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On primitive permutation groups with small suborbits and their orbital graphs

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Abstract

In this paper, we study finite primitive permutation groups with a small suborbit. Based on the classification result of Quirin [Math. Z. 122 (1971) 267] and Wang [Comm. Algebra 20 (1992) 889], we first produce a precise list of primitive permutation groups with a suborbit of length 4. In particular, we show that there exist no examples of such groups with the point stabiliser of order $2^4 3^6$, clarifying an uncertain question (since 1970s). Then we analyse the orbital graphs of primitive permutation groups with a suborbit of length 3 or of length 4. We obtain a complete classification of vertex-primitive arc-transitive graphs of valency 3 and valency 4, and we prove that there exist no vertex-primitive half-arc-transitive graphs of valency less than 10. Finally, we construct vertex-primitive half-arc-transitive graphs of valency $2k$ for infinitely many integers k, with 14 as the smallest valency.

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1. Introduction

Let G be a finite transitive permutation group on a set Ω. An orbit Δ of G on $\Omega \times \Omega$ is called an *orbital* of G, while for $\alpha \in \Omega$, the set $\Delta(\alpha) = \{\beta \in \Omega \mid (\alpha, \beta) \in \Delta\}$ is an orbit of the stabiliser G_α, called a *suborbit* of G at α. Then an *orbital graph* of G is a digraph with vertex set Ω such that the arc set is an orbital of G. (We remark that orbital graphs are directed graphs and sometimes called *orbital digraphs*, although *graphs* studied in this paper are undirected graphs.)

The well-known Sims conjecture, proved in [3], says that for a primitive permutation group G, the order $|G_\alpha|$ is bounded above in terms of the length $|\Delta(\alpha)|$ of a suborbit $\Delta(\alpha)$. It is then a natural problem to determine G_α for a primitive permutation group G in terms of a suborbit length. The principle purpose of this paper is to investigate this problem for those with a small suborbit.

The problem of determining the stabiliser G_α for a primitive permutation group G which has a small suborbit $\Delta(\alpha)$ was actually the original motivation for the Sims conjecture (see [18]). It is easily shown that if $|\Delta(\alpha)| = 1$ or 2, then G is cyclic or dihedral of prime degree, see [15, Theorem 5]. Sims [18] determined G_α for the case where $|\Delta(\alpha)| = 3$, that is, $G_\alpha \cong \mathrm{Z}_3$, S_3, D_{12}, S_4, or $\mathrm{S}_4 \times \mathrm{Z}_2$. For the case where $|\Delta(\alpha)| = 4$, the candidates for G_α are essentially given in [16,22]. It was widely believed that there exist primitive permutation groups with a suborbit of length 4 and point stabiliser of order $2^4 3^6$: for example, it was claimed by Quirin [16, p. 273] that "Sims and Thompson (in work as yet not published) have established an upper bound of $2^4 3^6$ on the order of vertex-stabiliser G_α, and Thompson has shown that this bound is sharp." The upper bound $2^4 3^6$ was published in [6], but no proof was published for the sharpness. The claim of Quirin was also accepted in [22, p. 897]. However, it is shown in the following corollary that the upper bound $2^4 3^6$ is not reachable, and a precise list for G_α is given.

Corollary 1.1. *Let G be a finite primitive permutation group on Ω with a suborbit of length* 4. *Then G_α is explicitly listed in the following table*:

G_α	$\lvert G_\alpha\rvert$	Examples of groups G
Z_4	2^2	$\mathrm{Z}_p{:}\mathrm{Z}_4$
D_8	2^3	$\mathrm{Z}_p^2{:}\mathrm{D}_8$
D_{16}	2^4	$\mathrm{PGL}_2(9)$
$\mathrm{Z}_8{:}\mathrm{Z}_2$	2^4	M_{10}
$[2^5]$	2^5	$\mathrm{Aut}(\mathrm{A}_6)$
A_4	$2^2 3$	$\mathrm{PSL}_2(11)$
S_4	$2^3 3$	$\mathrm{PSL}_3(3)$
$\mathrm{A}_4 \times \mathrm{Z}_3$	$2^2 3^2$	$\mathrm{P\Sigma L}_2(27)$
$(\mathrm{A}_4 \times \mathrm{Z}_3){:}\mathrm{Z}_2$	$2^3 3^2$	A_7
$\mathrm{S}_4 \times \mathrm{S}_3$	$2^4 3^2$	S_7

The candidates for G_α with $|\Delta(\alpha)| = 4$ given by Wang [22] were obtained by a classification of such groups G. Several groups in Wang's list will be proved not to have suborbits of length 4, and a precise list of such groups is given in Theorem 3.4. The result of Sims [18] and Corollary 1.1 motivates the following problem.

Problem 1.2. Let G be a primitive permutation group on Ω that has a 'small' suborbit. Find the possibilities for the point stabiliser G_α where $\alpha \in \Omega$.

The other main motivation for this paper is a problem in algebraic graph theory. Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. An ordered pair of adjacent vertices is called an *arc*, and the set of arcs of Γ is denoted by $A\Gamma$. If a subgroup $G \leqslant \mathrm{Aut}\,\Gamma$ is transitive on $V\Gamma$, $E\Gamma$, or $A\Gamma$, then the graph Γ is said to be *G-vertex-transitive*, *G-edge-transitive*, or *G-arc-transitive*, respectively; in particular, Γ is simply called *vertex-transitive*, *edge-transitive*, or *arc-transitive* if $G = \mathrm{Aut}\,\Gamma$. Further, if Γ is vertex-transitive and edge-transitive but not arc-transitive, then Γ is called a *half-arc-transitive graph*.

It was proved in [21] that the valency of a finite half-arc-transitive graph is even. In [2] Bouwer gave a construction of a half-arc-transitive graph of valency $2k$ for every $k \geqslant 2$. The study of half-arc-transitive graphs has currently been an active topic (see [10,12,13] for references). In [10], given any prime $p \geqslant 5$, infinitely many half-arc-transitive graphs of valency $2k$ and p-power order were constructed for each non-trivial factor k of $p-1$. By a classical result of Dirichlet (1837, see [17, p. 205]), for any positive integer k, there exists a prime p such that k divides $p-1$. We thus have the following result.

Theorem A. *For each positive integer $k \geqslant 2$, there exist infinitely many half-arc-transitive graphs of valency $2k$ and prime power order.*

However, although considerable attention has been paid to the existence problem of vertex-primitive half-arc-transitive graphs, only a few values of $2k$ have been known to be the valencies of such graphs until now (see, for example, [5,14,24]). It is quite easily shown that there exist no vertex-primitive half-arc-transitive graphs of valency 4 (see [12], for example). Here we propose to study the following problem.

Problem 1.3. Find all positive integers k such that there exist vertex-primitive half-arc-transitive graphs of valency $2k$.

In this paper we construct vertex-primitive half-arc-transitive graphs of valency $2k$ for infinitely many integers k, with 14 being the smallest valency. Moreover, we prove that there are no vertex-primitive half-arc-transitive graphs of valency less than 10 in the following theorem.

Theorem 1.4.

(1) *There exist no vertex-primitive half-arc-transitive graphs of valency less than* 10.
(2) *For each $m \geqslant 1$, there exists a vertex-primitive half-arc-transitive graph of valency* $2(2^{2m+1}-1)$.

We have been unable to determine whether there exist vertex-primitive half-arc-transitive graphs of valencies 10 and 12. It was proposed in [24] to determine the smallest valency $2k$ of a vertex-primitive half-arc-transitive graph; see also [12]. By Theorem 1.4, we know that $2k \in \{10, 12, 14\}$. Also it would be interesting to find out if for every positive even integer $2k \geqslant 14$, there exists a vertex-primitive half-arc-transitive graph of valency $2k$.

A graph Γ is called a *Cayley graph* of a group R if $\mathsf{Aut}\,\Gamma$ contains a subgroup which is isomorphic to R and acts regularly on $V\Gamma$. For an integer $r \geqslant 2$, an $(r+1)$-tuple $(v_0, v_1, \ldots, v_r)$ of vertices of Γ is called an r-*arc* if v_i is adjacent to v_{i+1} for $0 \leqslant i \leqslant r-1$, and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant r-1$. If a subgroup $G \leqslant \mathsf{Aut}\,\Gamma$ is transitive on the set of s-arcs of Γ, then Γ is said to be (G, s)-*arc-transitive*. A (G, s)-arc-transitive graph is called (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a graph Γ is said to be s-*transitive* if it is $(\mathsf{Aut}\,\Gamma, s)$-transitive.

By Theorem 1.4, finite vertex-primitive edge-transitive graphs with valency less than 10 are arc-transitive. All vertex-primitive arc-transitive graphs of valency 3 and valency 4 are classified in the next theorem.

Theorem 1.5. *Let Γ be a vertex-primitive arc-transitive graph of valency l, where $l = 3$ or* 4. *Then the following two statements are true, where p is a prime, and n is the number of vertices of Γ*:

(1) *If $l = 3$, then Γ is an s-transitive graph such that s, n, $\mathsf{Aut}\,\Gamma$, and Γ are as in Table* 1; *further, Γ is a Cayley graph if and only if $\Gamma \cong \mathsf{K}_4$.*
(2) *If $l = 4$, then Γ is one of m non-isomorphic s-transitive graphs such that s, m, n, and $\mathsf{Aut}\,\Gamma$ are as in Table* 2; *further, Γ is a Cayley graph of a group R if and only if $\mathsf{Aut}\,\Gamma = \mathsf{Z}_p{:}\mathsf{Z}_4$, $\mathsf{Z}_p^2{:}\mathsf{D}_8$, $\mathsf{PGL}_2(5)$, $\mathsf{PGL}_2(7)$, $\mathsf{PGL}_2(11)$, or $\mathsf{PSL}_2(23)$, and $R = \mathsf{Z}_p$, Z_p^2, Z_5, $\mathsf{Z}_7{:}\mathsf{Z}_3$, $\mathsf{Z}_{11}{:}\mathsf{Z}_5$, $\mathsf{Z}_{23}{:}\mathsf{Z}_{11}$, respectively.*

Remark. Theorem 1.5 tells us that there are only two vertex-primitive 3-arc-transitive graphs of valency 4; there is only one vertex-primitive 3-arc-transitive graph of valency 4 with an odd number of vertices; there are only a few vertex-primitive 2-arc-transitive Cayley graphs of valency at most 4. This indicates that graphs of these kinds are rare. In fact, vertex-primitive 4-arc-transitive graphs have been classified in [7]. It is shown in [8] that there exist no 4-arc-transitive graphs with an odd number of vertices. It is shown in [9] that 2-arc-transitive Cayley graphs are rare. These results motivate the following problem.

Table 1
Vertex-primitive arc-transitive graphs of valency 3

$\mathsf{Aut}\,\Gamma$	Stabiliser	s	n	Graph
S_4	S_3	2	4	Complete graph K_4
S_5	D_{12}	3	10	Petersen
$\mathsf{PGL}_2(7)$	D_{12}	3	28	Coxeter
$\mathsf{Aut}(\mathsf{PSL}_3(3))$	$\mathsf{S}_4 \times \mathsf{Z}_2$	5	234	Wong
$\mathsf{PSL}_2(p)$, $p \equiv \pm 1 \pmod{16}$	S_4	4	$(p(p^2-1))/48$	Γ is unique

Table 2
Vertex-primitive arc-transitive graphs of valency 4

Aut Γ	Vertex-stabiliser	s	n	m	Comments
$\mathrm{Z}_p{:}\mathrm{Z}_4$	Z_4	1	p	1	$p > 5$
$\mathrm{Z}_p^2{:}\mathrm{D}_8$	D_8	1	p^2	1	$p \geqslant 3$
$\mathrm{PSL}_2(p)$	S_4	2	$(p(p^2-1))/48$	1	$p \equiv \pm 1 \pmod 8$, $p \neq 7$
$\mathrm{PSL}_2(p)$	A_4	2	$(p(p^2-1))/24$	$[(p+\varepsilon)/12]$	$p \equiv \pm 3 \pmod 8$, $p \neq 5$, $\varepsilon = \pm 1$
					$3 \mid (p+\varepsilon)$, $p \not\equiv \pm 1 \pmod{10}$
$\mathrm{PGL}_2(p)$	S_4	2	$(p(p^2-1))/24$	1	$p \equiv \pm 3 \pmod 8$
$\mathrm{PGL}_2(7)$	D_{16}	1	21	1	Cayley
$\mathrm{Aut}(\mathrm{A}_6)$	$[2^5]$	1	45	1	non-Cayley
$\mathrm{PSL}_2(17)$	D_{16}	1	153	1	non-Cayley
S_7	$\mathrm{S}_4 \times \mathrm{S}_3$	3	35	1	odd graph
$\mathrm{PSL}_3(7)$	$(\mathrm{A}_4{:}\mathrm{Z}_3){:}\mathrm{Z}_2$	3	26 068	1	non-Cayley

Problem 1.6. Classify the vertex-primitive 3-arc-transitive graphs, and the vertex-primitive 2-arc-transitive Cayley graphs.

