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Advances in Mathematics 223 (2010) 1855–1884

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Discrete Morse theory for totally non-negative flag varieties ☆

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Received 1 July 2009; accepted 15 October 2009

Available online 30 October 2009

Communicated by Andrei Zelevinsky

Abstract

In a seminal 1994 paper Lusztig (1994) [26], Lusztig extended the theory of total positivity by introducing the totally non-negative part $(G/P)_{\geqslant 0}$ of an arbitrary (generalized, partial) flag variety G/P. He referred to this space as a "remarkable polyhedral subspace", and conjectured a decomposition into cells, which was subsequently proven by the first author Rietsch (1998) [33]. In Williams (2007) [40] the second author made the concrete conjecture that this cell decomposed space is the next best thing to a polyhedron, by conjecturing it to be a *regular* CW complex that is homeomorphic to a closed ball. In this article we use discrete Morse theory to prove this conjecture up to homotopy-equivalence. Explicitly, we prove that the boundaries of the cells are homotopic to spheres, and the closures of cells are contractible. The latter part generalizes a result of Lusztig's (1998) [28], that $(G/P)_{\geqslant 0}$ – the closure of the top-dimensional cell – is contractible. Concerning our result on the boundaries of cells, even the special case that the boundary of the top-dimensional cell $(G/P)_{>0}$ is homotopic to a sphere, is new for all G/P other than projective space.
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MSC: primary 57T99, 05E99; secondary 20G20, 14P10

Keywords: Total positivity; Partial flag varieties; Shellability; Reflection orders; Regular CW complexes; Discrete Morse theory

☆ The first author is funded by EPSRC grant EP/D071305/1. The second author is partially supported by the NSF.
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doi:10.1016/j.aim.2009.10.011

Contents

1. Introduction

The classical theory of total positivity studies matrices whose minors are all positive. Lusztig dramatically generalized this theory with a 1994 paper [26] in which he introduced the totally positive part of a reductive group G (totally positive matrices are recovered when G is a general linear group). Lusztig also defined the (totally) positive and (totally) non-negative parts $(G/P)_{>0}$ and $(G/P)_{\geqslant 0}$ of an arbitrary (generalized, partial) flag variety G/P. Lusztig referred to $(G/P)_{\geqslant 0}$ as a "remarkable polyhedral subspace" [26], and conjectured a decomposition into cells, which was subsequently validated by the first author [33]. This cell decomposition has a unique top-dimensional cell, the totally positive part $(G/P)_{>0}$; the totally non-negative part $(G/P)_{\geqslant 0}$ is the closure of this cell.

Lusztig [28] has proved that the totally non-negative part of the (full) flag variety is contractible, which implies the same result for any partial flag variety. More generally, in 1996 Lusztig asked whether the closure of each cell of $(G/P)_{\geqslant 0}$ is contractible [27], but this problem has remained open until now. By analogy with toric varieties, one might wonder whether even more is true – whether $(G/P)_{\geqslant 0}$ is homeomorphic to a ball, and in that case, whether there is a homeomorphism to a polyhedron mapping cells to faces: indeed, there is a notion of total positivity for toric varieties, and the non-negative part of a toric variety is homeomorphic – via the moment map – to its moment polytope [17]. It turns out that $(G/P)_{\geqslant 0}$ cannot be modeled by a polyhedron in the above sense: for example, the totally non-negative part of the Grassmannian $Gr_{2,4}(\mathbb{R})$ has one top-dimensional cell of dimension 4 and four 3-dimensional cells, but there is no 4-dimensional polytope with four facets. Nevertheless, in [40] the second author conjectured that $(G/P)_{\geqslant 0}$ together with its cell decomposition is the next best thing to a polyhedron, that is, it is a *regular* CW complex – the closure of each cell is homeomorphic to a closed ball and the boundary of each cell is homeomorphic to a sphere.

The goal of this paper is to apply combinatorial and topological methods in order to address this conjecture. Indeed, the past thirty years have seen a wealth of literature designed to facilitate the interplay between combinatorics and geometry (see [12,2,5,6,3]). In particular, in a 1984 paper [3], Bjorner recognized that regular CW complexes are combinatorial objects in the following sense: if Q is the poset of closed cells in a regular CW decomposition of a space X, then the order complex (or nerve) $\|Q\|$ is homeomorphic to X. Furthermore, he gave criteria [3] for recognizing when a poset is the face poset of a regular CW complex: for example, if a poset is *thin* and *shellable* then it is the face poset of some regular CW complex homeomorphic to a ball.

In [34], the first author described the poset Q of closed cells of $(G/P)_{\geqslant 0}$, and in [40], the second author applied techniques from poset topology to the poset of closed cells of $(G/P)_{\geqslant 0}$. In particular, she showed that the poset is thin and shellable. It follows that the order complex $\|Q\|$ is homeomorphic to a ball, and by Bjorner's results, Q is the poset of cells of a regular CW decomposition of a ball. These results were the motivation for her conjecture that the cell decomposition of $(G/P)_{\geqslant 0}$ is a regular CW decomposition of a ball.

While the statement that Q is the poset of cells of a regular CW decomposition of a ball is an extremely strong combinatorial result, one cannot use it to deduce any corresponding topological consequences for the original space $(G/P)_{\geqslant 0}$. Even to deduce results about the Euler characteristics of closures of cells requires further topological information about how cells are glued together, i.e. knowing that the cell complex is a *CW complex*. This was proved some ten years after the discovery of the cell decomposition, in [32,36].

To obtain new topological information about the CW complex $(G/P)_{\geqslant 0}$, we turn in this paper to another technique, namely Forman's discrete Morse theory [15]. The main theorem of discrete Morse theory is set up to provide a sequence of collapses for cells in a CW complex, which preserves the homotopy-type of the CW complex. To use it, one must input some combinatorial data – a *discrete Morse function*, which specifies the sequence of collapses – and check a number of topological hypotheses. Most notably, one must make sure that whenever one cell C_1 is collapsed into a cell C_2 whose closure contains C_1, C_1 is a *regular face* of C_2.

In this paper we use a blend of combinatorial and topological arguments to apply discrete Morse theory to $(G/P)_{\geqslant 0}$. Our main result is the following.

Theorem 1.1. *Let $(G/P)_{\geqslant 0}$ be an arbitrary (generalized, partial) flag variety. The closure of each cell of $(G/P)_{\geqslant 0}$ is collapsible, hence contractible. Furthermore, the boundary of each cell is homotopy-equivalent to a sphere. In particular, $(G/P)_{\geqslant 0}$ is contractible and its boundary is homotopy-equivalent to a sphere.*

While it was known already that $(G/P)_{\geqslant 0}$ is contractible by work of Lusztig, this theorem also identifies the homotopy type of its boundary, and of the closures of the smaller cells and their boundaries. Namely, we prove the conjecture that $(G/P)_{\geqslant 0}$ is a regular CW decomposition of a ball up to homotopy-equivalence.

We note that much of the technical difficulty of proving our main results stems from the fact that the attaching maps that we constructed for cells in [36] are defined in a non-explicit way in terms of Lusztig's canonical basis. Identifying enough pairs of cells (C_1, C_2) with C_1 a provably regular face of C_2, and then demonstrating regularity, requires an intricate analysis of parameterizations of cells and of what happens when parameters go to infinity. Our arguments rely in a fundamental way on positivity properties of the canonical basis.

The combinatorial component of our arguments is also nontrivial. For every cell C in $(G/P)_{\geqslant 0}$, we find a *Morse matching* on the poset of cells in the closure of C, with a unique critical cell of dimension 0, such that matched pairs of cells are regular. This requires us to identify appropriate Morse matchings of intervals in Bruhat order; the matchings we construct generalize certain *special matchings* found by Brenti [10] in the context of Kazhdan–Lusztig theory. An essential tool in our proofs is Dyer's notion of reflection orders and his EL-labeling of Bruhat order [14]. Along the way, we give a link between poset topology and discrete Morse theory, building on work of Chari to provide an algorithm for passing explicitly from an EL-labeling of a CW poset to a Morse matching.

For the time being, there is no simple strategy for proving that closures of cells are homeomorphic to balls. One might hope to use recent work of Hersh [19] on determining when an attaching map for a CW complex is a homeomorphism on its entire domain. However, there is only one known CW structure for $(G/P)_{\geqslant 0}$ (the one we gave in [36]), and its attaching maps are not homeomorphisms.

It is worth noting that to our knowledge this paper represents one of the first instances of the application of combinatorial tools (poset topology and discrete Morse theory) to a topological space which is not a simplicial or regular cell complex, and which arose outside the context of combinatorics. Indeed, most of the tools of poset topology are designed to analyze the order complex of a poset (a simplicial complex), e.g. the order complex of Bruhat order, the partition lattice, the lattice of subgroups of a finite group [39]. Similarly, discrete Morse theory is most readily applied to simplicial complexes and regular CW complexes (as opposed to general CW complexes), because in these situations one does not have to check extra topological hypotheses before collapsing cells. In light of this, it is not surprising that virtually all of the many applications of discrete Morse theory to date have been to simplicial complexes, e.g. complexes of t-colorable graphs, complexes of connected and biconnected graphs, complexes of not i-connected graphs; see [16] for an interesting survey.

Therefore we hope that this paper will be valuable not only in shedding light on the topology of $(G/P)_{\geqslant 0}$, but also in demonstrating the applicability of combinatorial tools to topological spaces outside the world of combinatorics.

2. Preliminaries on algebraic groups and flag varieties

We start with some preliminaries.

2.1. Pinnings

Let G be a semisimple, simply connected linear algebraic group over $\mathbb{C}$ split over $\mathbb{R}$, with split torus T. We identify G (and related spaces) with their real points and consider them with their real topology. Let $X(T) = \mathrm{Hom}(T, \mathbb{R}^*)$ and $\Phi \subset X(T)$ the set of roots. Choose a system of positive roots Φ^+. We denote by B^+ the Borel subgroup corresponding to Φ^+ and by U^+ its unipotent radical. We also have the opposite Borel subgroup B^- such that $B^+ \cap B^- = T$, and its unipotent radical U^-.

Denote the set of simple roots by $\Pi = \{\alpha_i \mid i \in I\} \subset \Phi^+$. For each $\alpha_i \in \Pi$ there is an associated homomorphism $\phi_i : \mathrm{SL}_2 \to G$. Consider the 1-parameter subgroups in G (landing in U^+, U^-, and T, respectively) defined by

$$x_i(m) = \phi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \qquad y_i(m) = \phi_i \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \qquad \alpha_i^\vee(t) = \phi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

where $m \in \mathbb{R}$, $t \in \mathbb{R}^*$, $i \in I$. The datum $(T, B^+, B^-, x_i, y_i; i \in I)$ for G is called a *pinning*. The standard pinning for SL_d consists of the diagonal, upper-triangular, and lower-triangular matrices, along with the simple root subgroups $x_i(m) = I_d + mE_{i,i+1}$ and $y_i(m) = I_d + mE_{i+1,i}$ where I_d is the identity matrix and $E_{i,j}$ has a 1 in position (i, j) and zeroes elsewhere.

2.2. Folding

If G is not simply laced, then one can construct a simply laced group $\dot{G}$ and an automorphism τ of $\dot{G}$ defined over $\mathbb{R}$, such that there is an isomorphism, also defined over $\mathbb{R}$, between G and the fixed point subset $\dot{G}^\tau$ of $\dot{G}$. Moreover the groups G and $\dot{G}$ have compatible pinnings. Explicitly we have the following.

Let $\dot{G}$ be simply connected and simply laced. We apply the same notations as in Section 2.1 for G, but with a dot, to our simply laced group $\dot{G}$. So we have a pinning $(\dot{T}, \dot{B}^+, \dot{B}^-, \dot{x}_i, \dot{y}_i, i \in \dot{I})$ of $\dot{G}$, and $\dot{I}$ may be identified with the vertex set of the Dynkin diagram of $\dot{G}$.

Let σ be a permutation of $\dot{I}$ preserving connected components of the Dynkin diagram, such that, if j and j' lie in the same orbit under σ then they are *not* connected by an edge. Then σ determines an automorphism τ of $\dot{G}$ such that

(1) $\tau(\dot{T}) = \dot{T}$,
(2) $\tau(x_i(m)) = x_{\sigma(i)}(m)$ and $\tau(y_i(m)) = y_{\sigma(i)}(m)$ for all $i \in \dot{I}$ and $m \in \mathbb{R}$.

In particular τ also preserves $\dot{B}^+, \dot{B}^-$. Let $\bar{I}$ denote the set of σ-orbits in $\dot{I}$, and for $\bar{i} \in \bar{I}$, let

$$x_{\bar{i}}(m) := \prod_{i\in\bar{i}} x_i(m),$$

$$y_{\bar{i}}(m) := \prod_{i\in\bar{i}} y_i(m).$$

The fixed point group $\dot{G}^\tau$ is a simply laced, simply connected algebraic group with pinning $(\dot{T}^\tau, \dot{B}^{+\tau}, \dot{B}^{-\tau}, x_{\bar{i}}, y_{\bar{i}}, \bar{i} \in \bar{I})$. There exists, and we choose, $\dot{G}$ and τ such that $\dot{G}^\tau$ is isomorphic to our group G via an isomorphism compatible with the pinnings.

2.3. Flag varieties

The Weyl group $W = N_G(T)/T$ acts on $X(T)$ permuting the roots Φ. We set $s_i := \dot{s}_i T$ where $\dot{s}_i := \phi_i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then any $w \in W$ can be expressed as a product $w = s_{i_1} s_{i_2} \dots s_{i_m}$ with $\ell(w)$ factors. This gives W the structure of a Coxeter group; we will assume some basic knowledge of Coxeter systems and Bruhat order as in [20]. We set $\dot{w} = \dot{s}_{i_1} \dot{s}_{i_2} \dots \dot{s}_{i_m}$. It is known that $\dot{w}$ is independent of the reduced expression chosen.

