ be defined by (7) and (9), then it can be written as*

$$\begin{aligned}\widetilde{W}_{n_1n_2}(x, y) &= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij}x^{m_1-i}y^{m_2-j}f_{n_1-m_1+i,n_2-m_2+j}(x, y)\\ &= \sum_{\alpha=0}^{n_1}\sum_{\beta=0}^{n_2} a_{\alpha\beta}x^{\alpha}y^{\beta}\end{aligned} \tag{10}$$

where

$$a_{\alpha\beta} = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} c_{i-m_1+\alpha,j-m_2+\beta} b_{ij}$$

and

$$f_{kl}(x, y) = \sum_{i=0}^{k}\sum_{j=0}^{l} c_{ij}x^i y^j. \tag{11}$$

Proof. It follows from (7) that

$$\begin{aligned}
W_{n_1n_2}(x, y) &= \hat{\phi}\left(\frac{\begin{Bmatrix} x^{n_1-m_1+1}y^{n_2-m_2+1}V_{m_1m_2}(x, y) + s^{n_1-m_1+1}t^{n_2-m_2+1}V_{m_1m_2}(s, t) \\ -x^{n_1-m_1+1}t^{n_2-m_2+1}V_{m_1m_2}(x, t) - s^{n_1-m_1+1}y^{n_2-m_2+1}V_{m_1m_2}(s, y)\end{Bmatrix}}{(s-x)(t-y)}\right) \\
&= \hat{\phi}\left(\frac{\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij}(x^{n_1-m_1+i+1} - s^{n_1-m_1+i+1})(y^{n_2-m_2+j+1} - t^{n_2-m_2+j+1})}{(s-x)(t-y)}\right) \\
&= \hat{\phi}\left(\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij} \sum_{\alpha=0}^{n_1-m_1+i}\sum_{\beta=0}^{n_2-m_2+j} s^\alpha t^\beta x^{n_1-m_1+i-\alpha} y^{n_2-m_2+j-\beta}\right) \\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij} \sum_{\alpha=0}^{n_1-m_1+i}\sum_{\beta=0}^{n_2-m_2+j} c_{\alpha\beta} x^{n_1-m_1+i-\alpha} y^{n_2-m_2+j-\beta}.
\end{aligned}$$

It is found from (9) that

$$\begin{aligned}
\widetilde{W}_{n_1n_2}(x, y) &= x^{n_1}y^{n_2}W_{n_1n_2}(x^{-1}, y^{-1}) \\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij} \sum_{\alpha=0}^{n_1-m_1+i}\sum_{\beta=0}^{n_2-m_2+j} c_{\alpha\beta} x^{m_1-i+\alpha} y^{m_2-j+\beta} \\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij}x^{m_1-i}y^{m_2-j} f_{n_1-m_1+i,n_2-m_2+j}(x, y) \\
&= \sum_{\alpha=0}^{n_1}\sum_{\beta=0}^{n_2}\left(\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} c_{i-m_1+\alpha,j-m_2+\beta} b_{ij}\right) x^\alpha y^\beta \\
&= \sum_{\alpha=0}^{n_1}\sum_{\beta=0}^{n_2} a_{\alpha\beta} x^\alpha y^\beta. \quad \square
\end{aligned} \tag{12}$$

Definition 3. Let $\widetilde{V}_{m_1m_2}(0, 0) \neq 0$, then the matrix-valued rational function

$$R_{n_1,n_2;m_1,m_2}(x, y) = \frac{\widetilde{W}_{n_1n_2}(x, y)}{\widetilde{V}_{m_1m_2}(x, y)} \tag{13}$$

is said to be a *BMPTA* of degree $(n_1, n_2/m_1, m_2)$ with the generating polynomial $V_{m_1m_2}(x, y)$ and is denoted by $(n_1, n_2/m_1, m_2)_f(x, y)$.

Remark 4. If $x = x_0$ or $y = y_0$, the 2-D approximants in (13) becomes the 1-D approximants.

The structure of $\widetilde{V}_{m_1m_2}(x, y)$ and $\widetilde{W}_{n_1n_2}(x, y)$ immediately implies the following Padé approximant property.

Theorem 5 (*Remainder Formula*). *Let* $\widetilde{V}_{m_1m_2}(x, y) \neq 0$, *then*

$$\widetilde{V}_{m_1,m_2}(x, y)f(x, y) - \widetilde{W}(x, y) = \sum_{\alpha=0}^{\infty}\sum_{\beta=n_2+1}^{\infty}\sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij}x^\alpha y^\beta + \sum_{\alpha=n_1+1}^{\infty}\sum_{\beta=0}^{n_2}\sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij}x^\alpha y^\beta$$

$$= \sum_{\alpha=0}^{n_1} \sum_{\beta=n_2+1}^{\infty} \sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij} x^\alpha y^\beta + \sum_{\alpha=n_1+1}^{\infty} \sum_{\beta=0}^{\infty} \sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij} x^\alpha y^\beta$$

$$= \sum_{\alpha=0}^{n_1} \sum_{\beta=n_2+1}^{\infty} \sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij} x^\alpha y^\beta + \sum_{\alpha=n_1+1}^{\infty} \sum_{\beta=0}^{n_2} \sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij} x^\alpha y^\beta + \sum_{\alpha=n_1+1}^{\infty} \sum_{\beta=n_2+1}^{\infty} \sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij} x^\alpha y^\beta, \tag{14}$$

where $\tilde{d}^{ij}_{\alpha\beta} = c_{i-m_1+\alpha, j-m_2+\beta}$, *and* $D = \{(i,j) \in \mathbb{N}^2 : 0 \le i \le m_1, 0 \le j \le m_2\}$ *with m elements:* $(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$, $Card(D) = m$.

We now give an error formula of $R_{n_1,n_2;m_1,m_2}(x, y)$ *with functional form.*

Theorem 6 (*Error Formula*).

$$\widetilde{W}_{n_1n_2}(x, y) - \widetilde{V}_{m_1m_2}(x, y) f(x, y)$$
$$= x^{n_1+1} y^{n_2+1} \phi \left(\frac{\begin{Bmatrix} s^{n_1-m_1+1} t^{n_2-m_2+1} \widetilde{V}_{m_1m_2}(s, t) - x^{m_1-n_1-1} t^{n_2-m_2+1} \widetilde{V}_{m_1m_2}(x^{-1}, t) \\ -s^{n_1-m_1+1} y^{m_2-n_2-1} \widetilde{V}_{m_1m_2}(s, y^{-1}) \end{Bmatrix}}{(1-sx)(1-ty)} \right). \tag{15}$$

Proof. It is obtained from (9) that

$$\widetilde{W}_{n_1n_2}(x, y) = x^{n_1} y^{n_2} W_{n_1n_2}(x^{-1}, y^{-1})$$
$$= x^{n_1} y^{n_2} \phi \left(\frac{\begin{Bmatrix} x^{-n_1+m_1-1} y^{-n_2+m_2-1} V_{m_1m_2}(x^{-1}, y^{-1}) - x^{-n_1+m_1-1} t^{n_2-m_2+1} V_{m_1m_2}(x^{-1}, t) \\ -s^{n_1-m_1+1} y^{-n_2+m_2-1} V_{m_1m_2}(s, y^{-1}) + s^{n_1-m_1+1} t^{n_2-m_2+1} V_{m_1m_2}(s, t) \end{Bmatrix}}{(s-x^{-1})(t-y^{-1})} \right)$$
$$= \phi \left(\frac{\begin{Bmatrix} x^{m_1} y^{m_2} V_{m_1m_2}(x^{-1}, y^{-1}) - x^{m_1} y^{n_2+1} t^{n_2-m_2+1} V_{m_1m_2}(x^{-1}, t) \\ -s^{n_1-m_1+1} x^{n_1+1} y^{m_2} V_{m_1m_2}(s, y^{-1}) + x^{n_1+1} y^{n_2+1} s^{n_1-m_1+1} t^{n_2-m_2+1} V_{m_1m_2}(s, t) \end{Bmatrix}}{(sx-1)(ty-1)} \right)$$
$$= \widetilde{V}_{m_1m_2}(x, y) f(x, y) + x^{n_1+1} y^{n_2+1} \phi \left(\frac{\begin{Bmatrix} s^{n_1-m_1+1} t^{n_2-m_2+1} V_{m_1m_2}(s, t) - x^{m_1-n_1-1} t^{n_2-m_2+1} V_{m_1m_2}(x^{-1}, t) \\ -s^{n_1-m_1+1} y^{m_2-n_2-1} V_{m_1m_2}(s, y^{-1}) \end{Bmatrix}}{(1-sx)(1-ty)} \right).$$