This paper is organized as follows. Section 2 collects some preliminary results. In Section 3, a precise list of primitive permutation groups with suborbits of length 4 is given, and then in Section 4, their orbital graphs of out-valency 4 are analyzed. Finally, in Section 5, Theorems 1.4 and 1.5 are proved.

2. Permutation groups, orbital graphs, and coset graphs

In this section, we collect some notation and results which will be used later.

Let G be a transitive permutation group on Ω. For an orbital $\Delta = (\alpha, \beta)^G$, the orbital $\Delta^* = (\beta, \alpha)^G$ is called the *paired orbital* of Δ. If $\Delta = \Delta^*$, then Δ is called *self-paired*, and $\Delta(\alpha)$ is called a *self-paired suborbit*. The digraph $\Sigma := (\Omega, \Delta)$ with vertex set Ω and arc set Δ is an orbital graph of G. Let Σ^* denote the orbital graph (Ω, Δ^*). Then $\Sigma \cup \Sigma^* := (\Omega, \Delta \cup \Delta^*)$, as an undirected graph with vertex set Ω and edge set $\Delta \cup \Delta^*$, is G-vertex-transitive and G-edge-transitive. Further, $\Sigma \cup \Sigma^*$ is G-arc-transitive if and only if Δ is self-paired, that is, $\Delta = \Delta^*$ and hence $\Sigma \cup \Sigma^* = \Sigma$. Conversely, for an arbitrary G-vertex-transitive graph with $G \leqslant \mathrm{Aut}\,\Gamma$, G is a transitive permutation group on the vertex set $V\Gamma$. Thus, if further Γ is G-edge-transitive, then there exists an orbital graph Σ of G such that $\Gamma \cong \Sigma \cup \Sigma^*$.

For an abstract group G, a subgroup $H \leqslant G$ is said to be *core free* if no non-trivial normal subgroup of G is contained in H. For a subset $S \subseteq G$ and a core free subgroup H of G, the *coset graph* $\Gamma = \mathrm{Cos}(G, H, HSH)$ is defined as the digraph with vertex set $V\Gamma = [G : H] = \{Hx \mid x \in G\}$ such that Hx is adjacent to Hy if and only if $yx^{-1} \in HSH$. It easily follows that each element $g \in G$ induces an automorphism of Γ by the *coset action*, that is,

$$g : Hx \mapsto Hxg \quad \text{for all } x \in G.$$

In the coset action, G is faithful on $V\Gamma$, and so we may assume that $G \leqslant \mathsf{Aut}\,\Gamma$. Then G acts transitively on $V\Gamma$, and Γ is G-vertex-transitive. If $HS^{-1}H = HSH$, then the adjacency relation of Γ is symmetric, and so Γ can be viewed as an undirected graph by identifying two arcs (Hx, Hy) and (Hy, Hx) with an edge $\{Hx, Hy\}$. The following lemma collects some basic properties about coset graphs.

Lemma 2.1. *Let $\Gamma = \mathsf{Cos}(G, H, HSH)$ be an undirected graph. Then*

(i) *Γ is connected if and only if $\langle H, S\rangle = G$;*
(ii) *Γ is G-edge-transitive if and only if $HSH = H\{g, g^{-1}\}H$ for some $g \in G$;*
(iii) *Γ is G-arc-transitive if and only if $HSH = HgH$ for some $g \in G$ such that $g^2 \in H$.*

Let $\mathsf{Aut}(G, H) = \{\sigma \in \mathsf{Aut}(G) \mid H^\sigma = H\}$. Some elements of $\mathsf{Aut}(G, H)$ induce automorphisms of Γ.

Lemma 2.2. *Suppose that $\sigma \in \mathsf{Aut}(G, H)$. Then $\Gamma = \mathsf{Cos}(G, H, HSH)$ is isomorphic to $\Sigma = \mathsf{Cos}(G, H, HS^\sigma H)$. Moreover, σ induces an automorphism of Γ if and only if $HS^\sigma H = HSH$.*

Proof. Each element $\sigma \in \mathsf{Aut}(G, H)$ induces a permutation on the vertex set $[G : H]$ by the natural action, that is, $(Hx)^\sigma = Hx^\sigma$. Further,

$$\begin{aligned}
Hx \text{ is adjacent to } Hy \text{ in } \Gamma \quad &\Longleftrightarrow \quad yx^{-1} \in HSH \\
&\Longleftrightarrow \quad \left(yx^{-1}\right)^\sigma \in (HSH)^\sigma \\
&\Longleftrightarrow \quad y^\sigma\left(x^\sigma\right)^{-1} \in HS^\sigma H \\
&\Longleftrightarrow \quad Hx^\sigma \text{ is adjacent to } Hy^\sigma \text{ in } \Sigma.
\end{aligned}$$

Thus σ induces an isomorphism from Γ to Σ, and σ induces an automorphism of Γ if and only if $\Gamma = \Sigma$, and this in turn is true if and only if $HS^\sigma H = HSH$. □

We need a criterion for determining isomorphic classes of certain coset graphs.

Lemma 2.3. *Let $\Gamma = \mathsf{Cos}(G, H, HSH)$ and $\Sigma = \mathsf{Cos}(G, H, HTH)$. Assume that $G = \mathsf{Aut}\,\Gamma = \mathsf{Aut}\,\Sigma$. Then Γ is isomorphic to Σ if and only if there exists $\sigma \in \mathsf{Aut}(G, H)$ such that $HS^\sigma H = HTH$.*

Proof. Let $\sigma \in \mathsf{Aut}(G, H)$ be such that $HS^\sigma H = HTH$. Then the following map is an isomorphism from Γ to Σ:

$$\phi : Hg \mapsto Hg^\sigma.$$

Conversely, suppose that $\Gamma \cong \Sigma$. Let $\psi \in \mathsf{Sym}(\Omega)$ be an isomorphism from Γ to Σ, where $\Omega = [G : H]$. Since G acts transitively on Ω, we may assume $\alpha^{\psi} = \alpha$, where α is the point $H \in \Omega$. For an arc $(\beta, \gamma) \in A\Sigma$, we have $(\beta', \gamma') := (\beta, \gamma)^{\psi^{-1}} \in A\Gamma$, and for $x \in \mathsf{Aut}\,\Gamma$, $(\beta'', \gamma'') := (\beta', \gamma')^{x} \in A\Gamma$. Then $(\beta''', \gamma''') := (\beta'', \gamma'')^{\psi} \in A\Sigma$, and hence $(\beta, \gamma)^{\psi^{-1}x\psi} = (\beta', \gamma')^{x\psi} = (\beta'', \gamma'')^{\psi} = (\beta''', \gamma''') \in A\Sigma$. Thus $\psi^{-1}x\psi$ is an automorphism of Σ, and so $\psi^{-1}(\mathsf{Aut}\,\Gamma)\psi \leqslant \mathsf{Aut}\,\Sigma$. Since $\Gamma \cong \Sigma$, $\mathsf{Aut}\,\Gamma \cong \mathsf{Aut}\,\Sigma$ and hence $\psi^{-1}(\mathsf{Aut}\,\Gamma)\psi = \mathsf{Aut}\,\Sigma$.

Assume further that $G = \mathsf{Aut}\,\Gamma = \mathsf{Aut}\,\Sigma$. Then $G^{\psi} = G$, that is, $\psi \in \mathsf{N}_{\mathsf{Sym}(\Omega)}(G)$. Thus ψ induces, by conjugation, an automorphism τ of G. Further, $H = G_{\alpha} = G_{\alpha^{\psi}} = (G_{\alpha})^{\psi} = \psi^{-1}H\psi = H^{\psi}$, and hence $\psi \in \mathsf{N}_{\mathsf{Sym}(\Omega)}(H)$. Then $H^{\tau} = H^{\psi} = H$. For $g \in G$, let $\omega \in \Omega$ be such that $\alpha^{g} = \omega$. Then $\omega = Hg$, and $\omega^{\psi} = \alpha^{g\psi} = \alpha^{\psi g^{\psi}} = \alpha^{g^{\psi}} = \alpha^{g^{\tau}}$. Therefore, $(Hg)^{\psi} = \omega^{\psi} = \alpha^{g^{\tau}} = Hg^{\tau}$. It follows that $HS^{\tau}H = (HSH)^{\tau} = (HSH)^{\psi} = HTH$. □

In the following, let $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$ with $g \in G \setminus H$. Then the neighborhood of the vertex H is the set $\{Hx \mid x \in \{g, g^{-1}\}H\}$, and its size is the valency of Γ and equals $|H\{g, g^{-1}\}H|/|H|$. The valency may be further written as the following form.

Lemma 2.4. *The valency of Γ is equal to $|H|/|H \cap H^{g}|$ if $HgH = Hg^{-1}H$, or $2|H|/|H \cap H^{g}|$ otherwise.*

The following lemma will be used for deciding whether a G-edge transitive graph is arc-transitive, in an extremal case.

Lemma 2.5. *Assume that $G \lhd \mathsf{Aut}\,\Gamma$. Then Γ is arc-transitive if and only if there exists $\sigma \in \mathsf{Aut}(G, H)$ such that $g^{\sigma} \in Hg^{-1}H$, or equivalently, σ interchanges HgH and $Hg^{-1}H$.*

Proof. Suppose that Γ is arc-transitive. Denote by α the vertex H of Γ. Then $G_{\alpha} = H$, both $\alpha^{g} = Hg$ and $\alpha^{g^{-1}} = Hg^{-1}$ are neighbors of α. Thus there exists some $\phi \in \mathsf{Aut}\,\Gamma$ such that $\alpha^{\phi} = \alpha$ and $(\alpha^{g})^{\phi} = \alpha^{g^{-1}}$. Since $G \lhd \mathsf{Aut}\,\Gamma$, ϕ induces, by conjugation, an automorphism σ of G. Thus $(\alpha^{x})^{\phi} = \alpha^{x\phi} = (\alpha^{\phi})^{x^{\phi}} = \alpha^{x^{\sigma}}$ for all $x \in G$. In particular, $\alpha^{g^{-1}} = (\alpha^{g})^{\phi} = \alpha^{g^{\sigma}}$, and $\alpha = (\alpha^{h})^{\phi} = \alpha^{h^{\sigma}}$ for all $h \in H$. It follows that $g^{\sigma}g, h^{\sigma} \in G_{\alpha} = H$, and hence $g^{\sigma} \in Hg^{-1} \subseteq Hg^{-1}H$ and $H^{\sigma} = H$.

On the other hand, if $g^{\tau} \in Hg^{-1}H$ for some $\tau \in \mathsf{Aut}(G, H)$, then $g^{\tau} = h_1g^{-1}h_2$ for some elements $h_1, h_2 \in H$. Thus we have $(Hg)^{\tau h_2^{-1}} = (Hg^{\tau})^{h_2^{-1}} = (Hg^{\tau})h_2^{-1} = (H(h_1g^{-1}h_2))h_2^{-1} = Hg^{-1}$. It follows that Γ is G-arc-transitive. □

The next two lemmas provide methods for constructing certain coset graphs.