We can identify the flag variety G/B^+ with the variety $\mathcal{B}$ of Borel subgroups, via

$$gB^+ \quad \Leftrightarrow \quad g \cdot B^+ := gB^+g^{-1}.$$

We have the Bruhat decompositions

$$\mathcal{B} = \bigsqcup_{w\in W} B^+\dot{w} \cdot B^+ = \bigsqcup_{w\in W} B^-\dot{w} \cdot B^+$$

of $\mathcal{B}$ into B^+-orbits called *Bruhat cells*, and B^--orbits called *opposite Bruhat cells*.

Definition 2.1. For $v, w \in W$ define

$$\mathcal{R}_{v,w} := B^+\dot{w} \cdot B^+ \cap B^-\dot{v} \cdot B^+.$$

The intersection $\mathcal{R}_{v,w}$ is non-empty precisely if $v \leqslant w$ in the Bruhat order, and in that case is irreducible of dimension $\ell(w) - \ell(v)$, see [22].

Let $J \subset I$. The parabolic subgroup $W_J \subset W$ corresponds to a parabolic subgroup P_J in G containing B^+. Namely, $P_J = \bigsqcup_{w \in W_J} B^+\dot{w}B^+$. Consider the variety $\mathcal{P}^J$ of all parabolic subgroups of G conjugate to P_J. This variety can be identified with the partial flag variety G/P_J via

$$gP_J \quad \Leftrightarrow \quad gP_Jg^{-1}.$$

We have the usual projection from the full flag variety to a partial flag variety which takes the form $\pi = \pi^J : \mathcal{B} \to \mathcal{P}^J$, where $\pi(B)$ is the unique parabolic subgroup of type J containing B.

3. Total positivity for flag varieties

3.1. The totally non-negative part of G/P_J and its cell decomposition

Definition 3.1. (See [26].) The totally non-negative part $U^-_{\geqslant 0}$ of U^- is defined to be the semi-group in U^- generated by the $y_i(t)$ for $t \in \mathbb{R}_{\geqslant 0}$.

The totally non-negative part of $\mathcal{B}$ (denoted by $\mathcal{B}_{\geqslant 0}$ or by $(G/B^+)_{\geqslant 0}$) is defined by

$$\mathcal{B}_{\geqslant 0} := \overline{\{u \cdot B^+ \mid u \in U^-_{\geqslant 0}\}},$$

where the closure is taken inside $\mathcal{B}$ in its real topology.

The totally non-negative part of a partial flag variety $\mathcal{P}^J$ (denoted by $\mathcal{P}^J_{\geqslant 0}$ or by $(G/P_J)_{\geqslant 0}$) is defined to be $\pi^J(\mathcal{B}_{\geqslant 0})$.

Lusztig [26,29] introduced natural decompositions of $\mathcal{B}_{\geqslant 0}$ and $\mathcal{P}^J_{\geqslant 0}$.

Definition 3.2. (See [26].) For $v, w \in W$ with $v \leqslant w$, let

$$\mathcal{R}_{v,w;>0} := \mathcal{R}_{v,w} \cap \mathcal{B}_{\geqslant 0}.$$

We write W^J (respectively $W^J_{\max}$) for the set of minimal (respectively maximal) length coset representatives of W/W_J.

Definition 3.3. (See [29].) Let $\mathcal{I}^J \subset W^J_{\max} \times W_J \times W^J$ be the set of triples (x, u, w) with the property that $x \leqslant wu$. Given $(x, u, w) \in \mathcal{I}^J$, we define $\mathcal{P}^J_{x,u,w;>0} := \pi^J(\mathcal{R}_{x,wu;>0}) = \pi^J(\mathcal{R}_{xu^{-1},w;>0})$.

The first author [33] proved that $\mathcal{R}_{v,w;>0}$ and $\mathcal{P}^J_{x,u,w;>0}$ are semi-algebraic cells of dimension $\ell(w) - \ell(v)$ and $\ell(wu) - \ell(x)$, respectively.

3.2. Parameterizations of cells

In [30], Marsh and the first author gave parameterizations of the cells $\mathcal{R}_{v,w;>0}$, which we now explain.

Let $v \leqslant w$ and let $\mathbf{w} = (i_1, \ldots, i_m)$ encode a reduced expression $s_{i_1} \ldots s_{i_m}$ for w. Then there exists a unique subexpression $s_{i_{j_1}} \ldots s_{i_{j_k}}$ for v in $\mathbf{w}$ with the property that, for $l = 1, \ldots, k$,

$$s_{i_{j_1}} \ldots s_{i_{j_l}} s_{i_r} > s_{i_{j_1}} \ldots s_{i_{j_l}} \quad \text{whenever } j_l < r \leqslant j_{l+1},$$

where $j_{k+1} := m$. This is the "rightmost reduced subexpression" for v in $\mathbf{w}$, and is called the "positive subexpression" in [30]. It was originally introduced by Deodhar [13]. We use the notation

$$\mathbf{v}_+ := \{j_1, \ldots, j_k\},$$
$$\mathbf{v}_+^c := \{1, \ldots, m\} \setminus \{j_1, j_2, \ldots, j_k\},$$

for this special subexpression for v in $\mathbf{w}$. Note that this notation only makes sense in the context of a fixed $\mathbf{w}$.

Now we can define the map

$$\phi_{\mathbf{v}_+,\mathbf{w}} : \left(\mathbb{C}^*\right)^{\mathbf{v}_+^c} \to \mathcal{R}_{v,w},$$
$$(t_r)_{r \in \mathbf{v}_+^c} \mapsto g_1 \ldots g_m \cdot B^+,$$

where

$$g_r = \begin{cases} \dot{s}_{i_r} & \text{if } r \in \mathbf{v}_+, \\ y_{i_r}(t_r) & \text{if } r \in \mathbf{v}_+^c. \end{cases}$$

Theorem 3.4. *(See [30, Theorem 11.3].) The restriction of $\phi_{\mathbf{v}_+,\mathbf{w}}$ to $(\mathbb{R}_{>0})^{\mathbf{v}_+^c}$ defines an isomorphism of semi-algebraic sets,*

$$\phi_{\mathbf{v}_+,\mathbf{w}}^{>0} : (\mathbb{R}_{>0})^{\mathbf{v}_+^c} \to \mathcal{R}_{v,w;>0}.$$

We note that this parameterization generalizes Lusztig's parametrization of totally nonnegative cells in $U^-_{\geqslant 0}$ from [26]. Namely $U^-_{\geqslant 0} = \bigsqcup_{w \in W} U^-_{>0}(w)$, for

$$U^-_{>0}(w) := \left\{ y_{i_1}(t_1) y_{i_2}(t_2) \ldots y_{i_m}(t_m) \;\middle|\; t_i \in \mathbb{R}_{>0} \right\},$$

where $\mathbf{w} = (i_1, \ldots, i_m)$ is a/any reduced expression of w. Clearly, $\mathcal{R}^{>0}_{1,w} = U^-_{>0}(w) \cdot B^+$.

3.3. Change of coordinates under braid relations

In the simply laced case there is a simple change of coordinates [26,35] which describes how two parameterizations of the same cell are related when considering two reduced expressions which differ by a commuting relation or a braid relation.

If $s_i s_j = s_j s_i$ then $y_i(a)y_j(b) = y_j(b)y_i(a)$ and $y_i(a)\dot{s}_j = \dot{s}_j y_i(a)$.

If $s_i s_j s_i = s_j s_i s_j$ then

(1) $y_i(a)y_j(b)y_i(c) = y_j(\frac{bc}{a+c})y_i(a+c)y_j(\frac{ab}{a+c})$,
(2) $y_i(a)\dot{s}_j y_i(b) = y_j(\frac{b}{a})y_i(a)\dot{s}_j x_j(\frac{b}{a})$,
(3) $\dot{s}_j\dot{s}_i y_j(a) = y_i(a)\dot{s}_j\dot{s}_i$.

In case (2) Lemma 11.4 from [30] implies that the factor $x_j(\frac{b}{a})$ disappears into B^+ without affecting the remaining parameters when this braid relation is applied in the parametrization of a totally non-negative cell.

The changes of coordinates have also been computed for more general braid relations and have been observed to be invertible, subtraction-free, homogeneous rational transformations [1,35].

3.4. Total positivity and canonical bases for simply laced G

Assume that G is simply laced. Let $\mathbf{U}$ be the enveloping algebra of the Lie algebra of G; this can be defined by generators e_i, h_i, f_i $(i \in I)$ and the Serre relations. For any dominant weight $\lambda \in X(T)$ embedded into $\mathfrak{h}^*$, there is a finite-dimensional simple $\mathbf{U}$-module $V(\lambda)$ with a non-zero vector η such that $e_i \cdot \eta = 0$ and $h_i \cdot \eta = \lambda(h_i)\eta$ for all $i \in I$. The pair $(V(\lambda), \eta)$ is determined up to unique isomorphism.

There is a unique G-module structure on $V(\lambda)$ such that for any $i \in I$, $a \in \mathbb{R}$ we have

$$x_i(a) = \exp(ae_i) : V(\lambda) \to V(\lambda), \qquad y_i(a) = \exp(af_i) : V(\lambda) \to V(\lambda).$$

Then $x_i(a) \cdot \eta = \eta$ for all $i \in I$, $a \in \mathbb{R}$, and $t \cdot \eta = \lambda(t)\eta$ for all $t \in T$. Let $\mathcal{B}(\lambda)$ be the canonical basis of $V(\lambda)$ that contains η [24]. We now collect some useful facts about the canonical basis.

Lemma 3.5. *(See [29, 1.7(a)].) For any $w \in W$, the vector $\dot{w} \cdot \eta$ is the unique element of $\mathcal{B}(\lambda)$ which lies in the extremal weight space $V(\lambda)^{w(\lambda)}$. In particular, $\dot{w} \cdot \eta \in \mathcal{B}(\lambda)$.*

We define $f_i^{(p)}$ to be $\frac{f_i^p}{p!}$.

Lemma 3.6. *Let $s_{i_1} \dots s_{i_n}$ be a reduced expression for $w \in W$. Then there exists $a \in \mathbb{N}$ such that $f_{i_1}^{(a)}\dot{s}_{i_2}\dot{s}_{i_3}\dots\dot{s}_{i_n} \cdot \eta = \dot{s}_{i_1}\dot{s}_{i_2}\dots\dot{s}_{i_n} \cdot \eta$. Moreover, $f_{i_1}^{(a+1)}\dot{s}_{i_2}\dot{s}_{i_3}\dots\dot{s}_{i_n} \cdot \eta = 0$.*

Proof. This follows from Lemma 3.5 and properties of the canonical basis, see e.g. the proof of [25, Proposition 28.1.4]. □

4. $(G/P)_{\geqslant 0}$ as a CW complex: Attaching maps using toric varieties

Recall that a *CW complex* is a union X of cells with additional requirements on how cells are *glued*: in particular, for each cell σ, one must define a (continuous) *attaching map* $h : B \to X$

where B is a closed ball, such that the restriction of h to the interior of B is a homeomorphism with image σ.

Even given the parameterizations of cells, it is not obvious how to define attaching maps. One needs to extend the domain of each map $\phi^{>0}_{\mathbf{v}_+,\mathbf{w}}$ from $(\mathbb{R}_{>0})^{\mathbf{v}^c_+}$ (an open ball) to a closed ball. However, a priori it is not clear how to let the parameters approach 0 and infinity. In this section we explain how, following earlier work for Grassmannians [32], the authors [36] defined attaching maps for the cells and proved that the cell decomposition of $(G/P_J)_{\geqslant 0}$ is a CW complex.

Lemma 4.1 below is the key to defining attaching maps. It says that one can compactify $(\mathbb{R}_{>0})^{\mathbf{v}^c_+}$ inside a toric variety related to the parameterization, obtaining a closed ball (the non-negative part of the toric variety). We refer to [17,37] for the basics on toric varieties and their non-negative parts. Let $S \in \mathbb{Z}^r$ be a finite set whose elements are ordered, $\mathbf{m_0}, \dots, \mathbf{m_K}$, and thought of as corresponding to monomials, $\mathbf{t}^{\mathbf{m_j}} = t_1^{m_{j,1}} t_2^{m_{j,2}} \dots t_r^{m_{j,r}}$. We let X_S denote the toric subvariety of $\mathbb{P}^K$ associated to S as in [37], and $X_S^{>0}$ and $X_S^{\geqslant 0}$ its positive and non-negative parts, respectively. Explicitly, X_S is the closure of the image of the associated map

$$\chi = \chi_S : \mathbf{t} = (t_1, \dots, t_r) \mapsto \left[\mathbf{t}^{\mathbf{m}_0}, \mathbf{t}^{\mathbf{m}_1}, \dots, \mathbf{t}^{\mathbf{m}_K}\right] \tag{4.1}$$

from $(\mathbb{C}^*)^r$ to $\mathbb{P}^K$, while $X_S^{>0}$, and $X_S^{\geqslant 0}$ are obtained as the image of $\mathbb{R}^r_{>0}$ and its closure.

A fact which is crucial here is that $X_S^{\geqslant 0}$ is homeomorphic to a closed ball. More specifically, X_S has a moment map which gives a homeomorphism from $X_S^{\geqslant 0}$ to the convex hull B_S of S.