Denote $g_{kl}(x) = \sum_{\alpha=0}^{k} a_{\alpha l} x^\alpha$, $k = 0, 1, \ldots, n_1$, $l = 0, 1, \ldots, n_2$. From Lemma 2 we can present a recursive algorithm for the numerator polynomials. □

Algorithm 7.

$$\begin{cases} g_{0l}(x) = a_{0l} \\ g_{1l}(x) = g_{0l}(x) + a_{1l} x \\ \cdots \\ g_{kl}(x) = g_{k-1,l}(x) + a_{kl} x^k \\ \widetilde{W}_{k0}(x, y) = g_{k0}(x) \\ \widetilde{W}_{k1}(x, y) = \widetilde{W}_{k0}(x, y) + g_{k1}(x) y \\ \cdots \\ \widetilde{W}_{kl}(x, y) = \widetilde{W}_{k,l-1}(x, y) + g_{kl} y^l. \end{cases} \tag{16}$$

Remark 8. Dividing every equality in (16) by $\widetilde{V}_{m_1m_2}(x, y)$, we can get a complete recursive algorithm for *BMPTA*.

Example 9. Find a *BMPTA* of degree (1, 1/1, 1) for

$$f(x, y) = \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ x & y \end{bmatrix} \Big/ (1-x)(1-y)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} y + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} xy + \cdots.$$

Note that $f(x, y)$ is singular if $x = 1$ or $y = 1$ and then choose $\widetilde{V}_{11}(x, y) = (1 - x)(1 - y)$ where $b_{00} = b_{11} = 1, b_{01} = b_{10} = -1$. By Algorithm 7 we have

$$g_{00}(x) = a_{00} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad g_{10}(x) = g_{00}(x) + a_{10}x = \begin{bmatrix} 1 & x \\ x & 0 \end{bmatrix},$$
$$g_{01}(x) = a_{01} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad g_{11}(x) = g_{01}(x) + a_{11}x = \begin{bmatrix} -x & 1 \\ 0 & 1 \end{bmatrix},$$
$$\widetilde{W}_{10}(x, y) = g_{10}(x), \qquad \widetilde{W}_{11}(x, y) = \widetilde{W}_{10}(x, y) + g_{11}(x)y = \begin{bmatrix} 1 - xy & x + y \\ x & y \end{bmatrix}.$$

Thus

$$(1, 1/1, 1)_f(x, y) = \begin{bmatrix} 1 - xy & x + y \\ x & y \end{bmatrix} \Big/ (1 - x)(1 - y).$$

We find that $(n_1, n_2/m_1, m_2)_f(x, y)$ depends on the generating polynomial $V_{m_1m_2}(x, y)$ which is given in advance. In the next section we will represent the determinantal expressions for *BMPTA*.

4. Determinantal expressions for *BMPTA*

In this section we will first discuss two choices of the index set E, which lead to different sequences of approximants, and then give the expressions for *BMPTA*.

4.1. The choices of the index set **E**

In the definition of *BMTPA* (13), the set of index E which indicates that the indices of the polynomials presents in the remainder term only needs to satisfy the conditions

$$\mathrm{Card}(E) = \mathrm{Card}(D) - 1, \tag{17}$$

where E is the set of indices of null terms in the expansion of the remainder with $\mathrm{Card}(E) = m - 1$. So E can be a very general set. We can choose E in such a way that:

(i) the two variables x and y are symmetric;
(ii) if $x = x_0$ or $y = y_0$, *BMPTA* will become the case of 1-D matrix-valued Padé approximant.

We consider from (14) three cases of E as follows.

Case 1.

$$\begin{aligned} &E = S_1 \cup S_2, \\ &S_1 = \{(\alpha, \beta) \in \mathbb{N}^2 : 0 \le \alpha \le m_1, n_2 + 1 \le \beta \le n_2 + m_2\}, \\ &S_2 = \{(\alpha, \beta) \in \mathbb{N}^2 : n_1 + 1 \le \alpha \le n_1 + m_1, \beta = 0\}. \end{aligned}$$

Case 2.

$$\begin{aligned} &E = S_1 \cup S_2, \\ &S_1 = \{(\alpha, \beta) \in \mathbb{N}^2 : \alpha = 0, n_2 + 1 \le \beta \le n_2 + m_2\}, \\ &S_2 = \{(\alpha, \beta) \in \mathbb{N}^2 : n_1 + 1 \le \alpha \le n_1 + m_1, 0 \le \beta \le m_2\}. \end{aligned}$$

Case 3.

$$\begin{aligned} &E = S_1 \cup S_2 \cup S_3, \\ &S_1 = \{(\alpha, \beta) \in \mathbb{N}^2 : n_1 + 1 \le \alpha \le n_1 + m_1, \beta = 0\}, \\ &S_2 = \{(\alpha, \beta) \in \mathbb{N}^2 : \alpha = 0, n_2 + 1 \le \beta \le n_2 + m_2, \}, \\ &S_3 = \{(\alpha, \beta) \in \mathbb{N}^2 : n_1 + 1 \le \alpha \le n_1 + m_1, n_2 + 1 \le \beta \le n_2 + m_2\}. \end{aligned}$$

Figs. 1–3 illustrate the case 1 case 2 and case 3 respectively. In the following the examples given are only discussed in terms of Case 1.

4.2. Determinantal expressions for ***BMPTA***

Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{C}^{p\times q}$. In [6] the matrix direct inner product is defined by

$$(A, B) = A \cdot B = \sum_{i=1}^{p}\sum_{j=1}^{q} a_{ij}b_{ij}. \tag{18}$$

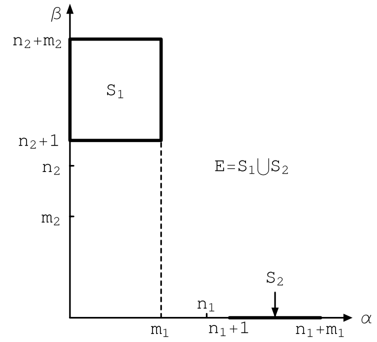

Fig. 1. Index set E for Case 1.

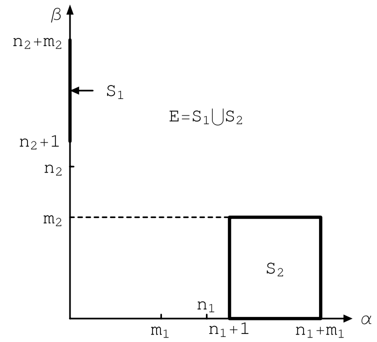

Fig. 2. Index set E for Case 2.