Lemma 2.6. *For $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$, let $P = H \cap H^{g}$. Then $P, P^{g^{-1}} \leqslant H$. Further, assume that P is conjugate to $P^{g^{-1}}$ in H. Then there exists $x \in G$ such that $P = H \cap H^{x}$, x normalises P, and $\Gamma = \mathsf{Cos}(G, H, H\{x, x^{-1}\}H)$.*

Proof. Since $P = H \cap H^g$, we have $P \leqslant H^g$, and thus $P^{g^{-1}} \leqslant H$. Let $h \in H$ be such that $P^h = P^{g^{-1}}$. Let $x = hg$. Then $P^x = P^{hg} = P$, $H\{g, g^{-1}\}H = H\{x, x^{-1}\}H$, and $\Gamma = \mathsf{Cos}(G, H, H\{x, x^{-1}\}H)$. Further, $P = H \cap H^x$, normalised by x. □

Lemma 2.7. *Suppose that $HgH = Hg^{-1}H$. Then there exists $x \in G$ such that $H \cap H^x = H \cap H^g =: P$, $x \in \mathsf{N}_G(P) \setminus H$, $x^2 \in H \cap H^x$, $gx \in H$, and $HgH = HxH$. In particular, $|\mathsf{N}_G(P) : P|$ is even.*

Proof. Since $HgH = Hg^{-1}H$, Γ is G-arc-transitive. Let $\alpha = H$ and $\beta = Hg$, two vertices of Γ. Then $\beta = \alpha^g$, $G_\alpha = H$ and $G_\beta = H^g$. Since (α, β) is an arc of Γ, there exists $x \in G \setminus H$ such that $\alpha^x = \beta$ and $\beta^x = \alpha$. Thus $\alpha^{x^2} = \beta^x = \alpha$, $\beta^{x^2} = \alpha^x = \beta$, $\alpha^{gx} = \beta^x = \alpha$, and $H^g = G_\beta = G_{\alpha^x} = G_\alpha^x = H^x$. It follows that $x^2, gx \in G_\alpha = H$, $x^2 \in G_\beta = H^g = H^x$. Hence $x^2 \in H \cap H^g = H \cap H^x$, $x^{-1}g^{-1} \in H$, and $HgH = Hg^{-1}H = Hx(x^{-1}g^{-1})H = HxH$. Further, $(H \cap H^x)^x = H^x \cap H^{x^2} = H^x \cap H$, and hence $x \in \mathsf{N}_G(H \cap H^x)$. □

Finally, we quote a known result about s-arc-transitive graphs, refer to [7] and [1, Chapter 17].

Proposition 2.8. *Let Γ be a (G, s)-transitive graph of valency k with $s \geqslant 2$. Then for a vertex α, the following statements are true*:

(i) *if $k = 3$, then $(s, G_\alpha) = (2, \mathsf{S}_3)$, $(3, \mathsf{D}_{12})$, $(4, \mathsf{S}_4)$, or $(5, \mathsf{S}_4 \times \mathsf{Z}_2)$;*
(ii) *if $k = 4$, then $s = 2$ and $\mathsf{A}_4 \leqslant G_\alpha \leqslant \mathsf{S}_4$; $s = 3$ and $\mathsf{A}_4 \times \mathsf{Z}_3 \leqslant G_\alpha \leqslant \mathsf{S}_4 \times \mathsf{S}_3$; $s = 4$ and $G_\alpha = \mathsf{Z}_3^2.\mathsf{Q}_8.\mathsf{S}_3$; or $s = 7$ and $G_\alpha = [3^5].\mathsf{Q}_8.\mathsf{S}_3$.*

3. Primitive permutation groups with a suborbit of length 4

Let G be a primitive permutation group on a set Ω. Assume that G has a suborbit $\Delta(\alpha)$ of length 4, where $\alpha \in \Omega$. Then G is classified by a collection of articles, see Quirin [16], Sims [18], and Wang [22]. Here we work out a precise list of such groups. Let Σ be the orbital graph corresponding $\Delta(\alpha)$, which is of out-valency 4. Then Σ may be represented as a coset graph:

$$\Sigma = \mathsf{Cos}(G, H, HgH),$$

where $H = G_\alpha$, and $g \in G \setminus H$. Denote by β the vertex $\alpha^g = Hg$. Then $G_{\alpha\beta} = H \cap H^g$, and as $|\Sigma(\alpha)| = 4$, $|H : H \cap H^g| = |G_\alpha : G_{\alpha\beta}| = 4$.

3.1. Non-examples

We here prove that three groups in Wang's list do not have suborbits of length 4. The first group is $\mathsf{PSL}_2(7)$, as a permutation of degree 7, has no suborbits of length 4. In the following lemmas, we deal with the other two groups. For a group M and a subgroup N of M, by $N \operatorname{char} M$ we mean that N is a characteristic subgroup of M.

Lemma 3.1. *The group* $\mathsf{PSL}_3(7).3$ *has no primitive permutation representation which has a suborbit of length* 4.

Proof. Suppose that $G := \mathsf{PSL}_3(7).3$ has a primitive permutation representation on Ω such that $\Delta(\alpha)$ is a suborbit of length 4 at α, where $\alpha \in \Omega$. Then by [4], we conclude that $G_\alpha^{\Delta(\alpha)} \cong \mathsf{S}_4$, and $H = G_\alpha \cong \mathsf{Z}_6^2{:}\mathsf{S}_3$. Take $g \in G$ such that $\alpha^g \in \Delta(\alpha)$, and let $P = H \cap H^g$. Then $|H : P| = 4$, and so $|P| = 2 \cdot 3^3$. Let P_3 be a Sylow 3-subgroup of P. Then $P_3 \lhd P$, $P = \mathsf{N}_H(P_3)$, and P_3 is also a Sylow 3-subgroup of G. Hence in particular, all subgroups of H which are isomorphic to P are conjugate (in H). By Lemma 2.6, we may assume $g \in \mathsf{N}_G(P) \setminus H$. Further, $P_3 \operatorname{char} P$, and hence $P_3 \lhd \mathsf{N}_G(P)$, so $P_3 \operatorname{char} \mathsf{N}_G(P)$. Let K be a maximal subgroup of G such that $\mathsf{N}_G(P) \leqslant K$. By the Atlas [4], either K is conjugate to H and $K \cong \mathsf{Z}_6^2{:}\mathsf{S}_3$, or $K \cong \mathsf{Z}_3^2{:}(2\mathsf{A}_4)$. Thus K has a normal subgroup Q such that $K/Q \cong \mathsf{S}_4$ or A_4. Since $\mathsf{N}_G(P) > P$, we have $1 < |\mathsf{N}_G(P) : P| \leqslant |K : P| = 4$. It follows that $|K : \mathsf{N}_G(P)| = 1$ or 2. Then $\mathsf{N}_G(P) \lhd K$, hence $P_3 \lhd K$. Thus K/Q has a normal subgroup $P_3Q/Q \cong \mathsf{Z}_3$, which is a contradiction. $\square$

We need the following lemma to prove $\mathsf{P\Omega}_8^+(2).\mathsf{Z}_3$ has no suborbits of length 4.

Lemma 3.2. *Let* $V = \mathsf{Z}_2 \wr \mathsf{S}_4$. *Suppose that* $W \cong \mathsf{Z}_2^3.\mathsf{S}_4$ *is a subgroup of* V. *If* X *is a Sylow* 3*-subgroup of* W, *then* $\mathsf{N}_W(X) = \mathsf{Z}_3.[4]$.

Proof. By the definition of the wreath product $\mathsf{Z}_2 \wr \mathsf{S}_4$, we may assume that $V = \langle a_1, a_2, a_3, a_4\rangle.\mathsf{S}_4$ such that S_4 transitively permutes $\{a_1, a_2, a_3, a_4\}$. Thus S_4 contains an element x such that $x : a_1 \to a_2$, $a_2 \to a_3$, $a_3 \to a_1$, and $a_4 \to a_4$. Let $X = \langle x\rangle \cong \mathsf{Z}_3$. It follows that $\mathsf{C}_V(x) = \langle a_1a_2a_3, a_4\rangle \times X \cong \mathsf{Z}_2^2 \times \mathsf{Z}_3$.

Let $W < V$ be such that $W = M.\mathsf{S}_4 \cong \mathsf{Z}_2^3.\mathsf{S}_4$, where $M \cong \mathsf{Z}_2^3$. Then by Sylow theorem, we may assume that $X < W$. Since $\mathsf{C}_W(x) \leqslant \mathsf{C}_V(x) \cong \mathsf{Z}_2^2 \times \mathsf{Z}_3$, we conclude that x does not centralise M. It then follows that $\mathsf{C}_M(x) \cong \mathsf{Z}_2$. Now $\mathsf{C}_W(x)/\mathsf{C}_M(x) \cong \mathsf{C}_W(x)M/M \leqslant \mathsf{C}_{W/M}(\bar{x}) = \langle \bar{x}\rangle$, where $\bar{x} = xM \in W/M$. Thus $\mathsf{C}_W(x) = \mathsf{C}_M(x).\langle \bar{x}\rangle \cong \mathsf{Z}_2 \times \mathsf{Z}_3$, and so $\mathsf{N}_W(X) = \mathsf{Z}_3.[4]$. $\square$

For a group G, denote by $\mathsf{soc}(G)$ the *socle* of G, which is the subgroup of G generated by all minimal normal subgroups of G; by $\mathsf{O}_p(G)$ we mean the largest normal p-subgroup of G, where p is a prime.

Lemma 3.3. *There is no primitive permutation representation of* $\mathsf{P\Omega}_8^+(2).\mathsf{Z}_3$ *that has a suborbit of length* 4.

Proof. Suppose that $G := \mathsf{P\Omega}_8^+(2).\mathsf{Z}_3$ has a primitive permutation representation on Ω with a suborbit $\Delta(\alpha)$ of length 4. Then by a result of Knapp [6], $|G_\alpha| \leqslant 2^4 3^6$, and thus by the Atlas [4], we have that $H := G_\alpha \cong 3_+^{1+4}{:}(2\mathsf{S}_4)$ such that $G_\alpha^{\Delta(\alpha)} \cong \mathsf{S}_4$. Take $g \in G$ such that $\beta := \alpha^g \in \Delta(\alpha)$. Then $G_{\alpha\beta} = H \cap H^g \cong 3_+^{1+4}{:}(2.\mathsf{S}_3) = (3_+^{1+4}.3){:}[4]$. Let $P = G_{\alpha\beta}$, and let $P_3 = \mathsf{O}_3(P)$. Then $P_3 \cong 3_+^{1+4}.3$, and P_3 is a Sylow 3-subgroup of G. Hence all subgroups of H which are isomorphic to P are conjugate (in H). By Lemma 2.6, we may choose $g \in \mathsf{N}_G(P)$. Then in particular, $P < \mathsf{N}_G(P) \leqslant \mathsf{N}_G(P_3)$.

Since P_3 is a Sylow 3-subgroup, $\mathsf{N}_G(P_3)$ is contained in a maximal subgroup of G of index coprime to 3. By the Atlas [4], either $\mathsf{N}_G(P_3) \leqslant H^x$ for some $x \in G$, or $\mathsf{N}_G(P_3) \leqslant I \cong (3^4{:}2^3.\mathsf{S}_4).3$, where I is a maximal subgroup of G of order $2^6 3^6$. Suppose that $\mathsf{N}_G(P_3) \leqslant H^x$. Then $|H^x : \mathsf{N}_G(P_3)|$ properly divides $|H^x : P| = 4$. Hence $|H^x : \mathsf{N}_G(P_3)| = 1$ or 2, and so $P_3 \,\mathsf{char}\, \mathsf{N}_G(P_3) \trianglelefteq H^x \cong H$, which is not possible. Therefore, $\mathsf{N}_G(P_3) \leqslant I$, and so $\mathsf{N}_G(P_3) = \mathsf{N}_I(P_3)$.

Let $S = \mathsf{soc}(G) = \mathsf{P}\Omega_8^+(2)$. By the Atlas [4], the intersection $I \cap S \cong 3^4{:}(2^3.\mathsf{S}_4)$, and $I \cap S$ is contained in a maximal subgroup J of $S.2$ such that $J \cong \mathsf{S}_3 \wr \mathsf{S}_4$. Then $M := \mathsf{O}_3(I \cap S) = \mathsf{O}_3(J) \cong \mathsf{Z}_3^4$, and thus $\mathsf{Z}_2^3.\mathsf{S}_4 \cong (I \cap S)/M < J/M \cong \mathsf{Z}_2 \wr \mathsf{S}_4$. By Lemma 3.2, the normaliser of the Sylow 3-subgroup $(P_3 \cap S)/M$ of $(I \cap S)/M$ is isomorphic to $\mathsf{Z}_3.[4]$. Thus the normaliser of $P_3 \cap S$ in $I \cap S$ is $(P_3 \cap S).[4]$. We observe that an element normalising P_3 also normalise $P_3 \cap S$. Hence $\mathsf{N}_I(P_3) \leqslant \mathsf{N}_I(P_3 \cap S)$. Let Q be a Sylow 2-subgroup of $\mathsf{N}_I(P_3)$. Then $Q \leqslant I \cap S$, and so $Q \leqslant \mathsf{N}_{I\cap S}(P_3 \cap S)$. Hence $|Q|$ is a divisor of 4. However, $P_3.[4] = P < \mathsf{N}_G(P) \leqslant \mathsf{N}_G(P_3) = \mathsf{N}_I(P_3) = P_3Q$, which is a contradiction. $\square$

After the above proof of Lemma 3.3 was obtained, the truth of the statement in the lemma was also confirmed by Magma. The authors are grateful to C. Schneider for implementing the computation.