Lemma 4.1. *(See [32].) Suppose we have a map $\phi : (\mathbb{R}_{>0})^r \to \mathbb{P}^N$ given by*

$$(t_1, \dots, t_r) \mapsto \left[p_1(t_1, \dots, t_r), \dots, p_{N+1}(t_1, \dots, t_r)\right],$$

where the p_i's are Laurent polynomials with positive coefficients. Let S be the set of all exponent vectors in $\mathbb{Z}^r$ which occur among the (Laurent) monomials of the p_i's, and let B_S be the convex hull of the points of S. Then the map ϕ factors through the totally positive part $X_S^{>0}$ of the toric variety, giving a map $\Phi_{>0} : X_S^{>0} \to \mathbb{P}^N$. Moreover $\Phi_{>0}$ extends continuously to the closure to give a well-defined map $\Phi_{\geqslant 0} : X_S^{\geqslant 0} \to \overline{\Phi_{>0}(X_S^{>0})}$. Note that if we precompose with the isomorphism $B_S \cong X_S^{\geqslant 0}$ given by the moment map, we can consider the domain of $\Phi_{\geqslant 0}$ to be the polytope B_S, a closed ball.

The following result constructs attaching maps for cells of $(G/P_J)_{\geqslant 0}$ [36].

Theorem 4.2. *(See [36].) For any G/B^+ we can construct a positivity preserving embedding $i : G/B^+ \to \mathbb{P}^N$, for some N, with the following property. For any totally non-negative cell $\mathcal{R}_{x,w;>0}$ and parameterization $\phi^{>0}_{\mathbf{x}_+,\mathbf{w}}$ as in Section 3.2, the composition*

$$i \circ \phi^{>0}_{\mathbf{x}_+,\mathbf{w}} : (\mathbb{R}_{>0})^{\mathbf{x}^c_+} \xrightarrow{\sim} \mathcal{R}_{x,w;>0} \hookrightarrow \mathbb{P}^N$$

takes the form

$$i \circ \phi^{>0}_{\mathbf{x}_+,\mathbf{w}} : \mathbf{t} = (t_r)_{r \in \mathbf{x}^c_+} \mapsto \left[p_1(\mathbf{t}), \dots, p_{N+1}(\mathbf{t})\right],$$

where the p_j's are polynomials with positive coefficients. Applying Lemma 4.1 *to* $i \circ \phi^{>0}_{\mathbf{x}_+,\mathbf{w}}$, *we get an attaching map* $\Phi^{\geqslant 0}_{\mathbf{x}_+,\mathbf{w}} : X^{\geqslant 0}_{\mathbf{x}_+,\mathbf{w}} \to \overline{\mathcal{R}_{x,w;>0}}$ *where the non-negative toric variety* $X^{\geqslant 0}_{\mathbf{x}_+,\mathbf{w}}$ *is homeomorphic to its moment polytope* $B_{\mathbf{x}_+,\mathbf{w}}$.

In Theorem 4.2, the map i is defined as follows. When G is simply laced, we consider the representation $V = V(\rho)$ of G with a fixed highest weight vector η and corresponding canonical basis $\mathcal{B}(\rho)$. We let $i : \mathcal{B} \to \mathbb{P}(V)$ denote the embedding which takes $g \cdot B^+ \in \mathcal{B}$ to the line $\langle g \cdot \eta \rangle$. This is the unique $g \cdot B^+$-stable line in V. We specify points in the projective space $\mathbb{P}(V)$ using homogeneous coordinates corresponding to $\mathcal{B}(\rho)$. The theorem then follows using the positivity properties of the canonical basis in simply laced type.

If G is not simply laced, we use a folding argument to deduce the result from the simply laced case: the map i is given by i' from Lemma 4.3 as we will now explain. Let $\dot{G}$ be the simply laced group with automorphism corresponding to G. We identify G with $\dot{G}^\tau$ and use all of the notation from Section 2.2. For any $\bar{i} \in \bar{I}$ there is a simple reflection $s_{\bar{i}}$ in W, which is represented in $\dot{G}$ by

$$\dot{s}_{\bar{i}} := \prod_{i \in \bar{i}} \dot{s}_i.$$

In this way any reduced expression $\mathbf{w} = (\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_m)$ in W gives rise to a reduced expression $\dot{\mathbf{w}}$ in $\dot{W}$ of length $\sum_{k=1}^m |\bar{i}_k|$, which is determined uniquely up to commuting elements [38]. To a subexpression $\mathbf{v}$ of $\mathbf{w}$ we can then associate a unique subexpression $\dot{\mathbf{v}}$ of $\dot{\mathbf{w}}$ in the obvious way.

Lemma 4.3. *(See [36].) Let v, w be in W with $v \leqslant w$.*

(1) *We have*

$$\mathcal{R}_{v,w;>0} = \dot{\mathcal{R}}_{v,w;>0} \cap \mathcal{B}^\tau.$$

In particular the composition $i' : \mathcal{R}_{v,w} \hookrightarrow \dot{\mathcal{R}}_{v,w} \to \mathbb{P}(V(\dot{\rho}))$ is positivity preserving.

(2) *Suppose $\mathbf{w} = (\bar{i}_1, \ldots, \bar{i}_m)$ is a reduced expression for w in W, and $\mathbf{v}^c_+ = (h_1, \ldots, h_r)$ is the complement of the positive subexpression for v. Then we have a commutative diagram,*

$$\begin{array}{ccc} \mathcal{R}_{v,w;>0} & \xrightarrow{\iota} & \dot{\mathcal{R}}_{v,w;>0} \\ {\scriptstyle \phi^{>0}_{\mathbf{v}_+,\mathbf{w}}} \uparrow & & \uparrow {\scriptstyle \phi^{>0}_{\dot{\mathbf{v}}_+,\dot{\mathbf{w}}}} \\ \mathbb{R}^{\mathbf{v}^c_+}_{>0} & \xrightarrow{\bar{\iota}} & \mathbb{R}^{\dot{\mathbf{v}}^c_+}_{>0}, \end{array}$$

where the top arrow is the usual inclusion, the vertical arrows are both isomorphisms, and the map $\bar{\iota}$ has the form

$$(t_1, \ldots, t_r) \mapsto (t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_r),$$

where each t_l is repeated $|\bar{i}_{h_l}|$ times on the right-hand side.

Remark 4.4. For partial flag varieties we can also use Theorem 4.2 to construct an attaching map for each $\mathcal{P}^J_{x,u,w;>0}$. The projection $\pi^J : \mathcal{R}_{xu^{-1},w;>0} \to \mathcal{P}^J_{x,u,w;>0}$ is a homeomorphism so we take the composition $\Phi^{\geqslant 0}_{\mathbf{xu_+^{-1}},\mathbf{w}} \circ \pi^J$ as our attaching map.

Theorem 4.5. *(See [36].)* $(G/P_J)_{\geqslant 0}$ *is a CW complex.*

5. The poset $\mathcal{Q}^J$ of cells of $(G/P_J)_{\geqslant 0}$ and a regularity criterion

In this section we will review the description of the face poset of $(G/P_J)_{\geqslant 0}$ which was given by the first author [34]. We will then prove Theorem 5.6, giving a condition which ensures that a cell σ is a *regular face* of another cell τ with respect to the attaching map of τ.

Definition 5.1. Let K be a finite CW complex, and let Q denote its set of cells. The notation $\sigma^{(p)}$ indicates that σ is a cell of dimension p. We write $\tau > \sigma$ if $\sigma \neq \tau$ and $\sigma \subset \bar{\tau}$, where $\bar{\tau}$ is the closure of τ, and we say σ is a face of τ. This gives Q the structure of a partially ordered set, which we refer to as the *face poset* of K. Sometimes we will augment Q by adding a least element $\hat{0}$: in this case we will say that $\tau > \hat{0}$ for all τ, and we will call this the *augmented face poset* of K.

Remark 5.2. Our notion of face poset agrees with the notion used by Forman [15]. However, Bjorner [3] defines the face poset of a cell complex to be the poset of cells augmented by a least element $\hat{0}$ *and* a greatest element $\hat{1}$. In this paper we will never add a $\hat{1}$ to a poset because all posets we consider already have a unique greatest element.

A description of the face poset of $(G/P_J)_{\geqslant 0}$ was given in [34]. See also the paper [18] of Goodearl and Yakimov, who independently defined an isomorphic poset in their study of the T-orbits of symplectic leaves for a Poisson structure on G/P_J.

Theorem 5.3. *(See [34].) Fix W and W_J, the Weyl group and its parabolic subgroup corresponding to G/P_J. Let $\mathcal{Q}^J$ denote the augmented face poset of $(G/P_J)_{\geqslant 0}$ with its decomposition into totally non-negative cells. The elements of $\mathcal{Q}^J$ are indexed by $\mathcal{I}^J \cup \hat{0}$, where $\mathcal{I}^J$ is as in Definition* 3.3.

The order relations in $\mathcal{Q}^J$ are described in terms of Weyl group combinatorics by

$$\mathcal{P}^J_{x,u,w;>0} \leqslant \mathcal{P}^J_{x',u',w';>0}$$

if and only if there exist $u_1, u_2 \in W_J$ with $u_1u_2 = u$ and $\ell(u) = \ell(u_1) + \ell(u_2)$, such that $x'u'^{-1} \leqslant xu_2^{-1} \leqslant wu_1 \leqslant w'$. Moreover $\hat{0} < P^J_{x,u,w;>0}$ for all $(x, u, w) \in \mathcal{I}^J$.

Remark 5.4. When G/P_J is a (type A) Grassmannian, $\mathcal{Q}^J$ is the poset of cells of the totally non-negative Grassmannian, first studied by Postnikov [31].

When $P^J_{x,u,w;>0} < P^J_{x',u',w';>0}$ and $\dim P^J_{x',u',w';>0} = \dim P^J_{x,u,w;>0} + 1$, we will write $P^J_{x,u,w;>0} \lessdot P^J_{x',u',w';>0}$.

Suppose a cell $\sigma^{(p)}$ is a face of $\tau^{(p+1)}$. Let B be a closed ball of dimension $p+1$, and let $h : B \to K$ be the attaching map for τ, i.e. h is a continuous map that maps $\mathrm{Int}(B)$ homeomorphically onto τ. The following definition is essential to discrete Morse theory for general CW complexes, as collapses of cells must take place along *regular edges*.

Definition 5.5. (See [15, Definition 1.1].) We say that $\sigma^{(p)}$ is a regular face of $\tau^{(p+1)}$ (with respect to the attaching map h for τ) and that (σ, τ) is a regular edge, if

(1) $h : h^{-1}(\sigma) \to \sigma$ is a homeomorphism,
(2) $\overline{h^{-1}(\sigma)}$ is a closed p-ball.

To use discrete Morse theory in our situation we must find enough regular edges. However, the toric varieties and attaching maps in Theorem 4.2 are constructed using the canonical basis, and hence are not at all explicit. Thus at first glance it might seem hopeless to deduce whether a cell σ is a regular face of τ with respect to an attaching map h for τ. Fortunately, by the following result we do have a situation in which we can prove regularity of a pair of faces. We will first prove Theorem 5.6 in the case of complete flag varieties, and then generalize it to partial flag varieties.

Theorem 5.6. *Consider $\mathcal{P}^J_{x,u,w;>0} \gtrdot \mathcal{P}^J_{x',u',w;>0}$ in $(G/P_J)_{\geqslant 0}$ and let $\mathbf{w} = (i_1, \dots, i_m)$ be a reduced expression for w. We call this pair of cells* **good** *with respect to* $\mathbf{w}$ *if the positive subexpression $\mathbf{x}'\mathbf{u}'^{-1}_+$ is equal to $\mathbf{xu}^{-1}_+ \cup \{k\}$ and moreover $\mathbf{xu}^{-1}_+$ contains $\{k+1, \dots, m\}$. In this case $\mathcal{P}^J_{x',u',w;>0}$ is a regular face of $\mathcal{P}^J_{x,u,w;>0}$ with respect to the attaching map $\Phi^{\geqslant 0}_{\mathbf{xu}^{-1}_+,\mathbf{w}} \circ \pi^J$.*

When the choice of reduced expression $\mathbf{w}$ and the attaching map are clear from context, we will sometimes omit the phrase *with respect to* $\mathbf{w}$ or *with respect to the attaching map*.

Proposition 5.7. *Choose a reduced expression $\mathbf{w} = (i_1, \dots, i_m)$ for w, and suppose that the pair $\mathcal{R}_{v,w;>0} \gtrdot \mathcal{R}_{v',w;>0}$ is good with respect to $\mathbf{w}$. Suppose that $\mathbf{v}_+$ and $\mathbf{v}'_+$ are related by $\mathbf{v}'_+ = \mathbf{v}_+ \cup \{k\}$. Then $X_{\mathbf{v}'_+,\mathbf{w}}$ can be identified with a sub-toric variety of $X_{\mathbf{v}_+,\mathbf{w}}$, and its moment polytope $B_{\mathbf{v}'_+,\mathbf{w}}$ is a facet of the moment polytope $B_{\mathbf{v}_+,\mathbf{w}}$ of $X_{\mathbf{v}_+,\mathbf{w}}$. Moreover, the attaching map $\Phi^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}} : X^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}} \to \overline{\mathcal{R}_{v,w;>0}}$ restricts to $X^{\geqslant 0}_{\mathbf{v}'_+,\mathbf{w}}$ to give the attaching map $\Phi^{\geqslant 0}_{\mathbf{v}'_+,\mathbf{w}}$ for $\overline{\mathcal{R}_{v',w;>0}}$.*

Proof. Let us first consider the case that G is simply-laced. By our assumptions the parameterizations of the two cells take the form

$$\phi^{>0}_{\mathbf{v}_+,\mathbf{w}} = g_1 \dots g_{k-1} y_{i_k}(t_k) \dot{s}_{i_{k+1}} \dots \dot{s}_{i_m} \cdot B^+,$$
$$\phi^{>0}_{\mathbf{v}'_+,\mathbf{w}} = g_1 \dots g_{k-1} \dot{s}_{i_k} \dot{s}_{i_{k+1}} \dots \dot{s}_{i_m} \cdot B^+.$$

If we compose the parameterization $\phi^{>0}_{\mathbf{v}_+,\mathbf{w}}$ with the inclusion $i : \mathcal{R}_{v,w} \hookrightarrow \mathbb{P}^N$ from Theorem 4.2 we get a map

$$\mathbf{t} = (t_{h_1}, \dots, t_{h_r}, t_k) \mapsto \big[p_1(\mathbf{t}), \dots, p_{N+1}(\mathbf{t})\big],$$

where the p_j's are polynomials with positive coefficients. We note that by the definition of the map i, which we recalled just after Theorem 4.2,

$$\big[p_1(\mathbf{t}),\dots,p_{N+1}(\mathbf{t})\big]=\big\langle g_1\dots g_{k-1}y_{i_k}(t_k)\dot{s}_{i_{k+1}}\dots\dot{s}_{i_m}\cdot\eta\big\rangle.$$

Here we have identified $\mathbb{P}^N$ with $\mathbb{P}(V(\rho))$ using the canonical basis.