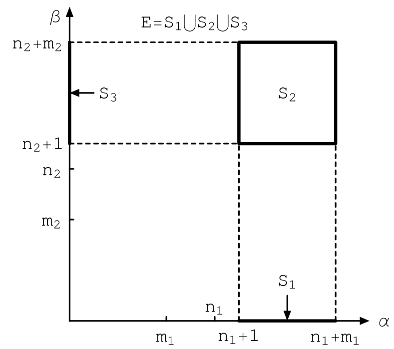

Fig. 3. Index set E for Case 3.

For convenience, we let $N = \{(i, j) \in \mathbb{N}^2 : 0 \leq i \leq m_1, 0 \leq j \leq m_2\}$ with $\text{Card}(N) = n$ and reorder E as $\{(\alpha_k, \beta_k) \in \mathbb{N}^2 : k = n + 1, n + 2, \ldots, n + m - 1\}$. To obtain higher approximants, we cancel the coefficients $\sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij}$, $(\alpha, \beta) \in E$ and solve the following matrix-valued system:

$$\sum_{(i,j)\in D} \tilde{d}^{ij}_{\alpha\beta} b_{ij} = 0, \quad (\alpha, \beta) \in E. \tag{19}$$

Let $e = (e_{ij}) \in \mathbb{C}^{p\times q}$ be the matrix whose elements are equal to 1, that is, $e_{ij} = 1$. We take the direct inner product to both sides of the system of matrix equations (19) such that it becomes the scalar system in the form

$$\sum_{(i,j)\in D} d^{ij}_{\alpha\beta} b_{ij} = 0, \quad (\alpha, \beta) \in E, \tag{20}$$

where $d^{ij}_{\alpha\beta} = (e, \tilde{d}^{ij}_{\alpha\beta}) = e \cdot \tilde{d}^{ij}_{\alpha\beta}$.

We now construct the *Hankel-like* matrix

$$\mathcal{H}_{1,l}^{1,k}=\begin{bmatrix} d^{i_1j_1}_{\alpha_{n+1}\beta_{n+1}} & d^{i_2j_2}_{\alpha_{n+1}\beta_{n+1}} & \cdots & d^{i_{k-1}j_{k-1}}_{\alpha_{n+1}\beta_{n+1}} & d^{i_kj_k}_{\alpha_{n+1}\beta_{n+1}}\\ d^{i_1j_1}_{\alpha_{n+2}\beta_{n+2}} & d^{i_2j_2}_{\alpha_{n+2}\beta_{n+2}} & \cdots & d^{i_{k-1}j_{k-1}}_{\alpha_{n+2}\beta_{n+2}} & d^{i_kj_k}_{\alpha_{n+2}\beta_{n+2}}\\ \vdots & \vdots & & \vdots & \vdots\\ d^{i_1j_1}_{\alpha_{n+l}\beta_{n+l}} & d^{i_2j_2}_{\alpha_{n+l}\beta_{n+l}} & \cdots & d^{i_{k-1}j_{k-1}}_{\alpha_{n+l}\beta_{n+l}} & d^{i_kj_k}_{\alpha_{n+l}\beta_{n+l}} \end{bmatrix}, \tag{21}$$

where for convenience, fix $(i_m, j_m) = (m_1, m_2)$. On the basis of Cramer rule we can obtain the following determinantal expressions.

Theorem 10 (*Determinantal Expression*). *If the Hankel-like matrix $\mathcal{H}_{1,m-1}^{1,m-1}$ is nonsingular, the solution of the system (20) exists, and the numerator and denominator of BMPTA can be expressed as*

$$\widetilde{V}_{m_1m_2}(x,y)=\begin{vmatrix} d^{i_1j_1}_{\alpha_{n+1}\beta_{n+1}} & d^{i_2j_2}_{\alpha_{n+1}\beta_{n+1}} & \cdots & d^{i_{m-1}j_{m-1}}_{\alpha_{n+1}\beta_{n+1}} & d^{i_mj_m}_{\alpha_{n+1}\beta_{n+1}}\\ d^{i_1j_1}_{\alpha_{n+2}\beta_{n+2}} & d^{i_2j_2}_{\alpha_{n+2}\beta_{n+2}} & \cdots & d^{i_{m-1}j_{m-1}}_{\alpha_{n+2}\beta_{n+2}} & d^{i_mj_m}_{\alpha_{n+2}\beta_{n+2}}\\ \vdots & \vdots & & \vdots & \vdots\\ d^{i_1j_1}_{\alpha_{n+m-1}\beta_{n+m-1}} & d^{i_2j_2}_{\alpha_{n+m-1}\beta_{n+m-1}} & \cdots & d^{i_{m-1}j_{m-1}}_{\alpha_{n+m-1}\beta_{n+m-1}} & d^{i_mj_m}_{\alpha_{n+m-1}\beta_{n+m-1}}\\ x^{m_1-i_1}y^{m_2-j_1} & x^{m_1-i_2}y^{m_2-j_2} & \cdots & x^{m_1-i_{m-1}}y^{m_2-j_{m-1}} & 1 \end{vmatrix}, \tag{22}$$

and

$$\widetilde{W}_{m_1m_2}(x,y)=\begin{vmatrix} d^{i_1j_1}_{\alpha_{n+1}\beta_{n+1}} & d^{i_2j_2}_{\alpha_{n+1}\beta_{n+1}} & \cdots & d^{i_{m-1}j_{m-1}}_{\alpha_{n+1}\beta_{n+1}} & d^{i_mj_m}_{\alpha_{n+1}\beta_{n+1}}\\ d^{i_1j_1}_{\alpha_{n+2}\beta_{n+2}} & d^{i_2j_2}_{\alpha_{n+2}\beta_{n+2}} & \cdots & d^{i_{m-1}j_{m-1}}_{\alpha_{n+2}\beta_{n+2}} & d^{i_mj_m}_{\alpha_{n+2}\beta_{n+2}}\\ \vdots & \vdots & & \vdots & \vdots\\ d^{i_1j_1}_{\alpha_{n+m-1}\beta_{n+m-1}} & d^{i_2j_2}_{\alpha_{n+m-1}\beta_{n+m-1}} & \cdots & d^{i_{m-1}j_{m-1}}_{\alpha_{n+m-1}\beta_{n+m-1}} & d^{i_mj_m}_{\alpha_{n+m-1}\beta_{n+m-1}}\\ \varphi_{i_1j_1}(x,y) & \varphi_{i_2j_2}(x,y) & \cdots & \varphi_{i_{m-1}j_{m-1}}(x,y) & \varphi_{i_mj_m}(x,y) \end{vmatrix}, \tag{23}$$

where

$$\varphi_{i_kj_k}(x,y)=\sum_{(\alpha,\beta)\in N} c_{\alpha-m_1+i_k,\beta-m_2+j_k}x^{\alpha}y^{\beta}=\sum_{l=1}^{n}\tilde{d}^{i_kj_k}_{\alpha_l\beta_l}x^{\alpha_l}y^{\beta_l}.$$

Remark 11. Here the condition that $\det(\mathcal{H}_{m-1}^{1,m-1}) \neq 0$ is only just sufficient. In fact, if $\operatorname{rank}(\mathcal{H}_{1,m-1}^{1,m-1}) = \operatorname{rank}(\mathcal{H}_{1,m-1}^{1,m}) < m-1$, we can also obtain the $(n_1, n_2/m_1, m_2)_f(x, y)$.