3.2. The classification

Let G be a primitive permutation group on a set Ω which has a suborbit of length 4. By the results of [16,18,22], we have a list of candidates for the pair (G, G_α) where $\alpha \in \Omega$. Among them, $\mathsf{PSL}_2(7)$ of degree 7, $\mathsf{PSL}_3(7).\mathsf{Z}_3$ and $\mathsf{P}\Omega^+(8,2).\mathsf{Z}_3$ have no suborbit of length 4, see Lemmas 3.1 and 3.3. A precise list of such pairs (G, G_α) with G insoluble is now given as follows.

Theorem 3.4. *Let G be an insoluble primitive permutation group on Ω which has a suborbit of length* 4. *Then for a point $\alpha \in \Omega$, one of the following holds*:

- (i) $G = \mathsf{PGL}_2(p)$, *and* $G_\alpha \cong \mathsf{S}_4$, *where p is a prime and* $p \equiv \pm 3 \pmod 8$;
- (ii) $G = \mathsf{PSL}_2(p)$, *and* $G_\alpha \cong \mathsf{S}_4$, *where* $p > 7$ *is a prime and* $p \equiv \pm 1 \pmod 8$;
- (iii) $G = \mathsf{PSL}_2(p)$, *and* $G_\alpha \cong \mathsf{A}_4$, *where* $p \geqslant 5$ *is a prime,* $p \equiv \pm 3 \pmod 8$, *and* $p \not\equiv \pm 1 \pmod{10}$;
- (iv) $G = \mathsf{PSL}_2(3^t)$, *and* $G_\alpha \cong \mathsf{A}_4$, *where t is an odd prime,* $3^t \equiv \pm 3 \pmod 8$, *and* $3^t \not\equiv \pm 1 \pmod{10}$;
- (v) *G and G_α lie in Table* 3.

The soluble primitive permutation groups with a suborbit of length 4 were classified by Wang [22], see Theorem 5.4(ii). The primitive permutation groups with suborbits of length

Table 3

G	G_α	$\|\Omega\|$
$\mathrm{PGL}_2(7)$	D_{16}	21
$\mathrm{PGL}_2(9)$	D_{16}	45
M_{10}	$\mathrm{Z}_8{:}\mathrm{Z}_2$	45
$\mathrm{Aut}(\mathrm{A}_6)$	$[2^5]$	45
$\mathrm{PSL}_2(17)$	D_{16}	153
$\mathrm{P\Sigma L}_2(27)$	$\mathrm{A}_4 \times \mathrm{Z}_3$	819
$\mathrm{PSL}_3(3)$	S_4	234
$\mathrm{PSL}_3(7)$	$(\mathrm{A}_4 \times \mathrm{Z}_3){:}\mathrm{Z}_2$	26 068
A_7	$(\mathrm{A}_4 \times \mathrm{Z}_3){:}\mathrm{Z}_2$	35
S_7	$\mathrm{S}_4 \times \mathrm{S}_3$	35

3 were classified by Wong [23], which consists of soluble groups, and insoluble groups given in the following theorem.

Theorem 3.5 (Wong [23]). *Let G be an insoluble primitive permutation group on Ω which has a suborbit of length* 3. *Then G and G_α, where $\alpha \in \Omega$, lie in the following table*:

G	A_5	S_5	$\mathrm{PGL}_2(7)$	$\mathrm{PSL}_2(11)$	$\mathrm{PSL}_2(13)$	$\mathrm{PSL}_3(3)$	$\mathrm{Aut}(\mathrm{PSL}_3(3))$	$\mathrm{PSL}_2(p)$
G_α	S_3	D_{12}	D_{12}	D_{12}	D_{12}	S_4	$\mathrm{S}_4 \times \mathrm{Z}_2$	S_4

3.3. Automorphism groups of certain graphs

Theorems 3.4 and 3.5 enable us to determine the automorphism groups of certain vertex-primitive graphs.

Lemma 3.6. *Suppose that G is an insoluble primitive permutation group on Ω which has an orbital graph Σ of out-valency l, where $l = 3$ or* 4. *Let $\Gamma = \Sigma \cup \Sigma^*$. Then $G \leqslant \mathrm{Aut}\,\Sigma \leqslant \mathrm{Aut}\,\Gamma \leqslant \mathrm{Aut}(\mathrm{soc}(G))$.*

Proof. Let $A = \mathrm{Aut}\,\Gamma$. Then $G \leqslant A \leqslant \mathrm{Sym}(\Omega)$. Since G is primitive, A is primitive. By Theorems 3.5 and 3.4, we have that $\mathrm{soc}(G) = \mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(3)$, A_7, or $\mathrm{PSL}_3(7)$.

Suppose that $\mathrm{soc}(G) \neq \mathrm{soc}(A)$. Then there exists a pair of subgroups K and L of A such that $G \leqslant K < L \leqslant A$, $\mathrm{soc}(G) = \mathrm{soc}(K) \neq \mathrm{soc}(L)$, and K is maximal in L. Such pairs (K, L) are classified in [11]. Since Γ is of valency l or $2l$, for $\alpha \in \Omega$, every prime divisor of $|A_\alpha|$ is smaller than $2l \leqslant 8$. Inspecting the pairs (K, L) given in [11], calculation shows that $(\mathrm{soc}(G), \mathrm{soc}(L))$ lies in the following table:

$\mathrm{soc}(G)$	$\mathrm{soc}(L)$	Ω	$\|\Omega\|$	Comments
A_5	A_6	3, 3-partitions	10	$l = 3$
A_7	A_8	4, 4-partitions	35	$l = 4$
$\mathrm{PSL}_2(7)$	A_8	2-sets	28	$G > \mathrm{PSL}_2(7),\ l = 3$
$\mathrm{PSL}_2(9)$	A_{10}	2-sets	45	$G > \mathrm{PSL}_2(9),\ l = 4$
$\mathrm{PSL}_2(7)$	$\mathrm{U}_3(3)$	singular 1-spaces	28	
$\mathrm{PSL}_2(11)$	M_{11}	$[\mathrm{M}_{11} : \mathrm{M}_9.2]$	55	
$\mathrm{PSL}_3(3)$	$\mathrm{P\Omega}_6^+(3)$	orbit of non-singular points	117	$\mathrm{P\Omega}_6^+(3) \cong \mathrm{PSL}_4(3)$

Further, since Γ is a G-edge-transitive graph of valency l or $2l$, $\mathsf{soc}(L)$ has a suborbit of length at most 8. If $\mathsf{soc}(L) = \mathsf{A}_6$ or $\mathsf{U}_3(3)$, then L is 2-transitive on Ω, and so the valency of Γ equals $|\Omega| - 1 > 8$, which is a contradiction. Thus $\mathsf{soc}(L) \neq \mathsf{A}_6, \mathsf{U}_3(3)$.

Suppose now that $(\mathsf{soc}(G), \mathsf{soc}(L)) = (\mathsf{A}_7, \mathsf{A}_8)$. Calculation shows that $\mathsf{soc}(L)$, acting on "4, 4"-partitions, has exactly three suborbits, of length 1, 16, and 18, which is a contradiction.

Suppose that the pair $(\mathsf{soc}(G), \mathsf{soc}(L))$ is one of $(\mathsf{PSL}_2(7), \mathsf{A}_8)$, $(\mathsf{PSL}_2(9), \mathsf{A}_{10})$, and $(\mathsf{PSL}_2(11), \mathsf{M}_{11})$. Note that $\mathsf{soc}(L)$ is a 4-transitive permutation of degree d, where $d = 8, 10$, or 11, respectively. Then $\mathsf{soc}(L)$, acting on 2-sets, has exactly three suborbits with length 1, $2(d-2)$, and $((d-2)(d-3))/2$, respectively. This contradicts that $\mathsf{soc}(L)$ has a suborbit of length at most 8.

Thus $(\mathsf{soc}(G), \mathsf{soc}(L)) = (\mathsf{PSL}_3(3), \mathsf{PSL}_4(3))$. Then $G \cong \mathsf{PSL}_3(3).2$ and the stabiliser of α in $\mathsf{soc}(L)$ is isomorphic to $\mathsf{U}_4(2){:}\mathsf{Z}_2$. By the Atlas [4], we know that $\mathsf{U}_4(2){:}\mathsf{Z}_2$ has no permutation representation of degree less than 27, which contradicts the fact that Γ is of valency at most 8.

Therefore, we have that $\mathsf{soc}(A) = \mathsf{soc}(G)$. Thus $G \leqslant \mathsf{Aut}\,\Gamma \leqslant \mathsf{Aut}(\mathsf{soc}(G))$, and further, $G \leqslant \mathsf{Aut}\,\Sigma \leqslant \mathsf{Aut}\,\Gamma \leqslant \mathsf{Aut}(\mathsf{soc}(G))$, as claimed. □

4. Graphs with insoluble automorphism groups

This section treats insoluble primitive permutation groups with suborbits of length 4.

Let G be a primitive permutation group on Ω which has a suborbit $\Delta(\alpha)$ of length 4. Let Σ be the corresponding orbital graph of G, which is of out-valency 4, and let $\Gamma = \Sigma \cup \Sigma^*$. Then Γ is a G-edge-transitive undirected graph of valency 4 or 8. Take $g \in G$ such that $\beta := \alpha^g \in \Delta(\alpha)$. Let $H = G_\alpha$ and $P = G_{\alpha\beta} = H \cap H^g$. Then $|H : P| = 4$, and Γ may be represented as a coset graph

$$\Gamma = \mathsf{Cos}\bigl(G, H, H\{g, g^{-1}\}H\bigr).$$

Then Γ is G-arc-transitive if and only if $H\{g, g^{-1}\}H = HgH$, and this in turn is true if and only if $HgH = HfH$ for some $f \in G$ such that $f^2 \in H$. The notation defined here will be used throughout this section.

We use a series of lemmas to analyze the graph Γ for each of the groups G listed in Theorem 3.4.

4.1. Infinite families of groups

In this subsection, we treat the infinite families of groups given in Theorem 3.4.

Lemma 4.1. *Suppose that either $G = \mathsf{PGL}_2(p)$ for $p \equiv \pm 3 \pmod 8$, or $G = \mathsf{PSL}_2(p)$ for $p \equiv \pm 1 \pmod 8$ and $p > 7$, where p is a prime, such that $H = G_\alpha = \mathsf{S}_4$. Then G has only one suborbit of length 4, and Γ is the corresponding orbital graph, which is undirected and has valency 4. Further,*

(i) $\mathsf{Aut}\,\Gamma = G$, *and* Γ *is* 2-*transitive*;
(ii) Γ *is a Cayley graph of a group* R *if and only if* $\mathsf{Aut}\,\Gamma = \mathsf{PGL}_2(5)$, $\mathsf{PGL}_2(11)$, *or* $\mathsf{PSL}_2(23)$ *and* $R = \mathsf{Z}_5$, $\mathsf{Z}_{11}{:}\mathsf{Z}_5$, *or* $\mathsf{Z}_{23}{:}\mathsf{Z}_{11}$, *respectively.*

Proof. Now $P \cong \mathsf{S}_3$ and $\mathsf{N}_G(P) \cong \mathsf{S}_3 \times \mathsf{Z}_2$. Since all subgroups of S_4 isomorphic to S_3 are conjugate, by Lemma 2.6, we may assume $g \in \mathsf{N}_G(P) \setminus H$. Then $H\{g, g^{-1}\}H = HxH$, where x is an involution. It follows that G has only one suborbit of length 4, which corresponds to HxH, and that Γ is the corresponding orbital graph. Thus Γ is an (undirected) arc-transitive graph of valency 4. By Theorem 3.4 and Lemma 3.6, $\mathsf{Aut}\,\Gamma = G$, and so Γ is 2-transitive.