If we take the limit as $t_k\to\infty$ we obtain a new map

$$\begin{aligned}\mathbf{t}'=(t_{h_1},\dots,t_{h_r})&\mapsto\big[p'_1(\mathbf{t}'),\dots,p'_{N+1}(\mathbf{t}')\big]\\&=\lim_{t_k\to\infty}\big\langle g_1\dots g_{k-1}y_{i_k}(t_k)\dot{s}_{i_{k+1}}\dots\dot{s}_{i_m}\cdot\eta\big\rangle=\langle g_1\dots g_{k-1}\dot{s}_{i_k}\dot{s}_{i_{k+1}}\dots\dot{s}_{i_m}\cdot\eta\rangle.\end{aligned}\tag{5.1}$$

Here the last equality follows by applying $g_1\dots g_{k-1}$ to the identity

$$\lim_{t_k\to\infty}\big\langle y_{i_k}(t_k)\dot{s}_{i_{k+1}}\dots\dot{s}_{i_m}\cdot\eta\big\rangle=\langle\dot{s}_{i_k}\dot{s}_{i_{k+1}}\dots\dot{s}_{i_m}\cdot\eta\rangle,$$

which comes from expanding the action of $y_{i_k}(t_k)=\exp(t_kf_{i_k})$, using that

$$\begin{aligned}f_{i_k}^{(a)}\dot{s}_{i_{k+1}}\dots\dot{s}_{i_m}\cdot\eta&=\dot{s}_{i_k}\dot{s}_{i_{k+1}}\dots\dot{s}_{i_m}\cdot\eta,\\ f_{i_k}^{(a+1)}\dot{s}_{i_{k+1}}\dots\dot{s}_{i_m}\cdot\eta&=0,\end{aligned}$$

for a positive integer a, by Lemma 3.6.

By the same argument, a as above is the highest power of t_k appearing in any p_j. Therefore to obtain homogeneous coordinates $p'_j(\mathbf{t}')$ for the limit point we may divide each $p_j(\mathbf{t})$ by t_k^a and take

$$p'_j(\mathbf{t}')=\lim_{t_k\to\infty}\frac{1}{t_k^a}p_j(\mathbf{t}).$$

The monomials of $p_j(\mathbf{t})$ which don't vanish in this limit are precisely those which are multiples of this maximal power, t_k^a.

It follows that the toric variety $X_{\mathbf{v}'_+,\mathbf{w}}$ is the sub-toric variety of $X_{\mathbf{v}_+,\mathbf{w}}$ which is given precisely by those monomials which are multiples of t_k^a (and other coordinates set to zero). Its moment polytope can be identified with the face of $B_{\mathbf{v}_+,\mathbf{w}}$ cut out by the hyperplane $x_k=a$. Moreover from (5.1) it follows that the attaching map $\Phi^{\geqslant 0}_{\mathbf{v}'_+,\mathbf{w}}:X^{\geqslant 0}_{\mathbf{v}'_+,\mathbf{w}}\to\overline{\mathcal{R}_{v',w;>0}}$ is the restriction of $\Phi^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}}$. This also implies that $B_{\mathbf{v}'_+,\mathbf{w}}$, which is isomorphic to $X^{\geqslant 0}_{\mathbf{v}'_+,\mathbf{w}}$, has codimension 1 in $B_{\mathbf{v}_+,\mathbf{w}}$, making it a facet.

In the non-simply-laced case the proof is analogous. However now the attaching map for $\overline{\mathcal{R}_{v,w;>0}}$ is obtained from $\phi^{>0}_{\dot{\mathbf{v}}_+,\dot{\mathbf{w}}}\circ\bar{\iota}$ as in Lemma 4.3, where $\phi^{>0}_{\dot{\mathbf{v}}_+,\dot{\mathbf{w}}}$ is the corresponding parameterization in the related simply laced group $\dot{G}$. So we are looking at parameterizations of $\mathcal{R}_{v,w;>0}$ and $\mathcal{R}_{v',w;>0}$ embedded into $\dot{\mathcal{R}}_{v,w}$ and $\dot{\mathcal{R}}_{v',w}$, respectively, which take the form

$$\begin{aligned}\mathbf{t}&\mapsto\bar{g}_1\dots\bar{g}_{k-1}y_{i_{k,1}}(t_k)y_{i_{k,2}}(t_k)\dots y_{i_{k,l}}(t_k)\dot{s}_{\bar{i}_{k+1}}\dots\dot{s}_{\bar{i}_m}\cdot B^+,\\ \mathbf{t}'&\mapsto\bar{g}_1\dots\bar{g}_{k-1}\dot{s}_{\bar{i}_k}\dot{s}_{\bar{i}_{k+1}}\dots\dot{s}_{\bar{i}_m}\cdot B^+,\end{aligned}$$

where $\dot{s}_{\bar{i}_k}=\dot{s}_{i_{k,1}}\dot{s}_{i_{k,2}}\dots\dot{s}_{i_{k,l}}$.

As before, there are unique positive integers $a_1, \dots, a_l$ such that

$$f_{i_{k,1}}^{(a_1)} \dots f_{i_{k,l}}^{(a_l)} \dot{s}_{\bar{i}_{k+1}} \dots \dot{s}_{\bar{i}_m} \cdot \eta = \dot{s}_{i_{k,1}} \dot{s}_{i_{k,2}} \dots \dot{s}_{i_{k,l}} \dot{s}_{\bar{i}_{k+1}} \dots \dot{s}_{\bar{i}_m} \cdot \eta,$$

and for each $1 \leqslant h \leqslant l$, if we increase the corresponding exponent by 1, we have

$$f_{i_{k,h}}^{(a_h+1)} \dots f_{i_{k,l}}^{(a_l)} \dot{s}_{\bar{i}_{k+1}} \dots \dot{s}_{\bar{i}_m} \cdot \eta = 0.$$

Now the composition $i' \circ \phi_{\mathbf{v}_+,\mathbf{w}}^{>0}$ for i' as in Lemma 4.3 takes the form

$$\mathbf{t} \mapsto \left[p_1(\mathbf{t}), \dots, p_{N+1}(\mathbf{t})\right],$$

for polynomials p_j with positive coefficients. And by the observation about the $f_{i_{k,h}}$'s, the maximal power of t_k in any of the p_j's is $t_k^{a_1+\dots+a_l}$.

Finally we look at what happens if t_k tends to infinity and repeat the arguments from the simply-laced case. In this case $X_{\mathbf{v}'_+,\mathbf{w}}$ is the sub-toric variety of $X_{\mathbf{v}_+,\mathbf{w}}$, which is given by those monomials which are multiples of $t_k^{a_1+\dots+a_l}$ (and other coordinates set to zero), and its moment polytope can be identified with the face of $B_{\mathbf{v}_+,\mathbf{w}}$ cut out by the equations $x_1 = a_1, \dots, x_l = a_l$. Moreover from the analogue of (5.1) it follows that the attaching map $\Phi_{\mathbf{v}'_+,\mathbf{w}}^{\geqslant 0} : X_{\mathbf{v}'_+,\mathbf{w}}^{\geqslant 0} \to \overline{\mathcal{R}_{v',w}^{>0}}$ is the restriction of $\Phi_{\mathbf{v}_+,\mathbf{w}}^{\geqslant 0}$. This also implies that $B_{\mathbf{v}'_+,\mathbf{w}}$, which is isomorphic to $X_{\mathbf{v}'_+,\mathbf{w}}^{\geqslant 0}$, has codimension 1 in $B_{\mathbf{v}_+,\mathbf{w}}$, making it a facet. $\square$

Remark 5.8. We note that in the situation of Proposition 5.7 we have also shown that for any point $\chi(t_1, \dots, t_r)$ in $X_{\mathbf{v}_+,\mathbf{w}}^{>0}$, with χ as in Section 4 (4.1), and any positive integer c, the limit $\lim_{z\to\infty} \chi(t_1, \dots, t_{r-1}, z^c t_r)$, lies in $X_{\mathbf{v}'_+,\mathbf{w}}^{>0}$.

Remark 5.9. Proposition 5.7 is a big step towards proving Theorem 5.6 for $(G/B)_{\geqslant 0}$. To relate our notation to Definition 5.5, let $\tau = \mathcal{R}_{v,w;>0}$, $\sigma = \mathcal{R}_{v',w;>0}$, $h = \Phi_{\mathbf{v}_+,\mathbf{w}}^{\geqslant 0}$, and $h' = \Phi_{\mathbf{v}'_+,\mathbf{w}}^{\geqslant 0}$. By Proposition 5.7, $h^{-1}(\sigma)$ contains $X_{\mathbf{v}'_+,\mathbf{w}}^{>0}$. If we could show that this is an equality, then because $h|_{X_{\mathbf{v}'_+,\mathbf{w}}^{\geqslant 0}}$ is the attaching map h', the restriction $h|_{h^{-1}(\sigma)} = h|_{X_{\mathbf{v}'_+,\mathbf{w}}^{>0}}$ would be a homeomorphism, proving that Definition 5.5(1) is satisfied. Furthermore, $\overline{h^{-1}(\sigma)} = \overline{X_{v'_+,w}^{>0}}$ is a closed ball of appropriate dimension, verifying (2).

Proposition 5.10. *Suppose that $w > v' > v$ and we have a reduced expression $\mathbf{w} = (i_1, \dots, i_j, \dots, i_m)$ such that $v' = s_{i_{j+1}} s_{i_{j+2}} \dots s_{i_m}$, and v is obtained from v' by removing a unique factor s_{i_k} for $j+1 \leqslant k \leqslant m$. Suppose furthermore that we have a sequence $(c_1, c_2, \dots, c_j, c_k) \in \mathbb{Z}^{j+1}$ such that for $z > 0$ and some/any fixed $t_1, t_2, \dots, t_j$, $t_k \in \mathbb{R}_{>0}$ the 1-parameter family*

$$g_z \cdot B^+ := y_{i_1}\left(z^{c_1} t_1\right) \dots y_{i_j}\left(z^{c_j} t_j\right) \dot{s}_{i_{j+1}} \dots \dot{s}_{i_{k-1}} y_{i_k}\left(z^{c_k} t_k\right) \dot{s}_{i_{k+1}} \dots \dot{s}_{i_m} \cdot B^+$$

in $\mathcal{R}_{v,w;>0}$ tends as $z \to \infty$ to an element of $\mathcal{R}_{v',w;>0}$. Then we must have $c_1 = \dots = c_j = 0$ and $c_k > 0$.

Proof. Recall that since w ends in v' (that is, $\ell(wv'^{-1}) = \ell(w) - \ell(v')$) we have a continuous map

$$\pi = \pi^{w}_{wv'^{-1}} : B^+\dot{w} \cdot B^+ \to B^+\dot{w}\dot{v}'^{-1} \cdot B^+.$$

See for example Section 4.3 of [30]. In terms of our parameterizations, if $x < w$ and we consider an element

$$\phi_{\mathbf{x}_+,\mathbf{w}}(\mathbf{t}) = g_1 \dots g_m \cdot B^+ \in \mathcal{R}_{x,w}$$

then $\pi(g_1 \dots g_m \cdot B^+)$ is just given by deleting the last $m - j$ factors spelling out the v':

$$\pi\big(g_1 \dots g_m \cdot B^+\big) = g_1 \dots g_j \cdot B^+.$$

Note that in particular π preserves total non-negativity and takes both $\mathcal{R}_{v',w;>0}$ and $\mathcal{R}_{v,w;>0}$ to the same cell, namely $\mathcal{R}_{e,wv'^{-1};>0}$.

Now since the limit of $g_z \cdot B^+$ is assumed to lie in $\mathcal{R}_{v',w;>0}$ everything is taking place in $B^+\dot{w} \cdot B^+$, the domain of π, and we can apply π to $g_z \cdot B^+$ before and after taking the limit $z \to \infty$:

$$\lim\big(\pi\big(g_z \cdot B^+\big)\big) = \pi\big(\lim\big(g_z \cdot B^+\big)\big) \in \mathcal{R}_{e,wv'^{-1};>0}.$$

So we see that

$$\pi\big(g_z \cdot B^+\big) = y_{i_1}\big(z^{c_1}t_1\big) \dots y_{i_j}\big(z^{c_j}t_j\big) \cdot B^+$$

is a 1-parameter family in $\mathcal{R}_{e,wv'^{-1};>0}$ whose limit point as $z \to \infty$ again lies in $\mathcal{R}_{e,wv'^{-1};>0}$. However, suppose that one of the $c_1, \dots, c_j$ is non-zero. Then taking the limit would certainly give something that left the cell $\mathcal{R}_{e,wv'^{-1};>0}$ and went to a smaller one. So all of these c_i must be zero.