Example 12 (*[16]*). Find a *BMPTA* of order $(1, 2/1, 2)$ for

$$f(x,y)=\begin{bmatrix}\cos(x+y) & \sin(x+y)\\ x & y\end{bmatrix}$$

$$=\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&1\\0&1\end{bmatrix}y+\begin{bmatrix}-1/2&0\\0&0\end{bmatrix}y^2+\begin{bmatrix}0&1\\1&0\end{bmatrix}x+\begin{bmatrix}-1&0\\0&0\end{bmatrix}xy+\begin{bmatrix}0&-1/2\\0&0\end{bmatrix}xy^2+\cdots.$$

For convenience, we rearrange the elements in D and E here as follows

$$D=\left\{(i_{k(m_2+1)+l+1}, j_{k(m_2+1)+l+1})=(k,l), k=0,1,\ldots,m_1, l=0,1,\ldots,m_2\right\},$$
$$S_1=\left\{(\alpha_{n+km_2+l}, \beta_{n+km_2+l})=(k, n_2+l), k=0,1,\ldots,m_1, l=1,2,\ldots,m_2\right\},$$
$$S_2=\left\{(\alpha_{n+m_2(m_1+1)+k}, \beta_{n+m_2(m_1+1)+k})=(n_1+k, 0), k=1,2,\ldots,m_1\right\},$$

and $E = S_1 \cup S_2$. Then the corresponding *Hankel-like* matrix is

$$\mathcal{H}_{1,m-1}^{1,m-1}=\mathcal{H}_{1,5}^{1,5}=\begin{bmatrix} 0 & 0 & 0 & 2 & -\frac{1}{2}\\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{6}\\ 2 & -\frac{1}{2} & -\frac{1}{6} & -1 & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{6} & \frac{1}{24} & -\frac{1}{2} & \frac{1}{6}\\ 0 & 0 & 2 & 0 & 0 \end{bmatrix},$$

where $\mathrm{rank}(\mathcal{H}_{1,5}^{1,5}) = 5$. Thus, the denominator and the numerator of $(1, 2/1, 2)_f(x, y)$ are given respectively by

$$\widetilde{V}_{12}(x,y) = \begin{vmatrix} 0 & 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{24} \\ 2 & -\frac{1}{2} & -\frac{1}{6} & -1 & -\frac{1}{2} & \frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{6} & \frac{1}{24} & -\frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 2 & 0 & 0 & -\frac{1}{2} \\ xy^2 & xy & x & y^2 & y & 1 \end{vmatrix},$$

and

$$\widetilde{W}_{12}(x,y) = \begin{vmatrix} 0 & 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{24} \\ 2 & -\frac{1}{2} & -\frac{1}{6} & -1 & -\frac{1}{2} & \frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{6} & \frac{1}{24} & -\frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 2 & 0 & 0 & -\frac{1}{2} \\ \varphi_{00} & \varphi_{01} & \varphi_{02} & \varphi_{10} & \varphi_{11} & \varphi_{12} \end{vmatrix},$$

where $\varphi_{00}, \varphi_{01}, \varphi_{02}, \varphi_{10}, \varphi_{11}, \varphi_{12}$ are respectively

$$\begin{bmatrix} xy^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} xy & xy^2 \\ 0 & xy^2 \end{bmatrix}, \quad \begin{bmatrix} x - \frac{xy^2}{2} & xy \\ 0 & xy \end{bmatrix},$$

$$\begin{bmatrix} y^2 & xy^2 \\ xy^2 & 0 \end{bmatrix}, \quad \begin{bmatrix} y - xy^2 & xy + y^2 \\ xy & y^2 \end{bmatrix}, \quad \begin{bmatrix} 1 - xy - \frac{y^2}{2} & x + y - \frac{xy^2}{2} \\ x & y \end{bmatrix}.$$

5. Two recursive algorithms for *BMPTA*

To avoid to compute the determinants of *BMPTA*, in this section, we present two recursive algorithms: Sylvester type algorithm and E-algorithm.

5.1. Sylvester type algorithm

We now only give the recursive algorithm for the computation of $\widetilde{V}_{m_1m_2}(x, y)$, then the computation of $\widetilde{W}_{n_1n_2}(x, y)$ follows the coefficients of $\widetilde{V}_{m_1m_2}(x, y)$ by means of the relation (10).

Denote

$$H_{l+1}^{k,h} = \begin{vmatrix} d_{\alpha_{n+h}\beta_{n+h}}^{i_kj_k} & d_{\alpha_{n+h}\beta_{n+h}}^{i_{m-l+1}j_{m-l+1}} & \cdots & d_{\alpha_{n+h}\beta_{n+h}}^{i_{m-l}j_{m-l}} & d_{\alpha_{n+h}\beta_{n+h}}^{i_mj_m} \\ d_{\alpha_{n+h+1}\beta_{n+h+1}}^{i_kj_k} & d_{\alpha_{n+h+1}\beta_{n+h+1}}^{i_{m-l+1}j_{m-l+1}} & \cdots & d_{\alpha_{n+h+1}\beta_{n+h+1}}^{i_{m-l}j_{m-l}} & d_{\alpha_{n+h+1}\beta_{n+h+1}}^{i_mj_m} \\ \vdots & \vdots & & \vdots & \vdots \\ d_{\alpha_{n+h+l}\beta_{n+h+l}}^{i_kj_k} & d_{\alpha_{n+h+l}\beta_{n+h+l}}^{i_{m-l+1}j_{m-l+1}} & \cdots & d_{\alpha_{n+h+l}\beta_{n+h+l}}^{i_{m-l}j_{m-l}} & d_{\alpha_{n+h+l}\beta_{n+h+l}}^{i_mj_m} \end{vmatrix}. \tag{24}$$

We will begin by giving the recursive relations satisfied by the quantities $H_{l+1}^{k,h}$. For this, we need the following lemma.

Lemma 13 (*Sylvester Theorem [14]*). *Let A be a matrix, and let A_{ru} be the matrix with the r-th row and u-th column deleted. Also let $A_{rs,uv}$ denote the matrix A with the r-th and s-th rows and the u-th and v-th columns deleted. Provided $r < s$ and $u < v$, then there holds*

$$\det(A)\det(A_{rs,uv}) = \det(A_{ru})\det(A_{sv}) - \det(A_{rv})\det(A_{su}).$$

Then we have the following theorem for $H_{l+1}^{k,h}$*:*

Theorem 14.

$$H_{l+1}^{k,h} = \frac{H_l^{m-l+1,h+1} H_l^{k,h} - H_l^{k,h+1} H_l^{m-l+1,h}}{H_{l-1}^{m-l+2,h+1}}, \tag{25}$$

with the initializations $H_0^{k,h} = 1,\ H_1^{k,1} = d_{\alpha_1\beta_1}^{i_k j_k},\ k \geq 1.$

Proof. Let us apply Lemma 13 to $H_{l+1}^{k,h}$ with $r = 1,\ s = h + l,\ u = 1,\ v = 2$:

$$H_{l+1}^{k,h} H_{l-1}^{m-l+2,h+1} = H_l^{m-l+1,h+1} H_l^{k,h} - H_l^{k,h+1} H_l^{m-l+1,h},$$

and the result is true. □

Denote

$$\widetilde{V}_l^k(x,y) = \begin{vmatrix} d_{\alpha_{n+1}\beta_{n+1}}^{i_k j_k} & d_{\alpha_{n+1}\beta_{n+1}}^{i_{m-l+1} j_{m-l+1}} & \cdots & d_{\alpha_{n+1}\beta_{n+1}}^{i_{m-1} j_{m-1}} & d_{\alpha_{n+1}\beta_{n+1}}^{i_m j_m} \\ \vdots & \vdots & & \vdots & \vdots \\ d_{\alpha_{n+l}\beta_{n+l}}^{i_k j_k} & d_{\alpha_{n+l}\beta_{n+l}}^{i_{m-l+1} j_{m-l+1}} & \cdots & d_{\alpha_{n+l}\beta_{n+l}}^{i_{m-1} j_{m-1}} & d_{\alpha_{n+l}\beta_{n+l}}^{i_m j_m} \\ x^{m_1-i_k} y^{m_2-j_k} & x^{m_1-i_{m-l+1}} y^{m_2-j_{m-l+1}} & \cdots & x^{m_1-i_{m-1}} y^{m_2-j_{m-1}} & 1 \end{vmatrix}, \tag{26}$$

where $k \leq m - l,\ l \geq 1$. It is clear that $\widetilde{V}_{m-1}^1(x,y) = \widetilde{V}_{m_1 m_2}(x,y)$.