Suppose that $\mathsf{Aut}\,\Gamma$ has a regular subgroup R. Then $|R| = |\mathsf{Aut}\,\Gamma|/|H| = |G|/24$. All subgroups of G are known (see [20, p. 417]), and inspecting these subgroups, it is easily shown that G is one of $\mathsf{PGL}_2(5)$, $\mathsf{PGL}_2(11)$, and $\mathsf{PSL}_2(23)$. These groups G indeed have a regular subgroup R, which is isomorphic to Z_5, $\mathsf{Z}_{11}{:}\mathsf{Z}_5$, or $\mathsf{Z}_{23}{:}\mathsf{Z}_{11}$, respectively. Therefore, Γ is a Cayley graph of R if and only if $\mathsf{Aut}\,\Gamma = \mathsf{PGL}_2(5)$, $\mathsf{PGL}_2(11)$, or $\mathsf{PSL}_2(23)$. □

The family of groups given in Theorem 3.4(iii) is treated in the next lemma.

Lemma 4.2. *Let* $G = \mathsf{PSL}_2(p)$, *where* p *is a prime,* $p \equiv \pm 3 \pmod 8$, *and* $p \not\equiv \pm 1 \pmod{10}$, *such that* $H = G_\alpha = \mathsf{A}_4$. *Then either*

(i) Γ *is an arc-transitive graph of valency* 8, *and* $\mathsf{Aut}\,\Gamma = \mathsf{PGL}_2(p)$; *or*
(ii) Γ *is* 2-*transitive of valency* 4, *and the following two statements hold*:
 (a) *either* $\mathsf{Aut}\,\Gamma = \mathsf{PGL}_2(p)$ *and* Γ *is unique, or* $\mathsf{Aut}\,\Gamma = \mathsf{PSL}_2(p)$ *and* Γ *is isomorphic to one of* $[(p+\varepsilon)/12]$ *non-isomorphic graphs, where* $\varepsilon = \pm 1$ *such that* $3 \mid (p+\varepsilon)$;
 (b) Γ *is a Cayley graph if and only if* $\mathsf{Aut}\,\Gamma = \mathsf{PGL}_2(5)$, *and* $\Gamma = \mathsf{K}_5$.

Proof. Since $|H : P| = 4$, $P = H \cap H^g = \langle z\rangle \cong \mathsf{Z}_3$. By Lemma 2.6, we may choose $g \in \mathsf{N}_G(P)$. Inspecting the subgroups of $\mathsf{PSL}_2(p)$ (see [20, p. 417]), we conclude that $\mathsf{N}_G(P) \cong \mathsf{D}_{p+\varepsilon}$, where $\varepsilon = \pm 1$ with $3 \mid (p+\varepsilon)$, $\mathsf{N}_{\mathsf{Aut}(G)}(H) \cong \mathsf{S}_4$, and $\mathsf{N}_{\mathsf{Aut}(G)}(P) \cong \mathsf{D}_{2(p+\varepsilon)}$. Thus there is an involution $\sigma \in \mathsf{N}_{\mathsf{Aut}(G)}(H) \setminus G$ such that $z^\sigma = z^{-1}$. It follows that $\mathsf{N}_{\mathsf{Aut}(G)}(P) = \langle\delta\rangle{:}\langle\sigma\rangle$ such that $o(\delta) = p+\varepsilon$ and $\delta^\sigma = \delta^{-1}$. Since $\mathsf{N}_{\mathsf{Aut}(G)}(P) \geqslant \mathsf{N}_G(P) \cong \mathsf{D}_{p+\varepsilon}$, we may write $\mathsf{N}_G(P) = \langle a, b\rangle$, where $a = \delta^2$, $b^2 = 1$ and $bab = a^{-1}$. So $\langle z\rangle = \langle \delta^{(p+\varepsilon)/3}\rangle = \langle a^{(p+\varepsilon)/6}\rangle$, and $\sigma = \delta^t b$ for some odd integer t.

Assume that $HgH \neq Hg^{-1}H$. Then Γ is of valency 8, and g is not an involution. Thus $g \in \langle a\rangle$, and so $g^\sigma = g^{-1}$. By Lemma 2.5, $\mathsf{Aut}\,\Gamma \geqslant \langle G, \sigma\rangle = \mathsf{PGL}_2(p)$, and Γ is arc-transitive. Further, by Lemma 3.6, we have $\mathsf{Aut}\,\Gamma = \mathsf{PGL}_2(p)$.

Assume next that $HgH = Hg^{-1}H$. Then Γ has valency 4 and is G-arc-transitive. Thus by Lemma 2.7, we may assume that $g \in \mathsf{N}_G(P)$ such that $g^2 \in P$. Since $P \cong \mathsf{Z}_3$, we may further assume that g is an involution. Then either $(p+\varepsilon)/2$ is even and $g = a^{(p+\varepsilon)/4}$, or $g = a^i b$, where $1 \leqslant i \leqslant (p+\varepsilon)/2$.

For the former, that is, $\Gamma = \mathsf{Cos}(G, H, Ha^{(p+\varepsilon)/4}H)$, since $(Ha^{(p+\varepsilon)/4}H)^\sigma = Ha^{(p+\varepsilon)/4}H$, we have $\mathsf{Aut}\,\Gamma = \langle G, \sigma\rangle = \mathsf{PGL}_2(p)$. Suppose that $Ha^{(p+\varepsilon)/4}H = Ha^i bH$ for some $1 \leqslant i \leqslant (p+\varepsilon)/2$. Then $ha^{(p+\varepsilon)/4}h' = a^i b$ for some $h, h' \in H$. Let $T =$

$\mathsf{O}_2(H) \cong \mathsf{Z}_2^2$. Since $H = T{:}\langle z\rangle$ and $z \in \langle a\rangle$, it follows that $ta^l t' = a^i b$, where $t, t' \in T$ and l is an integer. It is easily shown that $H \cap \langle a, b\rangle = P = \langle z\rangle$, and it then follows that both $t, t' \neq 1$. Since $\langle z\rangle$ acts by conjugation transitively on the 3 involutions of T, $t' = z^{-k} t z^k$ for some integer k. Thus $ta^l z^{-k} t z^k = a^i b$, and so $ta^{l'} t = a^{i'} b$, where l' and i' are integers. Since $a^{i'} b$ is an involution, $a^{l'}$ is an involution, and so $a^{l'} = a^{(p+\varepsilon)/4}$. It then follows that $\langle t, a^{(p+\varepsilon)/4}, a^{i'} b\rangle$ is a subgroup of order 8. This is a contradiction since $8 \nmid |G|$. Thus $Ha^{(p+\varepsilon)/4} H \neq Ha^i bH$.

Suppose that $g = a^i b$ with $1 \leqslant i \leqslant (p+\varepsilon)/2$. We claim that, for any integers $k < j$,

$$Ha^j bH = Ha^k bH \quad \text{if and only if} \quad \left.\frac{p+\varepsilon}{6}\,\right|\,(j-k). \tag{1}$$

Assume that $(p+\varepsilon)/6 \mid (j-k)$. Then $a^{j-k} \in \langle z\rangle$, and so $a^{j-k} = 1$, z, or z^{-1}. Hence $a^k = a^j$, $a^j z$, or $a^j z^{-1}$, and $a^k b = a^j b$, $a^j zb$, or $a^j z^{-1} b$, respectively. So $Ha^k bH = Ha^j bH$.

On the other hand, assume that $Ha^j bH = Ha^k bH$. Then $h_1 a^j b h_2 = a^k b$, where $h_1, h_2 \in H$. If $h_1 \in \langle z\rangle$ or $h_2 \in \langle z\rangle$, it follows since $z^b = z^{-1}$ and $az = za$, that $a^{j-k} = a^j b a^k b = h_1^{-1} h_2$ or $h_1 h_2^{-1}$, so $a^{j-k} \in H \cap \mathsf{N}_G(P) = \langle z\rangle$, as claimed. Suppose that $h_1 \notin \langle z\rangle$ and $h_2 \notin \langle z\rangle$. Since $\langle z\rangle$ acts by conjugation transitively on the 3 non-identity elements of T, we may write $h_1 = z_1 h$, and $h_2 = z_1' h$, where $h \in T$ and $z_1, z_1' \in \langle z\rangle$. Then calculation shows that $hz^{l'} a^k bh = z^l a^j b$ for some integers l' and l. Now $b_1 := z^l a^j b$ and $b_2 := z^{l'} a^k b$ are involutions of $\mathsf{N}_G(P)$, and $b_1 b_2 = z^{l'-l} a^{k-j} \in \langle a\rangle$. Since h is an involution, h interchanges b_1 and b_2. Hence $hb_1 b_2 h = b_2 b_1 = (b_1 b_2)^{-1}$; in particular, $h \in \mathsf{N}_G(\langle b_1 b_2\rangle) \geqslant \mathsf{N}_G(\langle a\rangle) = \mathsf{N}_G(P) \cong \mathsf{D}_{p+\varepsilon}$. Since $\mathsf{N}_G(P)$ is a maximal subgroup in G, we conclude that either $h \in \mathsf{N}_G(P)$, or $b_1 b_2 = 1$. If $h \in \mathsf{N}_G(P)$, then $z^h = z^{-1}$, not possible. Thus $z^{l'-l} a^{k-j} = b_1 b_2 = 1$, and so $a^{j-k} \in \langle z\rangle$ and $(p+\varepsilon)/6 \mid (j-k)$. Therefore, the claim in (1) is true.

It follows from claim (1) that $Ha^j bH \neq Ha^k bH$ for $1 \leqslant k < j \leqslant (p+\varepsilon)/6$. Thus we may assume that $1 \leqslant i \leqslant (p+\varepsilon)/6$. By Lemma 3.6, $\mathsf{PSL}_2(p) = G \leqslant \mathsf{Aut}\,\Gamma \leqslant \mathsf{Aut}(G) = \mathsf{PGL}_2(p)$. Since $\sigma = \delta^t b$ where t is an odd integer, we have

$$\left(Ha^i bH\right)^\sigma = H\left(\delta^{2i} b\right)^{\delta^t b} H = H\delta^{2t-2i} bH = Ha^{t-i} bH. \tag{2}$$

By Lemma 2.2, $\sigma \in \mathsf{Aut}\,\Gamma$ if and only if $Ha^i bH = (Ha^i bH)^\sigma = Ha^{t-i} bH$. Therefore, by claim (1), $\sigma \in \mathsf{Aut}\,\Gamma$ if and only if $(p+\varepsilon)/6$ divides $(t-i)-i$, in other words, $2i \equiv t \pmod{(p+\varepsilon)/6}$.

Suppose that $2i \equiv t \pmod{(p+\varepsilon)/6}$. Then since t is odd, we have that $(p+\varepsilon)/6$ is odd. In this case, $2i \equiv t \pmod{(p+\varepsilon)/6}$ has exactly one solution for $i \in \{1, 2, \ldots, (p+\varepsilon)/6\}$, and $\mathsf{Aut}\,\Gamma \geqslant \langle G, \sigma\rangle = \mathsf{Aut}(G)$, so $\mathsf{Aut}\,\Gamma = \mathsf{Aut}(G) = \mathsf{PGL}_2(p)$.

Suppose now that $2i \not\equiv t \pmod{(p+\varepsilon)/6}$. There are exactly $(p+\varepsilon)/6 - 1$ or $(p+\varepsilon)/6$ values of i satisfy this condition, depending on $(p+\varepsilon)/6$ is odd or even, respectively. Then $(Ha^i bH)^\sigma = Ha^{t-i} bH \neq Ha^i bH$. Let $i_t \leqslant (p+\varepsilon)/6$ be such that $i_t \equiv t-i\ ((p+\varepsilon)/6)$. Then $i_t \neq i$ but $\mathsf{Cos}(G, H, Ha^i bH) \cong \mathsf{Cos}(G, H, Ha^{i_t} bH)$. Thus there exists $[p+\varepsilon/12]$ non-isomorphic coset graphs, denoted by $\Sigma_1, \Sigma_2, \ldots, \Sigma_{[(p+\varepsilon)/12]}$. Since $\sigma \notin \mathsf{Aut}\,\Sigma_j$ and $\mathsf{Aut}\,\Sigma_j \leqslant \mathsf{Aut}(G)$, we conclude that $\mathsf{Aut}\,\Sigma_j = G$. So in this case $\mathsf{Aut}\,\Gamma = G$.