Given that the c_i are zero for $i \leqslant j$, it is clear that c_k must be positive for the limit of the original family to lie in $\mathcal{R}_{v',w;>0}$. □

Remark 5.11. Let $C = (c_1, \dots, c_j, c_k) \in \mathbb{Z}^{j+1}$ be the sequence from above. Then the 1-parameter family in Proposition 5.10 may also be written as

$$g_z \cdot B^+ = \phi^{>0}_{\mathbf{v}_+,\mathbf{w}}\big(z^C \cdot \mathbf{t}\big),$$

where $z > 0$ and $z^C \cdot \mathbf{t} = (z^{c_1}t_1, \dots, z^{c_j}t_j, z^{c_k}t_k)$, and where $\phi^{>0}_{\mathbf{v}_+,\mathbf{w}}$ is the parameterization from Section 3.2.

Proposition 5.12. *Suppose that $w_0 > v' \gtrdot v$, and choose any reduced expression $\mathbf{w}_0 = (i_1, \dots, i_n)$. Let $\phi^{>0}_{\mathbf{v}_+,\mathbf{w}_0}$ and $\mathbf{v}^c_+$ be as defined in Section 3.2, and write $\mathbf{v}^c_+ = \{h_1, \dots, h_r\}$ for $h_1 < \cdots < h_r$. There is a unique (up to positive scalar multiple) choice of sequence $C = (c_{h_1}, \dots, c_{h_r}) \in \mathbb{Z}^r$ such that, for $z > 0$ and for some/any fixed $t_{h_1}, \dots, t_{h_r} \in \mathbb{R}_{>0}$ the 1-parameter family in $\mathcal{R}_{v,w_0;>0}$,*

$$g_z \cdot B^+ = \phi^{>0}_{\mathbf{v}_+,\mathbf{w}_0}\big(z^C \cdot \mathbf{t}\big),$$

tends as $z \to \infty$ *to an element of* $\mathcal{R}_{v',w_0;>0}$. *Here* $z^C \cdot \mathbf{t} = (z^{c_{h_1}} t_{h_1}, \dots, z^{c_{h_r}} t_{h_r})$.

Proof. Let us first assume that $\mathbf{w}_0 = (i_1, \dots, i_n)$ ends with a reduced expression for v'. Then we are in the situation of Proposition 5.10, and so in terms of the coordinates of the parameterization $\phi^{>0}_{\mathbf{v}_+,\mathbf{w}'_0}$, there is a unique vector $C \in \mathbb{Z}^r$, up to positive scalar multiple, giving a 1-parameter family $g_z \cdot B^+ = \phi^{>0}_{\mathbf{v}_+,\mathbf{w}_0}(z^C \cdot \mathbf{t})$, whose limit point lies in $\mathcal{R}_{v',w_0;>0}$.

Recall that any two reduced expressions for w_0 can be related by braid and commuting relations, and suppose now that $\mathbf{w}_0$ is an arbitrary reduced expression for w_0. It suffices to prove that if the statement of the proposition holds for $\mathbf{w}_0$ then it also holds for any $\mathbf{w}'_0$ obtained from $\mathbf{w}_0$ by a braid relation or commuting relation.

This is obvious in the case of a commuting relation. Now suppose $\mathbf{w}_0$ and $\mathbf{w}'_0$ are related by a more general braid relation, and C is the vector associated (up to positive scalar multiple) to $\mathbf{w}_0$. The braid relation gives us a change of coordinates $\kappa(\mathbf{t}) = \mathbf{t}'$ which is rational, homogeneous and subtraction-free, see Section 3.3. We let $\mathbf{t}'_z := \kappa(z^C \cdot \mathbf{t})$. For example, if w_0 is the longest element of the symmetric group S_3, then applying formula (1) of Section 3.3 to $y_1(z^{c_1}t_1)y_2(z^{c_2}t_2)y_3(z^{c_3}t_3) \cdot B^+$, gives

$$y_2\left(\frac{z^{c_2+c_3}t_2t_3}{z^{c_1}t_1 + z^{c_3}t_3}\right) y_1\big(z^{c_1}t_1 + z^{c_3}t_3\big)\, y_2\left(\frac{z^{c_1+c_2}t_1t_2}{z^{c_1}t_1 + z^{c_3}t_3}\right) \cdot B^+,$$

and the entries are the components of $\mathbf{t}'_z$. Because the components of $\mathbf{t}'_z$ are subtraction-free, the maximal power of z in each one dominates the limit as $z \to \infty$. In this example, we have therefore

$$\begin{aligned}&\lim_{z\to\infty}\big(\phi^{>0}_{1,s_2s_1s_2}\big(\mathbf{t}'_z\big)\big)\\ &\quad= \lim_{z\to\infty}\big(\phi^{>0}_{1,s_2s_1s_2}\big(z^{c_2+c_3-\max(c_1,c_3)}q_1(\mathbf{t}), z^{\max(c_1,c_3)}q_2(\mathbf{t}), z^{c_1+c_2-\max(c_1,c_3)}q_3(\mathbf{t})\big)\big),\end{aligned}$$

where the $q_i(\mathbf{t})$ are new rational, subtraction-free functions in the t_j's. We define $C' := (c_2 + c_3 - \max(c_1, c_3), \max(c_1, c_3), c_1 + c_2 - \max(c_1, c_3))$.

The same procedure can be applied in the general case to define a C' out of the original C as well as new rational, subtraction-free functions q_i defining $q(\mathbf{t}) = (q_1(\mathbf{t}), \dots, q_r(\mathbf{t}))$ such that

$$\lim_{z\to\infty}\big(\phi^{>0}_{\mathbf{v}_+,\mathbf{w}'_0}\big(\mathbf{t}'_z\big)\big) = \lim_{z\to\infty}\big(\phi^{>0}_{\mathbf{v}_+,\mathbf{w}'_0}\big(z^{C'} \cdot q(\mathbf{t})\big)\big).$$

We note that C' is also in general related to the original C by a piecewise linear transformation, which one may compare to the *zones* and the map $R^{j'}_j$ from Sections 9.1 and 9.2 of [26].

We now see by testing out on a point $\phi^{>0}_{\mathbf{v}_+,\mathbf{w}'_0}(\mathbf{t}') \in \mathcal{R}_{v,w_0;>0}$, where $\mathbf{t}' = q(\mathbf{t})$, that

$$\lim_{z\to\infty}\big(\phi^{>0}_{\mathbf{v}_+,\mathbf{w}'_0}\big(z^{C'} \cdot \mathbf{t}'\big)\big) = \lim_{z\to\infty}\big(\phi^{>0}_{\mathbf{v}_+,\mathbf{w}'_0}\big(\mathbf{t}'_z\big)\big) = \lim_{z\to\infty}\big(\phi^{>0}_{\mathbf{v}_+,\mathbf{w}_0}\big(z^C \cdot \mathbf{t}\big)\big),$$

and lies in $\mathcal{R}_{v',w_0;>0}$. Therefore we have found a $C' \in \mathbb{Z}^r$ with the required property for our new reduced expression $\mathbf{w}'_0$.

To prove uniqueness, suppose $D' \in \mathbb{Z}^r$ is a different element such that

$$\lim_{z\to\infty}\left(\phi^{>0}_{\mathbf{v}_+,\mathbf{w}'_0}\left(z^{D'}\cdot \mathbf{t}'\right)\right) \in \mathcal{R}_{v',w_0;>0}$$

for some $\mathbf{t}' \in \mathbb{R}^r_{>0}$. Then we may apply the coordinate transformation back from the reduced expression $\mathbf{w}'_0$ to $\mathbf{w}_0$. Thus D' is transformed by a piecewise-linear transformation to a $D \in \mathbb{Z}^r$ such that

$$\lim_{z\to\infty}\left(\phi^{>0}_{\mathbf{v}_+,\mathbf{w}_0}\left(z^{D}\cdot \mathbf{t}\right)\right) \in \mathcal{R}_{v',w_0;>0}.$$

However this implies that D is a positive multiple of C, and by applying the original transformation again, that D' was a positive multiple of C'. $\square$

Proposition 5.13. *Choose $w > v' \gtrdot v$ and a reduced expression $\mathbf{w} = (i_1,\dots,i_m)$. Let $\phi^{>0}_{\mathbf{v}_+,\mathbf{w}}$ and $\mathbf{v}^c_+$ be as defined in Section 3.2, and write $\mathbf{v}^c_+ = \{h_1,\dots,h_r\}$ for $h_1 < \dots < h_r$. Suppose in addition that $\mathbf{v}'_+$ is equal to $\mathbf{v}_+ \cup \{h_r\}$. For $C = (c_{h_1},\dots,c_{h_r}) \in \mathbb{Z}^r$ and $z > 0$ we consider the 1-parameter family*

$$g_z \cdot B^+ = \phi^{>0}_{\mathbf{v}_+,\mathbf{w}}\left(z^C\cdot \mathbf{t}\right)$$

in $\mathcal{R}^{>0}_{v,w}$, where $z^C\cdot \mathbf{t} = (z^{c_{h_1}}t_{h_1},\dots,z^{c_{h_r}}t_{h_r})$. Then if $g_z\cdot B^+$ in $\mathcal{R}_{v,w;>0}$ tends as $z\to\infty$ to an element of $\mathcal{R}_{v',w;>0}$, we must have $c_{h_1} = \dots = c_{h_{r-1}} = 0$ and $c_{h_r} > 0$.

Remark 5.14. Note that the condition on $\mathbf{v}'_+$ in Proposition 5.13 is equivalent to the pair of cells $\mathcal{R}_{v,w;>0} \gtrdot \mathcal{R}_{v',w;>0}$ being good with respect to $\mathbf{w}$ in the sense of Theorem 5.6.

Proof. Suppose $C = (c_{h_1},\dots,c_{h_r}) \in \mathbb{Z}^r$ has the property,

$$g_z \cdot B^+ \text{ has limit in } \mathcal{R}_{v',w;>0} \text{ as } z\to\infty. \tag{5.2}$$

Choose a reduced expression $\mathbf{w}_0 = (j_1,\dots,j_{n-m},i_1,\dots,i_m)$ for w_0 ending with $\mathbf{w}$, and let us fix $u_1,\dots,u_{n-m} \in \mathbb{R}_{>0}$. Then we obtain a new one-parameter family,

$$y_{j_1}(u_1)\dots y_{j_{n-m}}(u_{n-m})g_z\cdot B^+,$$

which lies in $\mathcal{R}_{v,w_0;>0}$ for $z > 0$ and tends to an element in $\mathcal{R}_{v',w_0;>0}$ as $z\to\infty$. Now Proposition 5.12 is applicable and we have that $\tilde{C} = (0,\dots,0,c_{h_1},c_{h_2},\dots,c_{h_r})$ is the unique (up to positive scalar multiple) choice of $\tilde{C} \in \mathbb{Z}^{n-m+r}$ such that the corresponding 1-parameter family in $\mathcal{R}_{v,w_0;>0}$ tends to a point in $\mathcal{R}_{v',w_0;>0}$. It follows that the original r-tuple $(c_{h_1},\dots,c_{h_r})$ satisfying (5.2) is also uniquely determined up to positive scalar multiple.

Now it only remains to prove that (5.2) holds for $(c_{h_1},\dots,c_{h_{r-1}},c_{h_r}) = (0,\dots,0,1)$. But this is clear, by the same argument we used for (5.1) in the proof of Proposition 5.7. $\square$

We now turn to the proof of Theorem 5.6.

Proof of Theorem 5.6. We begin with the full flag variety case. Recall the natural inclusion $X^{>0}_{\mathbf{v}'_+,\mathbf{w}} \hookrightarrow X^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}}$, given by Proposition 5.7, for which $\Phi^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}}(X^{>0}_{\mathbf{v}'_+,\mathbf{w}}) = \mathcal{R}_{v',w;>0}$. By Remark 5.9, it suffices to prove the claim that

$$\left(\Phi^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}}\right)^{-1}(\mathcal{R}_{v',w;>0}) = X^{>0}_{\mathbf{v}'_+,\mathbf{w}}. \tag{5.3}$$

Suppose we have $x' \in X^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}}$ such that $\Phi^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}}(x') \in \mathcal{R}_{v',w;>0}$. We can approach x' from a point in the interior, $X^{>0}_{\mathbf{v}_+,\mathbf{w}}$, by a 1-parameter family. Namely,

$$x' = \lim_{z\to\infty} \chi\left(z^C \cdot \mathbf{t}\right) = \lim_{z\to\infty} \chi\left(z^{c_1}t_1, \dots, z^{c_r}t_r\right),$$

for some $\mathbf{t} \in \mathbb{R}^r_{>0}$, $C \in \mathbb{Z}^r$ and χ the map from (4.1) associated to $X_{\mathbf{v}_+,\mathbf{w}}$. Therefore

$$\Phi^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}}\left(x'\right) = \lim_{z\to\infty} \Phi^{>0}_{\mathbf{v}_+,\mathbf{w}}\left(\chi\left(z^C \cdot \mathbf{t}\right)\right) = \lim_{z\to\infty} \phi^{>0}_{\mathbf{v}_+,\mathbf{w}}\left(z^C \cdot \mathbf{t}\right),$$

where the 1-parameter subgroup on the right-hand side is as in Proposition 5.13. Since by our assumption $\Phi^{\geqslant 0}_{\mathbf{v}_+,\mathbf{w}}(x') \in \mathcal{R}_{v',w;>0}$, and we are in the 'good' situation, Proposition 5.13 tells us that $C = (0, \dots, 0, c)$, for positive c. But this implies that

$$x' = \lim_{z\to\infty} \chi\left(z^C \cdot \mathbf{t}\right) \in X^{>0}_{\mathbf{v}'_+,\mathbf{w}},$$

see Remark 5.8. Therefore the claim (5.3) holds and the theorem is true for the full flag variety.