Again applying Sylvester's theorem to $\widetilde{V}_l^k(x,y)$ with the choice of the last two rows and the first two columns, we get the following theorem.

Theorem 15.

$$\widetilde{V}_l^k(x,y) = \frac{\widetilde{V}_{l-1}^{m-l+1}(x,y) H_l^{k,1} - \widetilde{V}_{l-1}^k(x,y) H_l^{m-l+1,1}}{H_{l-1}^{m-l+2,1}}, \tag{27}$$

with the initialization $\widetilde{V}_1^k(x,y) = x^{m_1-i_k} y^{m_2-j_k}$.

Algorithm 16 (*Sylvester type algorithm for BMPTA*).

Step 1. for $l = 1$ to $m - 1$ do
  for $k = 1$ to $m - l + 1$ do
    compute $H_l^{k,1}$ using (25)
  end do
 end do
Step 2. for $l = 1$ to $m - 1$ do
  for $k = 1$ to $m - l + 1$ do
    compute $\widetilde{V}_l^k(x,y)$ using (27)
  end do
 end do
Step 3. for $k = 1$ to $m - 1$ do
  compute $b_{i_k j_k}$
 end do
Step 4. for $k = 1$ to n do
  compute $\widetilde{W}_{n_1 n_2}(x,y)$ using (10)
 end do

Example 17. Find a *BMPTA* of degree $(1, 1/1, 1)$ for

$$f(x,y) = \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ x & y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} y + \begin{bmatrix} -1/2 & 0 \\ 0 & 0 \end{bmatrix} y^2 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} xy + \begin{bmatrix} 0 & -1/2 \\ 0 & 0 \end{bmatrix} xy^2 + \cdots.$$

We first solve the system of equations (20) corresponding to $f(x, y)$ and then obtain the determinantal expression of $\widetilde{V}_{11}(x, y)$:

$$\widetilde{V}_{11}(x,y) = \widetilde{V}_3^1(x,y) = \begin{vmatrix} 0 & 0 & 2 & -1/2 \\ 2 & -1/2 & -1 & -1/2 \\ 0 & 2 & 0 & -1/2 \\ xy & x & y & 1 \end{vmatrix}.$$

Applying Algorithm 16 to the determinant, we have

$$\widetilde{V}_3^1(x,y) = \frac{\widetilde{V}_2^2(x,y)H_3^{1,1} - \widetilde{V}_2^1(x,y)H_3^{2,1}}{H_2^{3,1}},$$

where

$$\widetilde{V}_2^2(x,y) = \frac{\widetilde{V}_1^3(x,y)H_2^{2,1} - \widetilde{V}_1^2(x,y)H_2^{3,1}}{H_1^{4,1}} = 1 + \frac{y}{4} - \frac{3x}{2}, \qquad H_3^{1,1} = \frac{H_2^{3,2}H_2^{1,1} - H_2^{1,2}H_2^{3,1}}{H_1^{4,2}} = 2,$$

$$\widetilde{V}_2^1(x,y) = \frac{\widetilde{V}_1^3(x,y)H_2^{1,1} - \widetilde{V}_1^1(x,y)H_2^{3,1}}{H_1^{4,1}} = -4 - y - \frac{3xy}{2}, \qquad H_3^{2,1} = \frac{H_2^{3,2}H_2^{2,1} - H_2^{3,1}H_2^{3,1}}{H_1^{4,2}} = -\frac{7}{2}.$$

Thus

$$\widetilde{V}_3^1(x,y) = \frac{\left(1 + \frac{y}{4} - \frac{3}{2}x\right)2 - \left(-4 - y - \frac{3xy}{2}\right)\left(-\frac{7}{2}\right)}{-\frac{3}{2}} = 8 + 2x + 2y + \frac{7xy}{2}.$$

We normalize $V_3^1(x, y)$ as follows

$$\widetilde{V}_3^1(x,y) = 1 + \frac{x}{4} + \frac{y}{4} + \frac{7xy}{16},$$

and from (10) obtain

$$\widetilde{W}_{11}(x,y) = \begin{bmatrix} 1 + \frac{x}{4} + \frac{y}{4} - \frac{9xy}{16} & x + y + \frac{xy}{2} \\ x + \frac{xy}{4} & y + \frac{xy}{4} \end{bmatrix}.$$

5.2. ***E***-algorithm

In this part, we will give a different recursive process from the above Sylvester type algorithm. Rewrite (21) and (22) as follows:

$$\frac{\widetilde{W}_{n_1n_2}(x,y)}{\widetilde{V}_{m_1m_2}(x,y)} = \frac{\begin{vmatrix} \sum_{r=1}^{n} \tilde{d}^{i_1j_1}_{\alpha_r\beta_r} x^{\alpha_r} y^{\beta_r} & \sum_{r=1}^{n} \tilde{d}^{i_2j_2}_{\alpha_r\beta_r} x^{\alpha_r} y^{\beta_r} & \cdots & \sum_{r=1}^{n} \tilde{d}^{i_mj_m}_{\alpha_r\beta_r} x^{\alpha_r} y^{\beta_r} \\ d^{i_1j_1}_{\alpha_{n+1}\beta_{n+1}} & d^{i_2j_2}_{\alpha_{n+1}\beta_{n+1}} & \cdots & d^{i_mj_m}_{\alpha_{n+1}\beta_{n+1}} \\ d^{i_1j_1}_{\alpha_{n+2}\beta_{n+2}} & d^{i_2j_2}_{\alpha_{n+2}\beta_{n+2}} & \cdots & d^{i_mj_m}_{\alpha_{n+2}\beta_{n+2}} \\ \vdots & \vdots & & \vdots \\ d^{i_1j_1}_{\alpha_{n+m-1}\beta_{n+m-1}} & d^{i_2j_2}_{\alpha_{n+m-1}\beta_{n+m-1}} & \cdots & d^{i_mj_m}_{\alpha_{n+m-1}\beta_{n+m-1}} \end{vmatrix}}{\begin{vmatrix} x^{m_1-i_1}y^{m_2-j_1} & x^{m_1-i_2}y^{m_2-j_2} & \cdots & x^{m_1-i_m}y^{m_2-j_m} \\ d^{i_1j_1}_{\alpha_{n+1}\beta_{n+1}} & d^{i_2j_2}_{\alpha_{n+1}\beta_{n+1}} & \cdots & d^{i_mj_m}_{\alpha_{n+1}\beta_{n+1}} \\ d^{i_1j_1}_{\alpha_{n+2}\beta_{n+2}} & d^{i_2j_2}_{\alpha_{n+2}\beta_{n+2}} & \cdots & d^{i_mj_m}_{\alpha_{n+2}\beta_{n+2}} \\ \vdots & \vdots & & \vdots \\ d^{i_1j_1}_{\alpha_{n+m-1}\beta_{n+m-1}} & d^{i_2j_2}_{\alpha_{n+m-1}\beta_{n+m-1}} & \cdots & d^{i_mj_m}_{\alpha_{n+m-1}\beta_{n+m-1}} \end{vmatrix}}.$$