Therefore, in all the cases, $\mathsf{Aut}\,\Gamma = G$ or $\mathsf{Aut}(G)$. By Proposition 2.8, Γ is not 3-arc-transitive. Inspecting subgroups of $\mathsf{PSL}_2(p)$ and $\mathsf{PGL}_2(p)$ (see [20, p. 417]), it is easily shown that $\mathsf{Aut}\,\Gamma$ has a regular subgroup R if and only if $\mathsf{Aut}\,\Gamma = \mathsf{PGL}_2(5)$, $R \cong \mathsf{Z}_5$, and $\Gamma \cong \mathsf{K}_5$. So Γ is a Cayley graph if and only if $\Gamma \cong \mathsf{K}_5$. □

The next lemma analyses the groups in Theorem 3.4(iv).

Lemma 4.3. *Assume that* $\mathsf{soc}(G) = \mathsf{PSL}_2(3^t)$ *with t odd prime satisfying Theorem* 3.4(iv). *Then Γ is an arc-transitive graph of valency* 8.

Proof. By Theorem 3.4, in this case, $G = \mathsf{PSL}_2(3^t)$ or $\mathsf{P\Sigma L}_2(27)$.

Suppose first that $G = \mathsf{PSL}_2(3^t)$. Then $H \cong \mathsf{A}_4$, and hence $P = H \cap H^g = \langle z\rangle \cong \mathsf{Z}_3$. By Lemma 2.6, we may assume that $g \in \mathsf{N}_G(P)$. It is known that $\mathsf{N}_G(P) \cong \mathsf{Z}_3^t$ is a Sylow 3-subgroup of G (see [20, p. 417]), so g is of order 3. Write $\mathsf{N}_G(P) = P \times \widetilde{P}$, so that we may assume $g \in \widetilde{P}$. Since no element $x \in \mathsf{N}_G(P) \setminus H$ such that $x^2 \in P$, by Lemma 2.7, we have $HgH \neq Hg^{-1}H$, and Γ is a G-edge-transitive graph of valency 8. Inspecting subgroups of $\mathsf{PSL}_2(p)$ and $\mathsf{PGL}_2(p)$ (see [20, p. 417]), it is easily shown that $\mathsf{N}_{\mathsf{PGL}_2(3^t)}(H) = H{:}\langle\sigma\rangle \cong \mathsf{S}_4$, where σ is of order 2 such that $z^\sigma = z^{-1}$, and $\mathsf{N}_{\mathsf{PGL}_2(3^t)}(P) = \mathsf{N}_G(P){:}\langle\sigma\rangle \cong \mathsf{Z}_3^t{:}\mathsf{Z}_2$, where σ inverts all elements of $\mathsf{N}_G(P)$. In particular, $g^\sigma = g^{-1}$, and hence by Lemma 2.5, Γ is arc-transitive.

If $G = \mathsf{P\Sigma L}_2(27)$, then Γ is $\mathsf{soc}(G)$-edge-transitive. Thus by the previous paragraph, Γ has valency 8 and is arc-transitive. □

We next consider the groups listed in Table 3 which is not $\mathsf{P\Sigma L}_2(27)$.

4.2. Graphs arising from A_7, S_7, $\mathsf{PSL}_3(3)$, *and* $\mathsf{PSL}_3(7)$

Lemma 4.4. *If* $\mathsf{soc}(G) = \mathsf{A}_7$, *then* $\mathsf{Aut}\,\Gamma = \mathsf{S}_7$ *and Γ is a* 3*-transitive non-Cayley graph of valency* 4, *which is isomorphic to the odd graph* O_3.

Proof. Assume that $G = \mathsf{A}_7$. Then $H \cong (\mathsf{A}_4 \times \mathsf{Z}_3){:}\mathsf{Z}_2$, $P \cong \mathsf{Z}_3^2{:}\mathsf{Z}_2$, and $\mathsf{N}_G(P) \cong \mathsf{Z}_3^2{:}\mathsf{Z}_4$. It is easily shown that P and $P^{g^{-1}}$ are conjugate in H. By Lemma 2.6, we may choose $g \in \mathsf{N}_G(P) \setminus H$, so $g^2 \in P$. Thus Γ is an undirected arc-transitive graph of valency 4. It is actually known that Γ is the odd graph O_3 (see, for example, Biggs [1, p. 58, p. 137] or [8, p. 310]); further, $\mathsf{Aut}\,\Gamma = \mathsf{S}_7$, Γ is 3-transitive and is not a Cayley graph. It is now easily shown that the graph Γ is the only graph of valency 4 such that S_7 is primitive on the vertex set. □

Lemma 4.5. *Assume that* $G = \mathsf{PSL}_3(3)$ *and* $H = G_\alpha = \mathsf{S}_4$. *Then Γ is an arc-transitive graph of valency* 8, *and* $\mathsf{Aut}\,\Gamma = \mathsf{Aut}(G) \cong \mathsf{PSL}_3(3).2$.

Proof. Now $P = H \cap H^g \cong \mathsf{S}_3$. Since all subgroups of H isomorphic to S_3 are conjugate in H, by Lemma 2.6, we may choose $g \in \mathsf{N}_G(P) \setminus H$. Let z be an element of P of order 3. Then $\mathsf{Z}_3 \cong \langle z\rangle$ char P, and hence $\mathsf{N}_G(P) \leqslant \mathsf{N}_G(\langle z\rangle)$. By the Atlas [4], $|\mathsf{C}_G(z)| = 54$ or 9.

Suppose $|\mathsf{C}_G(z)| = 54$. Let Δ_0 be the set of elements of H of order 3 which are not z, z^{-1}, and let $\Delta = \bigcup_{x\in C} \Delta_0^x \cup \{z, z^{-1}\}$. Then all elements of Δ are conjugate to z, and so $|\Delta| \leqslant |z^G| = |G : \mathsf{C}_G(z)| = 104$. We next compute the size of $\bigcup_{x\in C} \Delta_0^x$. Arbitrarily take $x_1, x_2 \in C := \mathsf{C}_G(z)$. Suppose that $\Delta_0^{x_1} \cap \Delta_0^{x_2} \neq \emptyset$. Then $\Delta_0^{x_1x_2^{-1}} \cap \Delta_0$ contains an element $y \notin \langle z\rangle$. Then $\mathsf{A}_4 \cong \langle y, z\rangle \leqslant H^{x_1x_2^{-1}} \cap H$. It follows that $\langle y, z\rangle$ is normal in both $H^{x_1x_2^{-1}}$ and H. Since G is simple and H is maximal in G, we conclude that $\langle y, z\rangle \lhd \langle H^{x_1x_2^{-1}}, H\rangle = H$, hence $H^{x_1x_2^{-1}} = H$. Thus $x_1x_2^{-1} \in \mathsf{N}_G(H) \cap C = H \cap C = \langle z\rangle$, and $H^{x_1} = H^{x_2}$, so in particular, $\Delta_0^{x_1} = \Delta_0^{x_2}$. Therefore, either $\Delta_0^{x_1} \cap \Delta_0^{x_2} = \emptyset$, or $\Delta_0^{x_1} = \Delta_0^{x_2}$, and so there are exactly $|C : \langle z\rangle|$ different Δ_0^x with $x \in C$. Now $|C : \langle z\rangle| = 18$, and so

$$104 \geqslant |\Delta| = \left|\bigcup_{x\in C} \Delta_0^x\right| + 2 = \left|C : \langle z\rangle\right| |\Delta_0| + 2 = 18 \times 6 + 2 = 110,$$

which is a contradiction.

Thus $|\mathsf{C}_G(z)| = 9$. By the Atlas [4], G has no elements of order 9, and so $\mathsf{C}_G(z) \cong \mathsf{Z}_3^2$ and $\mathsf{N}_G(\langle z\rangle) \cong \mathsf{Z}_3^2{:}\mathsf{Z}_2$. Since $P < \mathsf{N}_G(P) \leqslant \mathsf{N}_G(\langle z\rangle)$, we have $\mathsf{N}_G(P) = \mathsf{N}_G(\langle z\rangle) = P \times Z \cong \mathsf{S}_3 \times \mathsf{Z}_3$. In particular, $|\mathsf{N}_G(P) : P| = 3$, and Z is the center of $\mathsf{N}_G(P)$. By Lemma 2.7, $HgH \neq Hg^{-1}H$, and Γ is of valency 8.

By the Atlas [4], $\mathsf{N}_{\mathsf{Aut}(G)}(H) = H \times \langle\sigma\rangle$ for an element $\sigma \in \mathsf{Aut}(G)$ of order 2. In particular, $P^\sigma = P$, and $(\mathsf{N}_G(P))^\sigma = \mathsf{N}_{G^\sigma}(P^\sigma) = \mathsf{N}_G(P)$. Hence σ normalises the center Z of $\mathsf{N}_G(P)$. Now $g = hz_1$ for some $h \in P$ and some $z_1 \in Z \setminus \{1\}$. It follows that $H\{g, g^{-1}\}H = H\{z_1, z_1^{-1}\}H$. If $z_1^\sigma = z_1$, then σ centralises both H and z_1, so σ centralises $\langle H, z_1\rangle = G$, which is not possible. Thus $z_1^\sigma = z_1^{-1}$, and hence Γ is arc-transitive graph, and $\mathsf{Aut}\,\Gamma = \langle G, \sigma\rangle = \mathsf{Aut}(G)$. □

Lemma 4.6. *Assume that $G = \mathsf{PSL}_3(7)$ and $H = G_\alpha = (\mathsf{A}_4 \times \mathsf{Z}_3){:}\mathsf{Z}_2$. Then Γ is a 3-transitive non-Cayley graph of valency 4, and $\mathsf{Aut}\,\Gamma = G$. Moreover, G has exactly three self-paired suborbits of length 4 and the three corresponding orbital graphs are isomorphic.*

Proof. Now $P \cong \mathsf{Z}_3^2{:}\mathsf{Z}_2$, and $\mathsf{N}_G(P) \cong \mathsf{Z}_3^2{:}\mathsf{Q}_8$. Obviously, all subgroups of H which are isomorphic to P are conjugate. By Lemma 2.6, we may assume $g \in \mathsf{N}_G(P)$. Further, $\mathsf{N}_G(P)/P \cong \mathsf{Z}_2^2$, and so $g^2 \in P$. Thus $H\{g, g^{-1}\}H = HgH$, and Γ has valency 4. By Proposition 2.8, Γ is $(G, 3)$-arc-transitive.

Let P_2 be a Sylow 2-subgroup of P, and let X_2 be a Sylow 2-subgroup of $\mathsf{N}_G(P)$ which contains P_2. Since $X_2 \cong \mathsf{Q}_8$, we may write $X_2 = \langle i, j \mid i^4 = j^4 = 1, i^j = i^3\rangle$. It follows that $HgH = HiH$, HjH, or $HijH$. By the Atlas [4], we conclude that $\mathsf{N}_{G.3}(H) = \mathsf{Z}_6^2{:}\mathsf{S}_3$ and $\mathsf{N}_{G.3}(P) = \mathsf{Z}_3^2{:}(\mathsf{Q}_8{:}\mathsf{Z}_3)$. It follows that there exists $z \in \mathsf{N}_{G.3}(H)$ such that $o(z) = 3$, and $\langle z\rangle$ transitively permutes (by conjugation) $\langle i\rangle$, $\langle j\rangle$, and $\langle ij\rangle$. Thus $\langle z\rangle$ transitively permutes HiH, HjH, and $HijH$, so up-to isomorphism, Γ is unique.

Since $\mathsf{Aut}\,\Gamma$ is also a primitive permutation group with a suborbit of length 4, by Lemma 3.1 and Theorem 3.4, $\mathsf{Aut}\,\Gamma = G = \mathsf{PSL}_3(7)$. By the Atlas [4], G has no maximal subgroup of order divisible by $|V\Gamma| = |G : H| = 26\,068$. Thus $\mathsf{Aut}\,\Gamma$ contains no subgroups acting regularly on $V\Gamma$, and so Γ is not a Cayley graph. $\square$

4.3. Three 1*-transitive graphs*

Here we treat the five groups in Table 3 for which the stabilisers are 2-subgroups. We need a simple lemma.