Now consider the case of G/P_J. We have that π^J gives an isomorphism from $\mathcal{R}_{xu^{-1},w;>0}$ to $\mathcal{P}^J_{x,u,w;>0}$ and from $\mathcal{R}_{x'u'^{-1},w;>0}$ to $\mathcal{P}^J_{x',u',w;>0}$. We've already proved that $\mathcal{R}_{x'u'^{-1},w;>0}$ is a regular face of $\mathcal{R}_{xu^{-1},w;>0}$ with respect to the attaching map $\Phi^{\geqslant 0}_{\mathbf{xu}_+^{-1},\mathbf{w}}$. Recall that the attaching map for $\mathcal{P}_{x,u,w;>0}$ is simply $\pi^J \circ \Phi^{\geqslant 0}_{\mathbf{xu}_+^{-1},\mathbf{w}}$.

When we restrict π^J to $\overline{\mathcal{R}_{xu^{-1},w;>0}}$, it is straightforward to check that the preimage of $\mathcal{P}^J_{x',u',w;>0}$ is $\mathcal{R}_{x'u'^{-1},w;>0}$. By Theorem 5.6 in the full flag variety case, we know that $(\Phi^{\geqslant 0}_{\mathbf{xu}_+^{-1},\mathbf{w}})^{-1}(\mathcal{R}_{x'u'^{-1},w;>0}) = X^{>0}_{\mathbf{x'u'}_+^{-1},\mathbf{w}}$ and $\Phi^{\geqslant 0}_{\mathbf{xu}_+^{-1},\mathbf{w}}$ is a homeomorphism from $X^{>0}_{\mathbf{x'u'}_+^{-1},\mathbf{w}}$ to $\mathcal{R}_{x'u'^{-1},w;>0}$. Therefore $(\pi^J \circ \Phi^{\geqslant 0}_{\mathbf{xu}_+^{-1},\mathbf{w}})^{-1}(\mathcal{R}_{x'u'^{-1},w;>0}) = X^{>0}_{\mathbf{x'u'}_+^{-1},\mathbf{w}}$ and $\pi^J \circ \Phi^{\geqslant 0}_{\mathbf{xu}_+^{-1},\mathbf{w}}$ is a homeomorphism from $X^{>0}_{\mathbf{x'u'}_+^{-1},\mathbf{w}}$ to $\mathcal{R}_{x'u'^{-1},w;>0}$. It follows that $\mathcal{P}^J_{x',u',w;>0}$ is a regular face of $P^J_{x,u,w;>0}$ with respect to the attaching map $\pi^J \circ \Phi^{\geqslant 0}_{\mathbf{xu}_+^{-1},\mathbf{w}}$. $\square$

6. Preliminaries on poset topology

6.1. Preliminaries

Poset topology is the study of combinatorial properties of a partially ordered set, or *poset*, which reflect the topology of an associated simplicial or cell complex. In this section we will review some of the basic definitions and results of poset topology.

Let P be a poset with order relation $<$. We will use the symbol $\lessdot$ to denote a *covering relation* in the poset: $x \lessdot y$ means that $x < y$ and there is no z such that $x < z < y$. Additionally, if $x < y$ then $[x, y]$ denotes the *closed interval* from x to y; that is, the set $\{z \in P \mid x \leqslant z \leqslant y\}$.

We will often identify a poset P with its *Hasse diagram*, which is the graph whose vertices represent elements of P and whose edges depict covering relations.

The natural geometric object associated to a poset P is the realization of its *order complex* (or *nerve*). The order complex $\Delta(P)$ is the simplicial complex whose vertices are the elements of P and whose simplices are the chains $x_0 < x_1 < \cdots < x_k$ in P.

A poset is called *bounded* if it has a least element $\hat{0}$ and a greatest element $\hat{1}$. The *atoms* of a bounded poset are the elements which cover $\hat{0}$. Dually, the *coatoms* are the elements which are covered by $\hat{1}$. A finite poset is said to be *pure* if all maximal chains have the same length, and *graded*, if in addition, it is bounded. An element x of a graded poset P has a well-defined *rank* $\rho(x)$ equal to the length of an unrefinable chain from $\hat{0}$ to x in P. A poset P is called *thin* if every interval of length 2 is a *diamond*, i.e. if for any $p < q$ such that $\mathrm{rank}(q) - \mathrm{rank}(p) = 2$, there are exactly two elements in the open interval (p, q).

6.2. *Shellability and edge-labelings*

A pure finite simplicial complex Δ is said to be *shellable* if its maximal faces can be ordered $F_1, F_2, \ldots, F_n$ in such a way that $F_k \cap (\bigcup_{i=1}^{k-1} F_i)$ is a non-empty union of maximal proper faces of F_k for $k = 2, 3, \ldots, n$. Certain edge-labelings of posets can be used to prove that the corresponding order complexes are shellable. These techniques were pioneered by Bjorner [2], and Bjorner and Wachs [5].

One technique that can be used to prove that an order complex $\Delta(P)$ is shellable is the notion of *lexicographic shellability*, or EL-shellability, which was first introduced by Bjorner [2]. Let P be a graded poset, and let $\mathcal{E}(P)$ be the set of edges of the Hasse diagram of P, i.e. $\mathcal{E}(P) = \{(x, y) \in P \times P \mid x \gtrdot y\}$. An *edge labeling* of P is a map $\lambda : \mathcal{E}(P) \to \Lambda$ where Λ is some poset (usually the integers). Given an edge labeling λ, each maximal chain $c = (x_0 \gtrdot x_1 \gtrdot \cdots \gtrdot x_k)$ of length k can be associated with a k-tuple $\sigma(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k))$. We say that c is an *increasing chain* if the k-tuple $\sigma(c)$ is increasing; that is, if $\lambda(x_0, x_1) \leqslant \lambda(x_1, x_2) \leqslant \cdots \leqslant \lambda(x_{k-1}, x_k)$. The edge labeling allows us to order the maximal chains of any interval of P by ordering the corresponding k-tuples lexicographically. If $\sigma(c_1)$ lexicographically precedes $\sigma(c_2)$ then we say that c_1 lexicographically precedes c_2 and we denote this by $c_1 <_L c_2$.

Definition 6.1. An edge labeling is called an *EL-labeling* (*edge lexicographical labeling*) if for every interval $[x, y]$ in P,

(1) there is a unique increasing maximal chain c in $[x, y]$, and
(2) $c <_L c'$ for all other maximal chains c' in $[x, y]$.

If one has an EL-labeling of P, it is not hard to see that the corresponding order on maximal chains gives a shelling of the order complex [2]. Therefore a graded poset that admits an EL-labeling is said to be *EL-shellable*.

Given an EL-labeling λ of P and $x \in P$, we define $\mathrm{Last}_\lambda(x)$ to be the set of elements $z \lessdot x$ such that $\lambda(z \lessdot x)$ is maximal among the set $\{\lambda(y \lessdot x) \mid y \lessdot x\}$.

6.3. Face posets of cell complexes

When analyzing a CW complex $\mathcal{K}$, it is sometimes useful to study its *face poset* $\mathcal{F}(\mathcal{K})$, as in Definition 5.1. The face poset is a natural poset to study particularly if the CW complex has the *subcomplex property*, i.e. if the closure of a cell is a union of cells.

The class of regular CW complexes is particularly nice. Recall that a CW complex is *regular* if the closure of each cell is homeomorphic to a closed ball and if additionally the closure minus the interior of a cell is homeomorphic to a sphere. In general, the order complex $\|\mathcal{F}(\mathcal{K})\|$ does not reveal the topology of $\mathcal{K}$. However, the following result shows that regular CW complexes are combinatorial objects in the sense that the incidence relation of cells determines their topology.

Proposition 6.2. *(See [4, Proposition 4.7.8].) Let $\mathcal{K}$ be a regular CW complex. Then $\mathcal{K}$ is homeomorphic to $\|\mathcal{F}(\mathcal{K})\|$.*

We will call a poset P a *CW poset* if it is the face poset of a regular CW complex.

There is a notion of shelling for regular cell complexes (which is distinct from the notion of shelling of the order complex), due to Bjorner and Wachs. Such a shelling is a certain ordering on the coatoms of the face poset. We don't need the precise definition, only the following result that an EL-labeling of the augmented face poset of a regular cell complex gives rise to a shelling.

Theorem 6.3. *(See [7, Theorem 5.11], [8, Theorem 13.2].) If P is the augmented face poset of a finite-dimensional regular CW complex K, then any EL-labeling of P gives rise to a shelling of K. To go from the EL-labeling to the shelling one chooses the ordering on coatoms which is specified by the order on edges between the unique greatest element and the coatoms.*

7. Discrete Morse theory for general CW complexes

In this section we review Forman's powerful *discrete Morse theory* [15]. The theory comes in three "flavors": for simplicial complexes, regular CW complexes, and general CW complexes. In each setting, one needs to find a certain *discrete Morse function*, and then the main theorem says that the space in question is homotopy equivalent to another simpler space obtained by collapsing non-critical cells.

The first two flavors of the theory are the simplest and most widely used, because in these two settings a result of Chari [11] implies that a discrete Morse function is equivalent to a matching on the face poset of the CW complex. To work with the third flavor of the theory, one must check some additional technical conditions: the *discrete Morse hypothesis*, as well as an extra topological condition included in the definition of discrete Morse function. However, as we will see in Theorem 7.6, it is enough to find a matching on the face poset of a CW complex with the *subcomplex property* such that matched edges are *regular*. Although this result will not be surprising to the experts, we could not find it in the literature and so we give an exposition here. The proof follows from an argument of Kozlov [23, Proof of Theorem 3.2].[1]

[1] Although Theorem 3.2 of [23] was in the more restricted setting of regular CW complexes, the proof still holds in our situation.

7.1. Forman's discrete Morse theorem for general CW complexes

Let K be a finite CW complex and let Q be its poset of cells. Recall the definition of regular face from Definition 5.5.

Definition 7.1. (See [15, p. 102].) A function $f : Q \to \mathbb{R}$ is a discrete Morse function if for every $\sigma^{(p)}$ of dimension p, the following conditions hold:

(1) $\#\{\tau^{(p+1)} \mid \tau^{(p+1)} > \sigma \text{ and } f(\tau) \leqslant f(\sigma)\} \leqslant 1$.
(2) $\#\{\upsilon^{(p-1)} \mid \upsilon^{(p-1)} < \sigma \text{ and } f(\upsilon) \geqslant f(\sigma)\} \leqslant 1$.
(3) If σ is an irregular face of $\tau^{(p+1)}$ then $f(\tau) > f(\sigma)$.
(4) If $\upsilon^{(p-1)}$ is an irregular face of σ then $f(\upsilon) < f(\sigma)$.

Note that a Morse function is a function which is "almost increasing". Indeed, one should think of a Morse function as a function which specifies the order in which to attach the cells of a homotopy-equivalent CW complex [23].

Definition 7.2. We say that a cell $\sigma^{(p)}$ is critical if

(1) $\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leqslant f(\sigma)\} = 0$, and
(2) $\#\{\upsilon^{(p-1)} < \sigma \mid f(\upsilon) \geqslant f(\sigma)\} = 0$.

Let $m_p(f)$ denote the number of critical cells of dimension p.

For each cell σ of a CW complex K, let $\mathrm{Carrier}(\sigma)$ denote the smallest subcomplex of K containing σ. If K has the subcomplex property (see Section 6.3), then for any σ, $\mathrm{Carrier}(\sigma)$ is its closure, and hence condition (1) of Definition 7.3 below is satisfied.

Definition 7.3. (See [15, p. 136].) Given a CW complex K and a discrete Morse function f, we say that (K, f) satisfies the *Discrete Morse Hypothesis* if:

(1) For every pair of cells σ and τ, if $\tau \subset \mathrm{Carrier}(\sigma)$ and τ is not a face of σ, then $f(\tau) \leqslant f(\sigma)$.
(2) Whenever there is a $\tau > \sigma^{(p)}$ with $f(\tau) < f(\sigma)$ then there is a $\tilde{\tau}^{(p+1)}$ with $\tilde{\tau} > \sigma$ and $f(\tilde{\tau}) \leqslant f(\tau)$.

The following is Forman's main theorem for general CW complexes.

Theorem 7.4. *(See [15, Theorem 10.2].) Let K be a CW complex satisfying the Discrete Morse Hypothesis, and f a discrete Morse function. Then K is homotopy equivalent to a CW complex with $m_p(f)$ cells of dimension p.*

7.2. Discrete Morse functions as matchings

Chari [11] pointed out that when the CW complex is regular, one can depict a Morse function f as a certain kind of matching on the Hasse diagram of the poset of cells. Given such an f, we define a matching $M(f)$ on the Hasse diagram of Q whose edges correspond to the pairs of cells in which we get equality in (1) or (2) of Definition 7.1.

Recall that a *matching* of a graph $G = (V, E)$ is a subset M of edges of G such that each vertex in V is incident to at most one edge of M. We define a *Morse matching* M on a poset Q to be a matching on the Hasse diagram such that if edges in M are directed from lower to higher-dimension elements and all other edges are directed from higher to lower-dimension elements, then the resulting directed graph $G(M)$ is acyclic. We refer to any elements of Q which are not matched by M as *critical elements* (or *critical cells*).

In the situation of *arbitrary* CW complexes, a Morse matching such that matched edges are regular gives rise to a discrete Morse function satisfying property (2) of the Discrete Morse Hypothesis, as the following lemma shows. The proof of this lemma follows an argument of Kozlov [23, Proof of Theorem 3.2].

Lemma 7.5. *Let M be a Morse matching on the face poset Q of a CW complex, such that each edge in M corresponds to a regular pair of faces in the CW complex. Then there exists a discrete Morse function f_M, satisfying property* (2) *of the Discrete Morse Hypothesis, whose critical cells are exactly the critical cells of M.*

Proof. We will inductively assign positive integer labels to each of the elements of Q, producing a function f_M. Moreover, f_M will have the property that if $x < y$, $f_M(x) \leqslant f_M(y)$, with $f_M(x) = f_M(y)$ if and only if $(x, y) \in M$; in the case that $(x, y) \in M$, we will label x and y at the same time.