Following the ideas developed in [17] for the recursive computation of multivariate matrix Padé-type approximants, we will do some row and column manipulation in the numerator and denominator. Let us divide the first column by $x^{m_1-i_1}y^{m_2-j_1}, \ldots,$ the m-th column by $x^{m_1-i_m}y^{m_2-j_m}$, and multiply the second row by $x^{\alpha_{n+1}}y^{\beta_{n+1}}, \ldots,$ the m-th row by $x^{\alpha_{n+m-1}}y^{\beta_{n+m-1}}$, then we obtain

$$\widetilde{W}_{n_1n_2}(x,y)=\begin{vmatrix} \dfrac{\sum_{r=1}^{n}\tilde{d}^{i_1j_1}_{\alpha_r\beta_r}x^{\alpha_r}y^{\beta_r}}{x^{m_1-i_1}y^{m_2-j_1}} & \dfrac{\sum_{r=1}^{n}\tilde{d}^{i_2j_2}_{\alpha_r\beta_r}x^{\alpha_r}y^{\beta_r}}{x^{m_1-i_2}y^{m_2-j_2}} & \cdots & \dfrac{\sum_{r=1}^{n}\tilde{d}^{i_mj_m}_{\alpha_r\beta_r}x^{\alpha_r}y^{\beta_r}}{x^{m_1-i_m}y^{m_2-j_m}} \\ d^{i_1j_1}_{\alpha_{n+1}\beta_{n+1}}\dfrac{x^{\alpha_{n+1}}y^{\beta_{n+1}}}{x^{m_1-i_1}y^{m_2-j_1}} & d^{i_2j_2}_{\alpha_{n+1}\beta_{n+1}}\dfrac{x^{\alpha_{n+1}}y^{\beta_{n+1}}}{x^{m_1-i_2}y^{m_2-j_2}} & \cdots & d^{i_mj_m}_{\alpha_{n+1}\beta_{n+1}}\dfrac{x^{\alpha_{n+1}}y^{\beta_{n+1}}}{x^{m_1-i_m}y^{m_2-j_m}} \\ d^{i_1j_1}_{\alpha_{n+2}\beta_{n+2}}\dfrac{x^{\alpha_{n+2}}y^{\beta_{n+2}}}{x^{m_1-i_1}y^{m_2-j_1}} & d^{i_2j_2}_{\alpha_{n+2}\beta_{n+2}}\dfrac{x^{\alpha_{n+2}}y^{\beta_{n+2}}}{x^{m_1-i_2}y^{m_2-j_2}} & \cdots & d^{i_mj_m}_{\alpha_{n+2}\beta_{n+2}}\dfrac{x^{\alpha_{n+2}}y^{\beta_{n+2}}}{x^{m_1-i_m}y^{m_2-j_m}} \\ \vdots & \vdots & & \vdots \\ d^{i_1j_1}_{\alpha_{n+m-1}\beta_{n+m-1}}\dfrac{x^{\alpha_{n+m-1}}y^{\beta_{n+m-1}}}{x^{m_1-i_1}y^{m_2-j_1}} & d^{i_2j_2}_{\alpha_{n+m-1}\beta_{n+m-1}}\dfrac{x^{\alpha_{n+1}}y^{\beta_{n+m-1}}}{x^{m_1-i_2}y^{m_2-j_2}} & \cdots & d^{i_mj_m}_{\alpha_{n+m-1}\beta_{n+m-1}}\dfrac{x^{\alpha_{n+m-1}}y^{\beta_{n+1}}}{x^{m_1-i_m}y^{m_2-j_m}} \end{vmatrix}$$

and

$$\widetilde{V}_{m_1m_2}(x,y)=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ d^{i_1j_1}_{\alpha_{n+1}\beta_{n+1}}\dfrac{x^{\alpha_{n+1}}y^{\beta_{n+1}}}{x^{m_1-i_1}y^{m_2-j_1}} & d^{i_2j_2}_{\alpha_{n+1}\beta_{n+1}}\dfrac{x^{\alpha_{n+1}}y^{\beta_{n+1}}}{x^{m_1-i_2}y^{m_2-j_2}} & \cdots & d^{i_mj_m}_{\alpha_{n+1}\beta_{n+1}}\dfrac{x^{\alpha_{n+1}}y^{\beta_{n+1}}}{x^{m_1-i_m}y^{m_2-j_m}} \\ d^{i_1j_1}_{\alpha_{n+2}\beta_{n+2}}\dfrac{x^{\alpha_{n+2}}y^{\beta_{n+2}}}{x^{m_1-i_1}y^{m_2-j_1}} & d^{i_2j_2}_{\alpha_{n+2}\beta_{n+2}}\dfrac{x^{\alpha_{n+2}}y^{\beta_{n+2}}}{x^{m_1-i_2}y^{m_2-j_2}} & \cdots & d^{i_mj_m}_{\alpha_{n+2}\beta_{n+2}}\dfrac{x^{\alpha_{n+2}}y^{\beta_{n+2}}}{x^{m_1-i_m}y^{m_2-j_m}} \\ \vdots & \vdots & & \vdots \\ d^{i_1j_1}_{\alpha_{n+m-1}\beta_{n+m-1}}\dfrac{x^{\alpha_{n+m-1}}y^{\beta_{n+m-1}}}{x^{m_1-i_1}y^{m_2-j_1}} & d^{i_2j_2}_{\alpha_{n+m-1}\beta_{n+m-1}}\dfrac{x^{\alpha_{n+1}}y^{\beta_{n+m-1}}}{x^{m_1-i_2}y^{m_2-j_2}} & \cdots & d^{i_mj_m}_{\alpha_{n+m-1}\beta_{n+m-1}}\dfrac{x^{\alpha_{n+m-1}}y^{\beta_{n+1}}}{x^{m_1-i_m}y^{m_2-j_m}} \end{vmatrix}.$$

Set

$$\begin{aligned} &\tilde{s}_u(v)=\sum_{r=1}^{v}\tilde{d}^{i_uj_u}_{\alpha_r\beta_r}x^{\alpha_r}y^{\beta_r}/x^{m_1-i_u}y^{m_2-j_u},\\ &s_u(v)=\mathrm{trace}(e\cdot\tilde{s}_u(v)),\\ &\Delta s_u(v)=s_u(v+1)-s_u(v),\quad u=1,\ldots,m,\ v\ge 1. \end{aligned} \tag{28}$$

Then we have

$$\frac{\widetilde{W}_{n_1n_2}(x,y)}{\widetilde{V}_{m_1m_2}(x,y)}=\frac{\begin{vmatrix} \tilde{s}_1(n) & \tilde{s}_2(n) & \cdots & \tilde{s}_m(n) \\ \Delta s_1(n) & \Delta s_2(n) & \cdots & \Delta s_m(n) \\ \Delta s_1(n+1) & \Delta s_2(n+1) & \cdots & \Delta s_m(n+1) \\ \vdots & \vdots & & \vdots \\ \Delta s_1(n+m-2) & \Delta s_2(n+m-2) & \cdots & \Delta s_m(n+m-2) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta s_1(n) & \Delta s_2(n) & \cdots & \Delta s_m(n) \\ \Delta s_1(n+1) & \Delta s_2(n+1) & \cdots & \Delta s_m(n+1) \\ \vdots & \vdots & & \vdots \\ \Delta s_1(n+m-2) & \Delta s_2(n+m-2) & \cdots & \Delta s_m(n+m-2) \end{vmatrix}}$$