Lemma 4.7. *Let G be a primitive permutation group on Ω with a suborbit $\Delta(\alpha)$ of length p^2 for a prime p. Assume that $H = G_\alpha$ is a p-group and $g \in G$ is such that $\alpha^g \in \Delta(\alpha)$. Then $\mathsf{N}_G(H \cap H^g)$ is not a p-group.*

Proof. By the assumption, H is a core free maximal subgroup of G. Let $P = H \cap H^g$. Then $|H : P| = |H^g : P| = p^2$. Since H is a p-group, we have $|H : \mathsf{N}_H(P)| \leqslant p$ and $|H^g : \mathsf{N}_{H^g}(P)| \leqslant p$. It follows that $\mathsf{N}_H(P) \lhd H$ and $\mathsf{N}_{H^g}(P) \lhd H^g$.

Suppose that $\mathsf{N}_G(P)$ is a p-subgroup of G. Since H is maximal in G, it follows that H is a Sylow p-subgroup of G. Thus $\mathsf{N}_H(P) \leqslant \mathsf{N}_G(P) \leqslant H^x$ for some $x \in G$, and hence $|\mathsf{N}_G(P) : \mathsf{N}_H(P)| \leqslant |H^x : \mathsf{N}_H(P)| = |H^x|/|\mathsf{N}_H(P)| = |H|/|\mathsf{N}_H(P)| \leqslant p$. Thus $\mathsf{N}_H(P)$ is normal in both of H and $\mathsf{N}_G(P)$, and so normal in $\langle H, \mathsf{N}_G(P)\rangle$. Since H is maximal in G, we have $\langle H, \mathsf{N}_G(P)\rangle = G$ or H. As H is core free in G and $\mathsf{N}_H(P)$ is normal in $\langle H, \mathsf{N}_G(P)\rangle$, we conclude that $\langle H, \mathsf{N}_G(P)\rangle = H$, and so $H \geqslant \mathsf{N}_G(P)$. Then $\mathsf{N}_{H^g}(P) \leqslant \mathsf{N}_G(P) \leqslant H$, and hence $|H : \mathsf{N}_{H^g}(P)| = |H|/|\mathsf{N}_{H^g}(P)| = |H^g|/|\mathsf{N}_{H^g}(P)| \leqslant p$. It follows $\mathsf{N}_{H^g}(P)$ is normal in H. Thus $\mathsf{N}_{H^g}(P) \lhd \langle H, H^g\rangle = G$, which is a contradiction. So $\mathsf{N}_G(P)$ is not a p-group. $\square$

Finally, we treat the groups $\mathsf{PGL}_2(7)$, $\mathsf{PGL}_2(9)$, M_{10}, $\mathsf{Aut}(\mathsf{A}_6)$, and $\mathsf{PSL}_2(17)$.

Lemma 4.8. *Let G be one of the groups $\mathsf{PGL}_2(7)$, $\mathsf{PGL}_2(9)$, M_{10}, $\mathsf{Aut}(\mathsf{A}_6)$, and $\mathsf{PSL}_2(17)$. Then the following statements hold*:

(i) *Γ is a* 1*-transitive graph of valency* 4;
(ii) *Γ is a Cayley graph of a group R if and only if $G = \mathsf{PGL}_2(7)$ and $R = \mathsf{Z}_7{:}\mathsf{Z}_3$*;
(iii) *either $\mathsf{Aut}\,\Gamma = G$, or $\mathsf{soc}(G) = \mathsf{A}_6$ and $\mathsf{Aut}\,\Gamma = \mathsf{Aut}(\mathsf{A}_6)$.*

Moreover, all suborbits of G of length 4 *are self-paired, $\mathsf{PSL}_2(17)$ has exactly two suborbits of length* 4*, and the others have exactly one suborbit of length* 4*.*

Proof. Let $A = \mathsf{Aut}\,\Gamma$. Then by Lemma 3.6, $G \leqslant A \leqslant \mathsf{Aut}(\mathsf{soc}(G))$. We note that $|V\Gamma| =$ 21, 45, or 153 for $\mathsf{soc}(G) = \mathsf{PSL}_2(7)$, A_6, or $\mathsf{PSL}_2(17)$, respectively. It follows that the vertex stabilisers G_α and A_α are Sylow 2-groups of G and A, respectively, and A has a regular subgroup R if and only if $G = \mathsf{PGL}_2(7)$ and $R = \mathsf{Z}_7{:}\mathsf{Z}_3$ (refer to the Atlas [4]). Thus Γ is not 2-arc-transitive, and Γ is a Cayley graph if and only if $G = \mathsf{PGL}_2(7)$.

As stated at the beginning of Section 4, $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$ where $H = G_\alpha$, and the subgroup $P := H \cap H^g$ has index 4 in H. Since H is a Sylow 2-subgroup of G, we have $|H : \mathsf{N}_H(P)| \leqslant 2$, and so $|\mathsf{N}_H(P) : P| \geqslant 2$. Thus by Lemma 4.7, $\mathsf{N}_G(P)$ is not a

2-group. It then follows from the Atlas [4] that either $\mathsf{N}_G(P) \cong \mathsf{S}_4$ and $\mathsf{N}_G(P) < \mathsf{soc}(G)$, or $G = \mathsf{Aut}(\mathsf{A}_6)$ and $\mathsf{N}_G(P) \cong \mathsf{S}_4 \times \mathsf{Z}_2$. Hence $P \cong \mathsf{Z}_2^2$ or Z_2^3, respectively. Let $N < \mathsf{N}_G(P)$ be such that $N \cong \mathsf{S}_3$. Then $\mathsf{N}_G(P) = P{:}N$, and $\mathsf{N}_H(P) = H \cap \mathsf{N}_G(P) = H \cap (P{:}N) = P{:}(H \cap N)$. Hence $|\mathsf{N}_H(P) : P| = |H \cap N|$. Noting that $|\mathsf{N}_H(P) : P| \geqslant 2$ and $N \cong \mathsf{S}_3$, we have $|H \cap N| = 2$. Let $H \cap N = \langle\sigma\rangle$, and let $z \in N$ be of order 3. Then $N = \langle z, \sigma\rangle$ and $z^\sigma = z^{-1}$. Thus $\mathsf{N}_G(P) = P{:}\langle z, \sigma\rangle = (P{:}\langle\sigma\rangle)\langle z\rangle$, and so

$$H\{y, y^{-1}\}H = H\{z, z^{-1}\}H = H\sigma z H, \quad \text{for each } y \in \mathsf{N}_G(P) \setminus H. \tag{3}$$

Assume that $G = \mathsf{PGL}_2(7)$, $\mathsf{PGL}_2(9)$, or $\mathsf{PSL}_2(17)$. Then we have $H \cong \mathsf{D}_{16}$ and $P = H \cap H^g \cong \mathsf{Z}_2^2$. Let $H = \langle a, b \mid a^8 = b^2 = 1,\ bab = a^{-1}\rangle$. Then H has exactly four subgroups isomorphic to Z_2^2: $\langle a^4, b\rangle$, $\langle a^4, b\rangle^a$, $\langle a^4, ab\rangle$, and $\langle a^4, ab\rangle^a$. Thus P is one of them.

Suppose that $G = \mathsf{PGL}_2(7)$ or $\mathsf{PGL}_2(9)$. Then $\mathsf{N}_G(P) \cong \mathsf{S}_4 \cong \mathsf{N}_G(P^{g^{-1}})$, and so by the Atlas [4], $\mathsf{N}_G(P), \mathsf{N}_G(P^{g^{-1}}) \leqslant \mathsf{soc}(G)$; in particular, $\mathsf{soc}(G)$ contains P and $P^{g^{-1}}$. Since $H \nleqslant \mathsf{soc}(G)$, we may choose $b \notin \mathsf{soc}(G)$. Then $\langle a^4, b\rangle, \langle a^4, b\rangle^a \nleqslant \mathsf{soc}(G)$, and thus $P, P^{g^{-1}} \in \{\langle a^4, ab\rangle, \langle a^4, ab\rangle^a\}$. So P and $P^{g^{-1}}$ are conjugate in H. By Lemma 2.6, we may assume $g \in \mathsf{N}_G(P) \setminus H$. Then $H\{g, g^{-1}\}H = H\sigma z H$ by (3). It follows that G has exactly one suborbit of length 4, which is self-paired, and Γ is the corresponding orbital graph, of valency 4.

Assume that $G = \mathsf{PSL}_2(17)$. If $\langle a^4, b\rangle$ and $\langle a^4, ab\rangle$ are conjugate in G, then it follows from the Sylow theorem that all subgroups of G isomorphic to Z_2^2 are conjugate, and so all subgroups of G isomorphic to S_4 are conjugate, which is a contradiction. Thus $\langle a^4, b\rangle$ and $\langle a^4, ab\rangle$ are not conjugate in G. By the Atlas [4], G has exactly two conjugate classes of subgroups isomorphic to S_4. Thus G has exactly two conjugacy classes of subgroups isomorphic to Z_2^2. Since $P, P^{g^{-1}}$ are conjugate in G and $P, P^{g^{-1}} \leqslant H$, it follows that P and $P^{g^{-1}}$ are conjugate in H. By Lemma 2.6, we may assume that $g \in \mathsf{N}_G(P) \setminus H$. Then $H\{g, g^{-1}\}H = H\sigma z H$ by (3), and so Γ is of valency 4, and the corresponding suborbit $\Gamma(\alpha)$ is self-paired. This particularly shows that G has a unique suborbit of length 4 corresponding to a given arc stabiliser P. It follows that G has at most two suborbits of length 4, corresponding to Hg_1H and Hg_2H, where $g_1 \in \mathsf{N}_G(\langle a^4, b\rangle)$ and $g_2 \in \mathsf{N}_G(\langle a^4, ab\rangle)$. It is known that $\mathsf{N}_{\mathsf{Aut}(G)}(H) \cong \mathsf{D}_{32}$, see [4]. Write $\mathsf{N}_{\mathsf{Aut}(G)}(H) = \langle\delta, b\rangle$ such that $\delta^2 = a$ and $\delta^b = \delta^{-1}$. Then $\langle a^4, b\rangle^{\delta^{-1}} = \langle a^4, ab\rangle$, and hence $(Hg_1H)^{\delta^{-1}} = Hg_2H$. By Lemma 2.2, δ^{-1} is an isomorphism from $\mathsf{Cos}(G, H, Hg_1H)$ to $\mathsf{Cos}(G, H, Hg_2H)$. Suppose that $Hg_1H = Hg_2H$. Then δ is an automorphism of $\mathsf{Cos}(G, H, Hg_1H)$, and so $\mathsf{Cos}(G, H, Hg_1H)$ is an orbital graph of $\langle G, \delta\rangle = \mathsf{Aut}(G)$ of valency 4. However, by Theorem 3.4, $\mathsf{PGL}_2(17)$ has no suborbits of length 4, which is a contradiction. Thus $Hg_1H \neq Hg_2H$, and G has exactly two suborbits of length 4 and $\mathsf{Aut}\,\Gamma = G$.

Assume now that $G = \mathsf{M}_{10}$. By the Atlas [4], $H \cong \mathsf{Z}_8{:}\mathsf{Z}_2$, and further, it is easily shown that H has a presentation $H = \langle a, b \mid a^8 = 1 = b^2,\ bab = a^3\rangle$. It follows that H has exactly two subgroups $\langle a^4, b\rangle$ and $\langle a^4, b\rangle^a$ isomorphic to Z_2^2. Thus P and $P^{g^{-1}}$ are conjugate in H. By Lemma 2.6, we may assume that $g \in \mathsf{N}_G(P) \setminus H$. Then $H\{g, g^{-1}\}H = H\sigma z H$ by (3). It follows that G has exactly one suborbit of length 4, which is self-paired.

Note that M_{10} and $\mathsf{PGL}_2(9)$ are two primitive subgroups of $\mathsf{Aut}(\mathsf{A}_6)$ of degree 45, and have no suborbits of length 2. It follows that $\mathsf{Aut}(\mathsf{A}_6)$ has exactly one suborbit of length 4. Therefore, these three groups have one common orbital graph Γ of valency 4. Further, by Lemma 3.6 and Theorem 3.4, $\mathsf{Aut}\,\Gamma = \mathsf{Aut}(\mathsf{A}_6)$. □

5. Proofs of theorems

Theorems 1.4 and 1.5 will be proved in this section.

It is known and easily shown that every edge-transitive Cayley graph of an abelian group is arc-transitive, see, for example, [12].

5.1. Graphs corresponding with suborbits of length 3

Let G be a primitive permutation group on Ω which has an orbital graph Σ of out-valency 3. Let $\Gamma = \Sigma \cup \Sigma^*$. Then Γ is G-edge-transitive and has valency 3 or 6.