At each step, let x be one of the elements of Q of minimal rank (dimension) among those not yet labeled, and let i be the smallest positive integer not yet appearing as a label in Q. If x is not in M and hence critical, label x with i. If x is not critical, then we must have $(x, y) \in M$, where $x \lessdot y$. If each $z < y$ in Q is labeled, then label both x and y with i. Otherwise, there exists $x_1 < y$ in Q where x_1 is not labeled; repeat the argument with x_1 taking the place of x. Either we will label x_1 or a pair $(x_1, y_1) \in M$, or, since $G(M)$ is acyclic, we will find $x_2 \neq x$, $x_2 \neq x_1$, $y_1 > x_2$, etc.

Since there are finitely many elements of Q, the process will terminate. Since we never label an element $y \in Q$ until we have labeled each $x < y$, f_M has the property that for $x < y$, $f_M(x) \leqslant f_M(y)$. Therefore condition (2) of the Discrete Morse Hypothesis is satisfied. The only case in which $f_M(x) = f_M(y)$ is when $(x, y) \in M$, i.e. $(x \lessdot y)$ is a regular pair of faces – and so conditions (3) and (4) of Definition 7.1 are satisfied. Conditions (1) and (2) of Definition 7.1 are satisfied because M is a matching. Finally, it is clear that the cells which are critical with respect to M are exactly those which are critical with respect to Definition 7.2. □

We now restate Forman's Morse Theorem for general CW complexes in terms of Morse matchings.

Theorem 7.6. *Let K be a CW complex with the subcomplex property. Suppose its face poset Q has a Morse matching M, such that whenever $(\sigma^{(p)}, \tau^{(p+1)}) \in M$, σ is a regular face of τ. Let $m_p(M)$ denote the number of critical cells of dimension p. Then K is homotopy equivalent to a CW complex with $m_p(M)$ cells of dimension p.*

Proof. By Lemma 7.5, we have a discrete Morse function f for K satisfying condition (2) of Definition 7.3. Since K has the subcomplex property, condition (1) of Definition 7.3 is satisfied. The result now follows from Theorem 7.4. □

7.3. From edge-labelings to Morse matchings

Both lexicographic shellability and discrete Morse theory are combinatorial tools which can be used to investigate the topology of a CW complex. In this section we will recall a result of Chari [11], which proves the existence of a certain Morse matching given a shelling of a regular CW complex. We will translate this into a statement about constructing a Morse matching from an EL-labeling, and note that one can gain some fairly explicit information about the Morse matching from the EL-labeling.

Recall the notion of pseudomanifold, e.g. from [11]. Note that by a result of Bing [4, Chapter 4], a shellable pseudomanifold is in particular a regular CW complex which is either a ball or a sphere.

Proposition 7.7. *(See [11, Proposition 4.1].) Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a shelling of a d-pseudomanifold Σ and let v be any vertex in $\overline{\sigma_1}$. Then the face poset of Σ admits a Morse matching M such that:*

- *If Σ is the d-sphere then v and σ_m are the only critical cells, while if Σ is a d-ball, then v is the only critical cell.*

By Theorem 6.3, an EL-labeling of the augmented face poset of a pseudomanifold gives rise to a shelling. Chari used induction to construct the Morse matching of Proposition 7.7. Chari's proof of Proposition 7.7, applied to a shelling which comes from an EL-labeling, implies the following.

Corollary 7.8. *Suppose that λ is an EL-labeling of the augmented face poset Q of a pseudomanifold Σ. Let M_λ be the Morse matching given by Proposition* 7.7*. Every $(\sigma \lessdot \tau) \in M$ has the following property: $\sigma \in \mathrm{Last}_\lambda(\tau)$.*

In fact, when the edge labels in λ come from a totally ordered set, Chari's proof of Proposition 7.7 gives the following algorithm for obtaining the Morse matching.

Corollary 7.9. *Suppose that λ is an EL-labeling of the augmented face poset Q of a pseudomanifold Σ. Then the Morse matching M_λ given by Proposition* 7.7 *can be constructed as follows. Set n equal to the rank of the poset Q and set $M = \emptyset$.*

(1) *Consider all unmatched elements σ of rank n, and for each, add $\mathrm{Last}_\lambda(\sigma) \lessdot \sigma$ to M.*
(2) *Decrease n by* 1 *and go to step* 1.

Remark 7.10. Chari also extends Proposition 7.7 to regular CW complexes [11, Theorem 4.2]. Corollaries 7.8 and 7.9 also hold in this situation.

Remark 7.11. Proposition 7.7 and Corollary 7.8 can be useful even when K is a CW complex not known to be regular. In particular, if the face poset Q of K is a CW poset, then there exists a regular CW complex K_{reg} whose face poset is Q. Therefore one can still use these results to construct a Morse matching of K.

8. The Bruhat order, shellability, and reduced expressions

Fix a Coxeter system (W, I) and let T be the set of reflections. In this section we will review some properties of the Bruhat order $\leqslant$ and prove a result (Proposition 8.8) about reduced expressions which will be needed for the proof of Proposition 9.5.

The first part of Theorem 8.1 below is due to Bjorner and Wachs [5]. The second part follows from the first together with Bjorner's result [3] characterizing CW posets.

Theorem 8.1. *(See [5,3].) The Bruhat order of a Coxeter group is thin and (CL)-shellable. Furthermore, an interval with at least two elements is the augmented face poset of a regular CW complex homeomorphic to a ball.*[2]

Theorem 8.1 together with Proposition 7.7 imply the following.

Corollary 8.2. *Let $v < w$ in the Bruhat order of a Coxeter group. Then if we remove v from the interval $[v, w]$ (which plays the role of $\hat{0}$), there is a Morse matching M on the resulting poset with one critical element u of minimal rank. Adding v back to the poset and adding the edge (v, u) to M, we get a Morse matching on $[v, w]$ with no critical elements.*

Dyer [14] subsequently strengthened the Bjorner–Wachs result by giving an EL-labeling of Bruhat order. Dyer's primary tool was his notion of "reflection orders", certain total orderings of T which can be characterized as follows.

Definition 8.3. (See [14, Proposition 2.13].) Let (W, I) be a finite Coxeter system with longest element w_0, and let $T = \{t_1, \dots, t_n\}$ $(n = \ell(w_0))$. Then the total order $t_1 \prec t_2 \prec \cdots \prec t_n$ on T is a reflection order if and only if there is a reduced expression $w_0 = s_{i_1} \dots s_{i_n}$ such that $t_j = s_{i_1} \dots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \dots s_{i_1}$, for $1 \leqslant j \leqslant n$.

Remark 8.4. (See [14, Remark 2.4].) The reverse of a reflection order is a reflection order.

Proposition 8.5. *(See [14].) Fix a reflection order $\preccurlyeq$ on T. Label each edge $x \gtrdot y$ of the Bruhat order by the reflection $x^{-1}y$. Then this edge labeling together with $\preccurlyeq$ is an EL-labeling; therefore the Bruhat order is EL-shellable.*

In what follows, the notation $\hat{s}_k$ indicates the omission of the factor s_k.

Definition 8.6. (See [19].) Consider a Coxeter system (W, I). Define a *deletion pair* in an expression $s_{i_1} \dots s_{i_d}$ to be a pair s_{i_r}, s_{i_t} (where $r < t$) such that the subexpression $s_{i_r} \dots s_{i_t}$ is not reduced but $\hat{s}_{i_r} \dots s_{i_t}$ and $s_{i_r} \dots \hat{s}_{i_t}$ are each reduced.

E.g. in type A the first s_1 and the last s_2 in $s_1 s_2 s_1 s_2$ comprise a deletion pair.

[2] Recall from Remark 5.2 that [5] augments the poset of cells with a $\hat{0}$ and also a greatest element $\hat{1}$. Using this convention, [5] considers intervals in Bruhat order to be posets associated to regular CW complexes homeomorphic to *spheres*.

Lemma 8.7. *(See [19, Lemma 3.31].) If $s_{i_r} \dots \hat{s}_{i_u} \dots s_{i_t}$ is reduced but $s_{i_r} \dots s_{i_t}$ is not, then s_{i_u} belongs to a deletion pair within $s_{i_r} \dots s_{i_t}$.*

Proposition 8.8. *Consider $x \leqslant w$ in a Coxeter group W, and fix a reduced expression $\mathbf{w} = (i_1, \dots, i_t)$ for w. Let $\mathbf{x}_+ = \{j_1, \dots, j_k\}$. For any $p \leqslant t$, consider the product $\gamma_1 \dots \gamma_t$, where*

$$\gamma_r = \begin{cases} s_{i_r}, & \text{if } r \in \mathbf{x}_+ \text{ or } r \geqslant p, \\ 1, & \text{otherwise.} \end{cases}$$

Then $\gamma_1 \dots \gamma_t$ is reduced.

Proof. We will prove this by induction. First consider $p = t$. If $i_t \in \mathbf{x}_+$ there is nothing to prove, since $s_{i_{j_1}} \dots s_{i_{j_k}}$ is reduced. If $i_t \notin \mathbf{x}_+$ then assume that $\gamma_1 \dots \gamma_t$ is not reduced. This means that $x s_{i_t} < x$, which contradicts the fact that $\mathbf{x}_+$ is the positive subexpression for x.

Now by induction assume the proposition holds for any p between some p' and t, where $p' \leqslant t$. We want to prove it for $p := p' - 1$. First suppose that $p' - 1 \in \mathbf{x}_+$. In this case the product $\gamma_1 \dots \gamma_t$ is the same for both $p = p'$ and $p = p' - 1$: in both cases, $\gamma_{p'-1} = s_{i_{p'-1}}$. Therefore by induction it follows that $\gamma_1 \dots \gamma_t$ is reduced.

Now suppose that $p' - 1 \notin \mathbf{x}_+$. In this case the induction hypothesis tells us only that $\gamma_1 \dots \gamma_{p'-2} \gamma_{p'} \dots \gamma_t$ is reduced; we need to prove that $\gamma_1 \dots \gamma_{p'-2} \gamma_{p'-1} \gamma_{p'} \dots \gamma_t = \gamma_1 \dots \gamma_{p'-2} s_{i_{p'-1}} s_{i_{p'}} \dots s_{i_t}$ is reduced. Assume it is not: then by Lemma 8.7, $s_{i_{p'-1}}$ belongs to a deletion pair within $\gamma_1 \dots \gamma_{p'-2} s_{i_{p'-1}} s_{i_{p'}} \dots s_{i_t}$. Note that $s_{i_{p'-1}} s_{i_{p'}} \dots s_{i_t}$ comprises a consecutive string of generators in a reduced expression and so must be reduced. Also note that by our argument in the first paragraph, $\gamma_1 \dots \gamma_{p'-2} s_{i_{p'-1}}$ must be reduced: otherwise $\gamma_1 \dots \gamma_{p'-2} s_{i_{p'-1}} < \gamma_1 \dots \gamma_{p'-2}$, which contradicts the fact that $\mathbf{x}_+$ is a positive subexpression and does not contain $s_{i_{p'-1}}$. But we've now shown that $s_{i_{p'-1}}$ cannot belong to a deletion pair within $\gamma_1 \dots \gamma_{p'-2} s_{i_{p'-1}} s_{i_{p'}} \dots s_{i_t}$, a contradiction. □

9. Morse matchings and the proof of contractibility

In this section we will construct a Morse matching on the face poset of the closure of an arbitrary cell of $(G/P)_{\geqslant 0}$, such that matched edges are provably regular. We will then use this to prove our main result: that the closure of each cell is contractible, and the boundary of each cell is homotopy equivalent to a sphere.

Recall the definition of the augmented face poset $\mathcal{Q}^J$ of $(G/P_J)_{\geqslant 0}$ from Section 5. Besides having a unique least element $\hat{0}$, $\mathcal{Q}^J$ also has a unique greatest element: This is $\hat{1} := \mathcal{P}^J_{u_0, u_0, w_0; >0}$, where u_0 and w_0^J are the longest elements in W_J and W^J, respectively.

The following was proved in [40].

Theorem 9.1. *(See [40].) $\mathcal{Q}^J$ is graded, thin, and EL-shellable. It follows that the face poset of $(G/P_J)_{\geqslant 0}$ is the face poset of a regular CW complex homeomorphic to a ball.*

It will be useful for us to classify the cover relations in Q^J. The following classification is analogous to the one used in [40], with the roles of x and w reversed.

Lemma 9.2. *The cover relations in $\mathcal{Q}^J$ fall into the following three categories.*

Type 1: $\mathcal{P}^J_{x',v,w;>0} \lessdot \mathcal{P}^J_{x,u,w;>0}$ *such that* $x < x'$. *It follows that* $xu^{-1} \lessdot x'v^{-1}$.
Type 2: $\mathcal{P}^J_{x,v,w';>0} \lessdot \mathcal{P}^J_{x,u,w;>0}$ *such that* $w' \leqslant w$. *It follows that* $w'v \lessdot wu$.
Type 3: $\hat{0} \lessdot \mathcal{P}^J_{x,u,w;>0}$ *where* $\mathcal{P}^J_{x,u,w;>0}$ *is a* 0*-cell. It follows that* $x = wu$.

Remark 9.3. If Q is a poset, then the *interval poset* $\mathrm{Int}(Q)$ is defined to be the poset of intervals $[x, y]$ of Q, ordered by containment. When G/P_J is the complete flag variety, i.e. when $J = \emptyset$, $\mathcal{Q}^J$ is simply the interval poset of the Bruhat order.