$$=\frac{\begin{vmatrix} \tilde{s}_1(n) & \tilde{s}_2(n)-\tilde{s}_1(n) & \cdots & \tilde{s}_m(n)-\tilde{s}_{m-1}(n) \\ \Delta s_1(n) & \Delta s_2(n)-\Delta s_1(n) & \cdots & \Delta s_m(n)-\Delta s_{m-1}(n) \\ \Delta s_1(n+1) & \Delta s_2(n+1)-\Delta s_1(n+1) & \cdots & \Delta s_m(n+1)-\Delta s_{m-1}(n+1) \\ \vdots & \vdots & & \vdots \\ \Delta s_1(n+m-2) & \Delta s_2(n+m-2)-\Delta s_1(n+m-2) & \cdots & \Delta s_m(n+m-2)-\Delta s_{m-1}(n+m-2) \end{vmatrix}}{\begin{vmatrix} 1 & 0 & \cdots & 0 \\ \Delta s_1(n) & \Delta s_2(n)-\Delta s_1(n) & \cdots & \Delta s_m(n)-\Delta s_{m-1}(n) \\ \Delta s_1(n+1) & \Delta s_2(n+1)-\Delta s_1(w+1) & \cdots & \Delta s_m(n+1)-\Delta s_{m-1}(n+1) \\ \vdots & \vdots & & \vdots \\ \Delta s_1(n+m-2) & \Delta s_2(n+m-2)-\Delta s_1(n+m-2) & \cdots & \Delta s_m(n+m-2)-\Delta s_{m-1}(n+m-2) \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \tilde{s}_1(n) & \tilde{s}_2(n)-\tilde{s}_1(n) & \cdots & \tilde{s}_m(n)-\tilde{s}_{m-1}(n) \\ \Delta s_1(n) & \Delta s_2(n)-\Delta s_1(n) & \cdots & \Delta s_m(n)-\Delta s_{m-1}(n) \\ \Delta s_1(n+1) & \Delta s_2(n+1)-\Delta s_1(n+1) & \cdots & \Delta s_m(n+1)-\Delta s_{m-1}(n+1) \\ \vdots & \vdots & & \vdots \\ \Delta s_1(n+m-2) & \Delta s_2(n+m-2)-\Delta s_1(n+m-2) & \cdots & \Delta s_m(n+m-2)-\Delta s_{m-1}(n+m-2) \end{vmatrix}}{\begin{vmatrix} \Delta s_2(n)-\Delta s_1(n) & \cdots & \Delta s_m(n)-\Delta s_{m-1}(n) \\ \Delta s_2(n+1)-\Delta s_1(n+1) & \cdots & \Delta s_m(n+1)-\Delta s_{m-1}(n+1) \\ \vdots & & \vdots \\ \Delta s_2(n+m-2)-\Delta s_1(n+m-2) & \cdots & \Delta s_m(n+m-2)-\Delta s_{m-1}(n+m-2) \end{vmatrix}}.$$

We set

$$\begin{aligned} &\tilde{g}_u(v) = \tilde{s}_{u+1}(v) - \tilde{s}_u(v), \\ &g_u(v) = s_{u+1}(v) - s_u(v), \quad u = 1, 2, \ldots, m-1, \\ &\Delta g_u(v) = g_u(v+1) - g_u(v), \quad v = 1, 2, \ldots, n+m-1. \end{aligned} \tag{29}$$

Then

$$\frac{\widetilde{W}_{n_1 n_2}(x,y)}{\widetilde{V}_{m_1 m_2}(x,y)} = \frac{\begin{vmatrix} \tilde{s}_1(n) & \tilde{s}_2(n)-\tilde{s}_1(n) & \cdots & \tilde{s}_m(n)-\tilde{s}_{m-1}(n) \\ \Delta s_1(n) & g_1(n+1)-g_1(n) & \cdots & g_{m-1}(n+1)-g_{m-1}(n) \\ \Delta s_1(n+1) & g_1(n+2)-g_1(n+1) & \cdots & g_{m-1}(n+1)-g_{m-1}(n+1) \\ \vdots & \vdots & & \vdots \\ \Delta s_1(n+m-2) & g_1(n+m-1)-g_1(n+m-2) & \cdots & g_{m-1}(n+m-1)-g_{m-1}(n+m-2) \end{vmatrix}}{\begin{vmatrix} g_1(n+1)-g_1(n) & \cdots & g_{m-1}(n+1)-g_{m-1}(n) \\ g_1(n+2)-g_1(n+1) & \cdots & g_{m-1}(n+2)-g_{m-1}(n+1) \\ \vdots & & \vdots \\ g_1(n+m-1)-g_1(n+m-2) & \cdots & g_{m-1}(n+m-1)-g_{m-1}(n+m-2) \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \tilde{s}_1(n) & \tilde{g}_1(n) & \cdots & \tilde{g}_{m-1}(n) \\ \Delta s_1(n) & \Delta g_1(n) & \cdots & \Delta g_{m-1}(n) \\ \Delta s_1(n+1) & \Delta g_1(n+1) & \cdots & \Delta g_{m-1}(n+1) \\ \vdots & \vdots & & \vdots \\ \Delta s_1(n+m-2) & \Delta g_1(n+m-2) & \cdots & \Delta g_{m-1}(n+m-2) \end{vmatrix}}{\begin{vmatrix} \Delta g_1(n) & \cdots & \Delta g_{m-1}(n) \\ \Delta g_1(n+1) & \cdots & \Delta g_{m-1}(n+1) \\ \vdots & & \vdots \\ \Delta g_1(n+m-2) & \cdots & \Delta g_{m-1}(n+m-2) \end{vmatrix}}. \tag{30}$$

It is well-known that this quotient of determinants can be computed by the E-algorithm [14] with the initializations:

$$\begin{cases} \widetilde{E}_0^{(v)} = \tilde{s}_1(v), & v = 1, 2, \ldots, n+m-1, \\ \tilde{g}_{0,u}^{(v)} = \tilde{g}_u(v), & u = 1, 2, \ldots, m-1. \end{cases} \tag{31}$$

Set $E_u^{(v)} = \operatorname{trace}\left(e \cdot \widetilde{E}_u^{(v)}\right)$, $g_{k,u}^{(v)} = \operatorname{trace}\left(e \cdot \tilde{g}_{k,u}^{(v)}\right)$,

$$\begin{cases} \widetilde{E}_k^{(v)} = \widetilde{E}_{k-1}^{(v)} - \dfrac{E_{k-1}^{(v+1)} - E_{k-1}^{(v)}}{g_{k-1,k}^{(v+1)} - g_{k-1,k}^{(v)}} \tilde{g}_{k-1,k}^{(v)}, & k = 1, 2, \ldots, m-1, u = k+1, k+2, \ldots, \\ \tilde{g}_{k,u}^{(v)} = \tilde{g}_{k-1,u}^{(v)} - \dfrac{g_{k-1,u}^{(v+1)} - g_{k-1,u}^{(v)}}{g_{k-1,k}^{(v+1)} - g_{k-1,k}^{(v)}} \tilde{g}_{k-1,k}^{(v)}, & v = 1, 2, \ldots, n+m-1. \end{cases}$$

Lut us define for $n \geq 1,\ m \geq 1$ the BMPTA

$$\widetilde{S}_{n,m-1}(x,y) = \widetilde{W}_{n_1 n_2}(x,y) / \widetilde{V}_{m_1 m_2}(x,y).$$

Using the E-algorithm with the initializations:

$$\begin{cases} \widetilde{E}_0^{(v)} = \dfrac{\sum_{r=1}^{v} \tilde{d}_{\alpha_r\beta_r}^{i_1 j_1} x^{\alpha_r} y^{\beta_r}}{x^{m_1-i_1} y^{m_2-j_1}}, & v = 1, 2, \ldots, \\ \tilde{g}_{0,u}^{(v)} = \dfrac{\sum_{r=1}^{v} \tilde{d}_{\alpha_r\beta_r}^{i_{u+1} j_{u+1}} x^{\alpha_r} y^{\beta_r}}{x^{m_1-i_{u+1}} y^{m_2-j_{u+1}}} - \dfrac{\sum_{r=1}^{v} \tilde{d}_{\alpha_r\beta_r}^{i_u j_u} x^{\alpha_r} y^{\beta_r}}{x^{m_1-i_u} y^{m_2-j_u}}, & u, v = 0, 1, \ldots, \end{cases}$$

we get the following result.