Assume first that G is soluble. Then $\mathsf{soc}(G)$ is abelian and regular on Ω, and Γ is a Cayley graph of $\mathsf{soc}(G)$. Thus Γ is arc-transitive of valency 3 or 6. Further, it is easily shown that Γ is of valency 3 if and only if $G = \mathsf{A}_4$ or S_4 and $\Gamma \cong \mathsf{K}_4$.

Assume next that G is insoluble. Take two vertices α, β of Γ such that $\beta \in \Gamma(\alpha)$. Then there exists $g \in G$ such that $\alpha^g = \beta$. Let $H = G_\alpha$, and $P = H \cap H^g = G_{\alpha\beta}$. Then $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$, and $|H|/|H \cap H^g| = 3$. By Theorem 3.5, H is isomorphic to S_3, D_{12}, S_4, or $\mathsf{S}_4 \times \mathsf{Z}_2$, and so P is a Sylow 2-subgroup of H. By Lemma 2.6, we may assume $g \in \mathsf{N}_G(P)$. We next analyse the groups listed in Theorem 3.5 one by one.

If $G = \mathsf{A}_5$ or S_5, then it is easily shown that Γ is the Petersen graph and $\mathsf{Aut}\,\Gamma = \mathsf{S}_5$. If $G = \mathsf{PGL}_2(7)$, then it is easily shown that Γ is the Coxeter graph and $\mathsf{Aut}\,\Gamma = \mathsf{PGL}_2(7)$. It is well known that both Petersen graph and Coxeter graph are 3-transitive non-Cayley graphs.

Assume now that $G = \mathsf{PSL}_2(11)$ or $\mathsf{PSL}_2(13)$. Then $H \cong \mathsf{D}_{12}$, and $P \cong \mathsf{Z}_2^2$. By the Atlas [4], we have $\mathsf{N}_G(P) \cong \mathsf{A}_4$. As g normalises P and $\langle H, g\rangle = G$, we conclude that g is of order 3, and $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$ has valency 6. Further, by the Atlas [4], there exists $\sigma \in \mathsf{Aut}(G)$ such that $H^\sigma = H$ and $g^\sigma = g^{-1}$. Therefore, by Lemma 2.5, Γ is arc-transitive, and $\mathsf{Aut}\,\Gamma = \mathsf{Aut}(G)$.

Assume next that $G = \mathsf{PSL}_3(3)$ or $\mathsf{Aut}(\mathsf{PSL}_3(3))$. Then $H \cong \mathsf{S}_4$ or $\mathsf{S}_4 \times \mathsf{Z}_2$. Thus $P \cong \mathsf{D}_8$ or $\mathsf{D}_8 \times \mathsf{Z}_2$, respectively. By the Atlas [4], we have that $\mathsf{N}_G(P)$ is of order 16 or 32. Hence $g \in \mathsf{N}_G(P)$ is such that $g^2 \in P$, and so Γ has valency 3. It is well-known that this graph is 5-transitive, and $\mathsf{Aut}\,\Gamma = \mathsf{Aut}(G) \cong \mathsf{PSL}_3(3).\mathsf{Z}_2$ (refer to [1, 18a]). Finally, by the Atlas [4], $\mathsf{Aut}(\mathsf{PSL}_3(3))$ has no maximal subgroup of order divisible by 234, and hence Γ is not a Cayley graph.

Assume finally that $G = \mathsf{PSL}_2(p)$, where $p \equiv \pm 1 \pmod{16}$ is a prime. Then $H \cong \mathsf{S}_4$ and $P \cong \mathsf{D}_8$. Since $p \equiv \pm 1 \pmod{16}$, a Sylow 2-subgroup of G has order 16. Inspecting the subgroups of G, see [20, p. 417], we conclude that $\mathsf{N}_G(P)$ is a Sylow 2-subgroup of G. Thus $g^2 \in P$, and Γ is cubic and arc transitive. It is known that $\mathsf{Aut}\,\Gamma = \mathsf{PSL}_2(p)$ and Γ is 4-transitive (refer to [1, 18b]). Since $\mathsf{Aut}\,\Gamma$ has no subgroups of order $(p(p^2 - 1))/48$ (see [20, p. 417]), Γ is not a Cayley graph.

In summary, we have proved the following result.

Proposition 5.1.

(1) *There exist no vertex-primitive half-arc-transitive graphs of valency* 6.
(2) *Vertex-primitive arc-transitive cubic graphs satisfy Theorem* 1.5(1).

5.2. An infinite family of half-arc-transitive graphs

Here we construct an infinite family of vertex-primitive half-arc-transitive graphs.

Let $G = \mathsf{Sz}(q)$ be a Suzuki group, where $q = 2^{2m+1} \geqslant 8$. By [19], we have the following conclusions:

(a) there exist maximal subgroups of G which are isomorphic to a dihedral group $\mathsf{D}_{2(q-1)}$ of order $2(q-1)$;
(b) if Q is a Sylow 2-subgroup of G, then $Q \cong \mathsf{Z}_2^e.\mathsf{Z}_2^e$ where $e = 2m+1$, $Q \cap Q^x = 1$, or Q for any $x \in G$;
(c) all involutions of G are conjugate, and each involution z is contained in the center of the Sylow 2-subgroup of G containing z;
(d) if g is an element of G of order 4, then g is not conjugate in $\mathsf{Aut}(G)$ to g^{-1};
(e) $\mathsf{Out}(G) \cong \mathsf{Z}_{2m+1}$.

Construction 5.2. Let $G = \mathsf{Sz}(q)$, and let H be a maximal subgroup of G such that $H = \langle a\rangle{:}\langle z\rangle \cong \mathsf{D}_{2(q-1)}$, where z is an involution. Let S be a Sylow 2-subgroup of G which contains z. Let g be an element of S of order 4 such that $g^2 \neq z$. Set $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$.

Then we have the following conclusion.

Proposition 5.3. *For each positive integer m, the graph constructed in Construction* 5.2 *is a vertex-primitive half-arc-transitive graph of valency* $2(2^{2m+1}-1)$.

Proof. Suppose that $HgH = Hg^{-1}H$. By Lemma 2.7, there exists $x \in G$ such that

$$x \in \mathsf{N}_G(H \cap H^g), \quad x^2 \in H \cap H^g, \quad \text{and} \quad gx \in H.$$

Noting that $H \cap H^g = \langle z\rangle$, we have $x^2 = z$ or 1. Since x normalises $H \cap H^g$, we have $xz = zx$. So $x, z \in Q$ for some Sylow 2-subgroup Q of G. By property (b), $Q = S$, and hence $gx \in S \cap H = \langle z\rangle$. Then $gx = z$ or 1, and $g^2 = (zx^{-1})^2 = z$ or 1, which is a contradiction. Thus $HgH \neq Hg^{-1}H$, and so the coset graph $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$ has valency $2(q-1)$ and admits a half-arc-transitive action of G.

Now $G \leqslant \mathsf{Aut}\,\Gamma \leqslant \mathsf{Sym}(V\Gamma)$, and both G and $\mathsf{Aut}\,\Gamma$ are primitive on $V\Gamma$. Suppose that $\mathsf{soc}(\mathsf{Aut}\,\Gamma) \neq G$. Then by [11], $\mathsf{soc}(\mathsf{Aut}\,\Gamma) = \mathsf{A}_{q^2+1}$ or $\mathsf{Sp}_4(q)$, which has point stabiliser isomorphic to S_{q^2-1} or $\mathsf{PSL}_2(q) \times \mathsf{PSL}_2(q)$, respectively. However, neither S_{q^2-1} nor $\mathsf{PSL}_2(q) \times \mathsf{PSL}_2(q)$ has a permutation representation of degree $q-1$ or $2(q-1)$, contradicting the fact Γ admits a half-arc-transitive group action and Γ has valency $2(q-1)$. Therefore, $\mathsf{soc}(\mathsf{Aut}\,\Gamma) = G$ and $\mathsf{Aut}\,\Gamma \leqslant \mathsf{Aut}(G)$. Further, since $\mathsf{Out}(G) \cong \mathsf{Z}_{2m+1}$ is of odd order, $(HgH)^\sigma \neq Hg^{-1}H$ for all $\sigma \in \mathsf{Aut}(G)$, so HgH and $Hg^{-1}H$ are not conjugate in $\mathsf{Aut}\,\Gamma$. By Lemma 2.5, Γ is not arc-transitive. □

5.3. *Proof of Theorems* 1.4 *and* 1.5

Let G be a primitive permutation group on Ω which has a suborbit $\Delta(\alpha)$ of length l, where $l = 3$ or 4. Let Σ be the corresponding orbital graph of G of out-valency l, and let $\Gamma = \Sigma \cup \Sigma^*$. Then Γ is a G-edge-transitive undirected graph of valency l or $2l$. Take two vertices α, β such that $\beta \in \Delta(\alpha)$. Then $\beta = \alpha^g$ for some $g \in G$. Let $H = G_\alpha$, and let $P = G_{\alpha\beta} = H \cap H^g$. Then $|H : P| = l$, and $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$.

Lemma 5.4. *Let G be a soluble primitive permutation group on Ω that has a suborbit of length* 4*. Then Γ is arc-transitive, and the following statements hold*:

(i) $G = \mathsf{Z}_p{:}\mathsf{Z}_4$, $\mathsf{Z}_p^2{:}\mathsf{Z}_4$, $\mathsf{Z}_p^2{:}\mathsf{D}_8$, $\mathsf{Z}_p^3{:}\mathsf{A}_4$, *or* $\mathsf{Z}_p^3{:}\mathsf{S}_4$, *where p is an odd prime.*
(ii) *If Γ is of valency* 4*, then* $\mathsf{Aut}(\Gamma) = \mathsf{S}_5$ $(\cong \mathsf{PGL}_2(5))$, $\mathsf{Z}_p{:}\mathsf{Z}_4$, *or* $\mathsf{Z}_p^2{:}\mathsf{D}_8$.

Proof. Since G is primitive, we have that $\mathsf{soc}(G) = \mathsf{Z}_p^d$ for some prime p and some integer $d \geqslant 1$, which is regular on Ω. It follows that Γ is a Cayley graph of Z_p^d, and hence Γ is arc-transitive. Identifying $V\Gamma$ with $\mathsf{soc}(G)$ and letting α be the identity of $\mathsf{soc}(G)$, it follows that $G_\alpha \leqslant \mathsf{Aut}(\mathsf{Z}_p^d)$ is faithful on the suborbit $\Delta(\alpha)$. Thus $G_\alpha \leqslant \mathsf{S}_4$. It then follows from Wang [22] that parts (i) and (ii) are true. □

Now we are ready to prove the two main theorems of this paper.

Proof of Theorem 1.4. The proof of part (2) of Theorem 1.4 follows from Proposition 5.3. Next we prove part (1) of Theorem 1.4.

Suppose that Γ is a vertex-primitive half-arc-transitive graph of valency less than 10. Then by Tutte's theorem (see [21]), the valency of Γ is 2, 4, 6, or 8. It is known that there exist no vertex-primitive half-arc-transitive graphs of valency 2 or 4. By Proposition 5.1(1), Γ is not of valency 6. Thus Γ has valency 8.

Let $G = \mathsf{Aut}\,\Gamma$. Then G is a primitive permutation group on the vertex set $V\Gamma$. Since Γ is edge-transitive but not arc-transitive, we have that G_α acting on $\Gamma(\alpha)$ has 2 orbits of equal length, which is 4. Thus G has a suborbit of length 4. Since an edge-transitive Cayley graph of an abelian group is arc-transitive, G is insoluble, and so G is a group listed in Theorem 3.4. Then by Lemmas 4.1–4.8, Γ is arc-transitive, which is a contradiction. This proves Theorem 1.4(1). □

Proof of Theorem 1.5. Part (1) of Theorem 1.5 is proved in Proposition 5.1.

Let Γ be a vertex-primitive arc-transitive graph of valency 4. Then $\mathsf{Aut}\,\Gamma$ is a primitive permutation group on the vertex set and has a self-paired suborbit of length 4. If G is soluble, then by Lemma 5.4, Γ is as in part (2) of Theorem 1.5. Assume that $\mathsf{Aut}\,\Gamma$ is insoluble. Then $\mathsf{Aut}\,\Gamma$ is a group listed in Theorem 3.4, and hence by Lemmas 4.1–4.8, the statements in part (2) of Theorem 1.5 are true. □

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