Theorem 9.4. *Choose any cell* $\mathcal{P}^J_{x,u,w;>0}$ *of* $(G/P_J)_{\geqslant 0}$. *Then there is a Morse matching on the face poset of* $\overline{\mathcal{P}^J_{x,u,w;>0}}$ *with a single critical cell of dimension* 0*, which restricts to a Morse matching on the face poset of the boundary* $\mathrm{bd}(\mathcal{P}^J_{x,u,w;>0})$ *with one additional critical cell of top dimension. Furthermore, all matched edges are good: that is, if* $\mathcal{P}^J_{x',u',w';>0} \lessdot \mathcal{P}^J_{x,u,w;>0}$ *are matched, then* $w' = w$ *and there is a reduced expression* $(i_1, \ldots, i_m)$ *of* w *such that the positive subexpression* $\mathbf{x}'\mathbf{u}'^{-1}_+$ *is equal to* $\mathbf{xu}^{-1}_+ \cup \{k\}$ *and moreover* $\mathbf{xu}^{-1}_+$ *contains* $\{k+1, \ldots, m\}$.

We will prove Theorem 9.4 in a series of steps. Define $S_x(w) := \{\mathcal{R}_{v,w;>0} \mid x \leqslant v \leqslant w\}$, and give this the poset structure inherited from $\mathcal{Q}^J$ (for $J = \emptyset$). This poset is isomorphic to the (dual of the) Bruhat interval between x and w.

Proposition 9.5. $S_x(w)$ *has a Morse matching* $M_x(w)$ *in which all matched edges are good. If* $x < w$*, then* $M_x(w)$ *has no critical cells. If* $x = w$, $M_x(w)$ *has one critical cell.*

Proof. We will construct $M_x(w)$ by using Dyer's EL-labeling of the Bruhat interval (Proposition 8.5) and Chari's observation that one can go from a shelling to a Morse matching (Proposition 7.7). To deduce that matched edges are good, we will choose our reflection order carefully and use Corollary 7.8.

Fix a reduced expression $\mathbf{w} = (i_1, \ldots, i_m)$ for w, and choose a reduced expression for w_0 which begins with $\mathbf{w}^{-1}$. By Definition 8.3, this gives a reflection order. Let $\prec$ be the reverse of this order; by Remark 8.4, $\prec$ is also a reflection order.

Label the edge $\mathcal{R}_{v',w;>0} \lessdot \mathcal{R}_{v,w;>0}$ (where $v' \gtrdot v$) in $S_x(w)$ with the reflection τ such that $v = v'\tau$. By Proposition 8.5, this gives an EL-labeling of $S_x(w)$. If $S_x(w)$ has at least two elements then by Corollary 8.2, there is a Morse matching $M_x(w)$ on $S_x(w)$ with no critical cells. Otherwise, if $S_x(w)$ has one element, i.e. if $x = w$, then we take $M_x(w)$ to be the empty matching with one critical cell.

We now need to show that all edges in $M_x(w)$ are good. By Corollary 7.8, if τ labels the edge $\mathcal{R}_{v,w;>0} \gtrdot \mathcal{R}_{v',w;>0}$ (for $v' \gtrdot v$) and this edge is in $M_x(w)$, then among all edge labels going from $\mathcal{R}_{v,w;>0}$ to lower-dimensional cells, τ is maximal in $\prec$. So we need to analyze cover relations corresponding to maximal labels.

Let $\mathbf{v}_+ = \{j_1, \ldots, j_r\}$. Let k be maximal ($1 \leqslant k \leqslant m$) such that $k \notin \{j_1, \ldots, j_r\}$. We first claim that $\mathbf{u} = \{j_1, \ldots, j_r\} \cup \{k\}$ is a reduced subexpression of $\mathbf{w}$, hence $\mathcal{R}_{u,w;>0} \lessdot \mathcal{R}_{v,w;>0}$, and that $\mathbf{u}$ is positive. Second, we claim that the label on the edge from $\mathcal{R}_{v,w;>0}$ to $\mathcal{R}_{u,w;>0}$ is maximal among all edge labels from $\mathcal{R}_{v,w;>0}$ down to a lower-dimensional cell.

Proposition 8.8 implies the first claim that $\{j_1, \ldots, j_r\} \cup \{k\}$ is a reduced subexpression of $\mathbf{w}$. Knowing that it is reduced, it is clear that it is positive.

To see that the second claim is true, note that by the choice of k, the label on the edge $\mathcal{R}_{v,w;>0} \gtrdot \mathcal{R}_{u,w;>0}$ is $u^{-1}v = s_{i_m}s_{i_{m-1}} \ldots s_{i_k} \ldots s_{i_{m-1}}s_{i_m}$. Furthermore, in our reflection order, $s_{i_m} \succ s_{i_m}s_{i_{m-1}}s_{i_m} \succ \cdots \succ s_{i_m}s_{i_{m-1}}\ldots s_{i_k}\ldots s_{i_{m-1}}s_{i_m} \succ \cdots$. Since k is maximal such that $k \notin \{j_1, \ldots, j_r\}$, if we define $v'' = vs_{i_m}s_{i_{m-1}}\ldots s_{i_\ell}\ldots s_{i_{m-1}}s_{i_m}$ for $\ell > k$, then $\ell \in \{j_1, \ldots, j_r\}$ so an expression for v'' is given by $\{j_1, \ldots, j_r\} \setminus \{\ell\}$. In particular, $v'' < v$ and so $\mathcal{R}_{v,w;>0}$ does not cover $\mathcal{R}_{v'',w;>0}$. On the other hand, we know that $\mathcal{R}_{v,w;>0}$ covers $\mathcal{R}_{u,w;>0}$, and that the label on this edge is $s_{i_m}s_{i_{m-1}}\ldots s_{i_k}\ldots s_{i_{m-1}}s_{i_m}$. By our choice of reflection order, this label is maximal among all edges from $\mathcal{R}_{v,w}$ down to lower-dimensional cells.

Finally, by the choice of k, and since $\{j_1, \ldots, j_r\} \cup \{k\} = \mathbf{u}$ is positive, this cover relation is good. Therefore every matched edge in $M_x(w)$ is good. □

Remark 9.6. Recently Brant Jones has constructed explicit matchings of the Hasse diagram of an interval in Bruhat order; he also proved that his matchings coincide with the matchings $M_x(w)$ that we constructed in Proposition 9.5 [21].

Remark 9.7. If x is the identity element in W, then the Morse matching constructed in Proposition 9.5 will actually be a multiplication matching by a Coxeter generator. This is a so-called *special matching*, and is relevant to Kazhdan–Lusztig theory [10]. Anders Bjorner suggested using special matchings to construct acyclic matchings, and realized that one could use them to obtain an acyclic matching for the face poset of the entire space $(G/B)_{\geqslant 0}$ [9]. We are grateful for his insights.

We now turn to the proof of Theorem 9.4.

Proof of Theorem 9.4. We partition the elements of the face poset of the closure of $\mathcal{P}^J_{x,u,w;>0}$ into subsets $S^J_{xu^{-1}}(y) = \{\mathcal{P}^J_{x',u',y;>0} \mid xu^{-1} \leqslant x'u'^{-1} \leqslant y\}$, for each $y \in W^J$ such that $xu^{-1} \leqslant y \leqslant w$. By Lemma 9.2, the restriction of the face poset $\mathcal{Q}^J$ to $S^J_{xu^{-1}}(y)$ is isomorphic to the (dual of the) Bruhat interval between xu^{-1} and y, so $S^J_{xu^{-1}}(y)$ and $S_{xu^{-1}}(y)$ are isomorphic as posets: we simply identify $\mathcal{P}^J_{a,b,y;>0}$ with $\mathcal{R}_{ab^{-1},y;>0}$.

We can now apply Proposition 9.5, which gives us a Morse matching $M^J_{xu^{-1}}(y)$ on $S^J_{xu^{-1}}(y)$ such that all matched edges are good. This matching has either zero or one critical cell, based on whether $xu^{-1} < y$ or $xu^{-1} = y$.

We now define

$$\mathcal{M}^J_{x,u,w} = \bigcup_{y \in W^J,\, xu^{-1} \leqslant y \leqslant w} M^J_{xu^{-1}}(y).$$

Since each $M^J_{xu^{-1}}(y)$ is a matching, and any two matched elements $\mathcal{P}^J_{a,b,y;>0}$ and $\mathcal{P}^J_{a',b',y;>0}$ in $\mathcal{M}^J_{x,u,w}$ have the same third factor y, $\mathcal{M}^J_{x,u,w}$ is also a matching.

Let us assume for the sake of contradiction that there is a cycle in $G(\mathcal{M}^J_{x,u,w})$. Since each $G(M^J_{xu^{-1}}(y))$ is acyclic, there must be some edges in the cycle which pass between two different $S^J_{xu^{-1}}(y)$'s. Each such edge must be directed from the higher-dimensional cell $\mathcal{P}^J_{a,b,y;>0}$ to the lower-dimensional cell $\mathcal{P}^J_{a',b',y';>0}$ for $y \neq y'$, so by Lemma 9.2, $y' < y$. So if we traverse the cycle and look at the sequence of poset elements $\mathcal{P}^J_{*,*,y;>0}$ that we encounter, the third factor can only decrease. Therefore it is impossible to return to the element of the cycle at which we started, which is a contradiction.

As y varies over elements of W^J between xu^{-1} and w, we have that $M^J_{xu^{-1}}(y)$ has no critical cells for $xu^{-1} < y$ and it has one critical cell $\mathcal{P}^J_{x,u,xu^{-1};>0}$ for $xu^{-1} = y$. Therefore $\mathcal{M}^J_{x,u,w}$ has a unique critical cell, the 0-dimensional cell $\mathcal{P}^J_{x,u,xu^{-1};>0}$.

Since the face poset of $\overline{\mathcal{P}^J_{x,u,w;>0}}$ has a unique cell of top dimension $\ell(w) - \ell(xu^{-1})$ which is matched in $M^J_{x,u,w}$, when we restrict $M^J_{x,u,w}$ to the boundary $\mathrm{bd}(\mathcal{P}^J_{x,u,w;>0})$, we will get a Morse matching with one additional critical cell of top dimension $\ell(w) - \ell(xu^{-1}) - 1$. This completes the proof of the theorem. □

Corollary 9.8. *Choose any cell $\mathcal{P}^J_{x,u,w;>0}$ of $(G/P_J)_{\geqslant 0}$. Then there is a Morse matching on the face poset of $\overline{\mathcal{P}^J_{x,u,w;>0}}$ with a single critical cell of dimension 0 in which all matched edges are regular; it restricts to a Morse matching on the face poset of the boundary $\mathrm{bd}(\mathcal{P}^J_{x,u,w;>0})$ with one additional critical cell of top dimension.*

Proof. This follows from Theorems 5.6 and 9.4. □

We now prove our main result.

Theorem 9.9. *The closure of each cell of $(G/P_J)_{\geqslant 0}$ is contractible, and the boundary of each cell of $(G/P_J)_{\geqslant 0}$ is homotopy equivalent to a sphere.*

Proof. Choose an arbitrary cell of $(G/P_J)_{\geqslant 0}$ and let K be its closure. Note that K is a CW complex with the subcomplex property because Theorems 4.5 and 5.3 imply that $(G/P_J)_{\geqslant 0}$ is. Let Q be the face poset of K. By Corollary 9.8, Q has a Morse matching with a unique critical cell of dimension 0, in which all matched edges are regular. Therefore Theorem 7.6 implies K is contractible.

Now let K' be the boundary of an arbitrary cell and let Q' be its face poset. By Corollary 9.8, Q' has a Morse matching with two critical cells, one of dimension 0 and one of top dimension, say p, in which all matched edges are regular. Therefore Theorem 7.6 implies that K' is homotopy equivalent to a CW complex with one 0-dimensional cell σ and one p-dimensional cell whose boundary is glued to σ. This is precisely a p-sphere. □

Remark 9.10. Since a Morse function actually gives rise to a concrete *collapsing* [15] of a CW complex, in fact we have shown that the closure of a cell is *collapsible.*

Remark 9.11. One can give a simpler proof that the closure of each cell in the totally nonnegative part of the type A Grassmannian $(Gr_{kn})_{\geqslant 0}$ is contractible. In that case, one can prove directly that whenever a cell σ has codimension 1 in the closure of τ, then σ is a regular face of τ. This follows from the technology of [31]: in particular, Theorem 18.3, Lemma 18.9, and Corollary 18.10. Then by Theorem 9.1, the poset of cells of $(Gr_{kn})_{\geqslant 0}$ is a CW poset with an EL-labeling (hence a shelling), so by Proposition 7.7, we have the requisite Morse matching.

Using Corollary 7.9, we see that there is a more concrete way to describe the matchings $M_x(w)$ of $S_x(w)$.

Remark 9.12. Fix a reduced expression $\mathbf{w} = (i_1, \ldots, i_m)$ for w. Start with the maximal element $\mathcal{R}_{x,w;>0}$. Let k be maximal such that $1 \leqslant k \leqslant m$ and $k \notin \mathbf{x}_+$. Then $\mathbf{x}_+ \cup \{k\}$ is the positive subexpression for an element $v > x$. We match $\mathcal{R}_{x,w;>0}$ to $\mathcal{R}_{v,w;>0}$. Now apply the same procedure to every element of dimension $\dim \mathcal{R}_{x,w;>0} - 1$ which has not been matched (the order in which we consider these elements does not matter). Continue in this fashion, from higher to lower-dimensional cells.

Acknowledgments

L.W.: It is a pleasure to thank Anders Bjorner for stimulating conversations about poset topology and discrete Morse theory at the Mittag–Leffler Institute in May 2005. I would also like to thank Sergey Fomin, Arun Ram, and David Speyer. Finally, I am grateful to Tricia Hersh for useful feedback and for drawing my attention to [23].

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