Theorem 18.

$$\widetilde{S}_{n,m-1}(x, y) = \widetilde{E}_{m-1}^{(n)} \quad \textit{for } m, n \geq 1. \tag{32}$$

Based on the discussion above we give the following algorithm:

Algorithm 19 (***E**-algorithm for BMPTA*).

Step 1. for $u = 1$ to m, $v = 1$ to $n + m - 1$ do
compute $\tilde{s}_u(v)$ using (28)
end do
Step 2. for $u = 1$ to m, $v = 1$ to $n + m - 1$ do
compute $\tilde{g}_u(v)$ using (29)
end do
Step 3. for $u = 1$ to $m - 1$, $v = 1$ to $n + m - 1$ do
compute $\widetilde{E}_0^{(v)}$ and $\tilde{g}_{0,u}^{(v)}$ using (31)
end do
Step 4. for $k = 1$ to $m - 2$ do
for $u = k + 1$ to $m - 1$, $v = 1$ to $n + m - 1$ do
compute $\tilde{g}_{k,u}^{(v)} = \tilde{g}_{k-1,u}^{(v)} - \frac{\mathrm{trace}\left(e\cdot\left(\tilde{g}_{k-1,u}^{(v+1)} - \tilde{g}_{k-1,u}^{(v)}\right)\right)}{\mathrm{trace}\left(e\cdot\left(\tilde{g}_{k-1,k}^{(v+1)} - \tilde{g}_{k-1,k}^{(v)}\right)\right)} \tilde{g}_{k-1,k}^{(v)}$
end do
end do
Step 5. for $k = 1$ to $m - 2$ do
for $u = k + 1$ to $m - 1$, $v = 1$ to $n + m - 1$ do
compute $\widetilde{E}_k^{(v)} = \widetilde{E}_{k-1}^{(v)} - \frac{\mathrm{trace}\left(e\cdot\left(\widetilde{E}_{k-1}^{(v+1)} - \widetilde{E}_{k-1}^{(v)}\right)\right)}{\mathrm{trace}\left(e\cdot\left(\tilde{g}_{k-1,k}^{(v+1)} - \tilde{g}_{k-1,k}^{(v)}\right)\right)} \tilde{g}_{k-1,k}^{(v)}$
end do
end do
Step 6. compute $\widetilde{W}_{n_1n_2}(x, y)/\widetilde{V}_{m_1m_2}(x, y)$ using (32).

Remark 20. If some $\Delta g_1(v) = 0$ for $v = n, \ldots, n + m - 2$, we call that the algorithm has a breakdown. In this case, we have to reorder the elements in D such that there is no breakdown in the process.

Example 21. Let $f(x, y)$ be the same as the matrix-valued function given by Example 17. Find a *BMPTA* of degree $(1, 1/1, 1)$ for $f(x, y)$.

Using Algorithm 19 we get

$$\frac{\widetilde{W}_{n_1n_2}(x, y)}{\widetilde{V}_{m_1m_2}(x, y)} = \widetilde{S}_{4,3}(x, y) = \widetilde{E}_3^{(4)} = \frac{\begin{bmatrix} 16 + 4x + 4y - 9xy & 16x + 16y + 8xy \\ 16x + 4xy & 16y + 4xy \end{bmatrix}}{16 + 4x + 4y + 7xy}.$$

6. An application to 2-D systems

Let $\mathbb{K}$ be the complex plane $\mathbb{C}$ or real plane $\mathbb{R}$. Let the state equation of 2-D linear discrete time invariable system for Fornasin–Marchesini model (F–M system) in [18,19] be $\Sigma = (A_0, A_1, A_2, B, B)$:

$$\begin{aligned} &X(h + 1, k + 1) = A_0X(h, k) + A_1X(h + 1, k) + A_2X(h, k + 1) + BU(h, k), \\ &Y(h, k) = CX(h, k), \end{aligned} \tag{33}$$

where $X(h,k) \in \mathbb{K}^q$ is the local state vector, $U(h,k)$ is the input vector, $Y(h,k)$ is the output vector, $A_0, A_1, A_2 \in \mathbb{K}^{q\times q}$, $B, C^T \in \mathbb{K}^{q\times l}$. Boundary conditions for (33) are given by $X(h,0), X(0,k), h, k = 0, 1, 2, \ldots$. By acting bivariate Z-transform on the both sides of (33) we obtain the transfer function of F–M system

$$G = G(x,y) = C(I - A_0xy - A_1x - A_2y)^{-1}B. \tag{34}$$

Assume that the matrix-valued polynomial $(I - A_0xy - A_1x - A_2y)$ have the inverse matrix given by

$$\begin{aligned} f(x,y) &= (I - A_0xy - A_1x - A_2y)^{-1} \\ &= \sum_{k=0}^{\infty}(A_0 + xy + A_1x + A_2y)^k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}x^iy^j, \end{aligned} \tag{35}$$

where $c_{ij} \in \mathbb{K}^{q\times q}$. Thus the output vector Y can be written as

$$Y(x,y) = C\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} X(h,k)x^hy^k = Cf(x,y)BU. \tag{36}$$

Definition 22. Let $(n_1, n_2/m_1, m_2)_f(x,y) = \widetilde{W}_{n_1n_2}(x,y)/\widetilde{V}_{m_1m_2}(x,y)$ be a *BMPTA* for $f(x,y)$ in (35). Replace $f(x,y)$ by $(n_1, n_2/m_1, m_2)_f(x,y)$ in (34) and (36) respectively, such that

$$\begin{aligned} &G_{(n_1,n_2/m_1,m_2)}(x,y) = C(n_1, n_2/m_1, m_2)_f(x,y)B, \\ &Y_{n_1,n_2/m_1,m_2}(x,y) = C(n_1, n_2/m_1, m_2)_f(x,y)BU. \end{aligned}$$

Thus $\Sigma_{m_1,m_1;n_1,n_2} = (A_0, A_1, A_2, B, C; (n_1, n_2/m_1, m_2)_f(x,y))$ is called a 2-D partial realization of type $(n_1, n_2/m_1, m_2)$ for F–M system.

Example 23 (*[16]*). Find a 2-D partial realization of type (1, 1/2, 2) for the bivariate matrix-valued function (35), where

$$cA_0 = 0, \qquad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 2 & 1 \end{bmatrix}^{\mathrm{T}}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Let us write the system of equations (20) as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 & 6 & 2 \\ 0 & 0 & 0 & 2 & 0 & -4 & 2 & 8 \\ 4 & 2 & 0 & 6 & 2 & 8 & 12 & 12 \\ 2 & 0 & -4 & 2 & 8 & 10 & 12 & 14 \\ 0 & 0 & 4 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 12 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{00} \\ b_{01} \\ b_{02} \\ b_{10} \\ b_{11} \\ b_{12} \\ b_{20} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -8 \\ -10 \\ -14 \\ 0 \\ -12 \\ -24 \end{bmatrix},$$

where $\mathrm{rank}(\mathcal{H}_{1,8}^{1,8}) = 8$.

Thus we obtain by Algorithm 16 or Algorithm 19 that

$$\begin{aligned} G_{1,1/2,2} &= C(1, 1/2, 2)_f(x,y)B = C\frac{\widetilde{W}_{11}(x,y)}{\widetilde{V}_{22}(x,y)}B \\ &= \frac{\begin{bmatrix} 1 - y - 32xy & 3x - 2y - 8xy \\ 1 - x - y - 28xy & -5y - xy \end{bmatrix}}{1 - 2x - 4y - 16xy - 156x^2y + 2y^2 + 5xy^2 + 85x^2y^2} \end{aligned}$$

and $Y_{1,1/2,2}(x,y) = C(1, 1/2, 2)_f BU$.

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