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Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces

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Abstract

Let $\mathcal{M}$ be a smooth and compact moduli space of stable coherent sheaves on a projective surface S with an effective (or trivial) anti-canonical line bundle. We find generators for the cohomology ring of $\mathcal{M}$, with integral coefficients. When S is simply connected and a universal sheaf $\mathcal{E}$ exists over $S \times \mathcal{M}$, then its class $[\mathcal{E}]$ admits a Künneth decomposition as a class in the tensor product $K^0_{\mathrm{top}}(S) \otimes K^0_{\mathrm{top}}(\mathcal{M})$ of the topological K-rings. The generators are the Chern classes of the Künneth factors of $[\mathcal{E}]$ in $K^0_{\mathrm{top}}(\mathcal{M})$. The general case is similar.

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1. Introduction

Let S be a smooth connected projective symplectic or Poisson surface. If symplectic, S is either a K3 or an abelian surface. Non-symplectic projective Poisson surfaces include all minimal rational surfaces, all del-Pezzo surfaces, and certain ruled surfaces over projective curves of arbitrary genus. Poisson surfaces are classified in [7].

Given an algebraic variety X, denote by $K^0_{\mathrm{alg}}(X)$ the Grothendieck K-ring of algebraic vector bundles on X and by $K^0_{\mathrm{top}}(X)$ its topological analogue. Let v be a class in $K^0_{\mathrm{top}}(S)$ of rank $r \geqslant 0$ with $c_1(v)$ of Hodge-type $(1,1)$. Assume that v is primitive, i.e., v is not a multiple of another class in $K^0_{\mathrm{top}}(S)$ by an integer $\geqslant 1$. If $r = 0$, assume for simplicity, that the anti-canonical line-bundle is either trivial or ample (the assumption is relaxed in Condition 7). Given an ample line bundle H on S, denote by $\mathcal{M} := \mathcal{M}_H(v)$ the moduli space of H-stable sheaves on S with class v. We use stability in the sense of Gieseker, Maruyama, and Simpson (Definition 4). For a generic choice of an ample line bundle H on S, called *v-generic* in Definition 5, the moduli space $\mathcal{M}_H(v)$ is either empty, or smooth of the expected dimension, connected, and projective, and it admits a holomorphic symplectic or Poisson structure (Section 2.1). The expected dimension of $\mathcal{M}_H(v)$ is $\epsilon - \chi(v^\vee \cup v)$, where $v^\vee$ is the class dual to v, $\cup$ is the product operation in $K^0_{\mathrm{top}}(S)$, $\epsilon = 2$ if S is symplectic, $\epsilon = 1$ if S is non-symplectic but Poisson, and χ is the Euler characteristic defined in Section 1.3.

A *universal sheaf* is a coherent sheaf $\mathcal{E}$ over $S \times \mathcal{M}$, flat over $\mathcal{M}$, whose restriction to $S \times \{m\}$, $m \in \mathcal{M}$, is isomorphic to the sheaf E_m on S in the isomorphism class m. The universal sheaf is canonical only up to tensorization by the pull-back of a line-bundle on $\mathcal{M}$. A universal sheaf exists, if there exists a class x in $K^0_{\mathrm{alg}}(S)$ satisfying $\chi(x \cup v) = 1$ ([34] or Section 3.1 below). Otherwise, there is a weaker notion of a *twisted* universal sheaf, denoted also by $\mathcal{E}$, where the twisting is encoded by a class θ in Čech cohomology $H^2(\mathcal{M}, \mathcal{O}^*_{\mathcal{M}})$, in the classical topology (Definition 27). For a triple S, v, H, as above, the class θ is always topologically trivial; it maps to 0 in $H^3(\mathcal{M}, \mathbb{Z})$ via the connecting homomorphism of the exponential sequence (Lemma 28). Consequently, $\mathcal{E}$ defines a class e in $K^0_{\mathrm{top}}(S \times \mathcal{M})$, canonical up to tensorization by the pull-back of the class of a topological line-bundle on $\mathcal{M}$ (Definition 26).

When $H^1(S, \mathbb{Z})$ does not vanish, we will need the odd K-groups of S and $\mathcal{M}$ as well. Given a complex algebraic variety X, let $K^1_{\mathrm{top}}(X)$ be its odd K-group and $K^*_{\mathrm{top}}(X) := K^0_{\mathrm{top}}(X) \oplus K^1_{\mathrm{top}}(X)$ its K-ring ([3,23] and Section 2.5). The K-ring $K^*(S)$ of a projective Poisson surface is known to be torsion free. As a consequence, the Künneth theorem yields an isomorphism

$$\left[K^0_{\mathrm{top}}(S) \otimes K^0_{\mathrm{top}}(\mathcal{M})\right] \oplus \left[K^1_{\mathrm{top}}(S) \otimes K^1_{\mathrm{top}}(\mathcal{M})\right] \to K^0_{\mathrm{top}}(S \times \mathcal{M}) \tag{1}$$

[3, Corollary 2.7.15]. Choose a basis $\{x_1, \dots, x_n\}$ of $K^*_{\mathrm{top}}(S)$, which is a union of bases of the summands $K^0_{\mathrm{top}}(S)$ and $K^1_{\mathrm{top}}(S)$. We get the Künneth decomposition

$$e = \sum_{i=1}^{n} x_i \otimes e_i, \tag{2}$$

of the class $e \in K^0_{\mathrm{top}}(S \times \mathcal{M}_H(v))$ of the universal sheaf, where each class e_i is either in $K^0_{\mathrm{top}}(\mathcal{M}_H(v))$ or in $K^1_{\mathrm{top}}(\mathcal{M}_H(v))$. The Chern classes $c_i(y) \in H^{2i}(\mathcal{M}, \mathbb{Z})$, for odd classes $y \in K^1_{\mathrm{top}}(\mathcal{M})$ and $i \geqslant 1/2$ a half-integer, are introduced in Definition 19.

The main result of this paper is the following.

Theorem 1. *Let S be a projective Poisson surface, $v \in K^0_{\mathrm{top}}(S)$ as above, and H a v-generic polarization.*

1. *The cohomology ring $H^*(\mathcal{M}_H(v), \mathbb{Z})$ is generated by the Chern classes $c_j(e_i)$, of the Künneth factors $e_i \in K^*_{\mathrm{top}}\mathcal{M}_H(v)$, which are given in Eq.* (2).
2. *The cohomology groups $H^i(\mathcal{M}_H(v), \mathbb{Z})$ are torsion free for all i.*
3. *If $H^1(S, \mathbb{Z}) = 0$, then $H^i(\mathcal{M}_H(v), \mathbb{Z})$ vanishes for odd i.*

The theorem is a summary of Propositions 12 and 20 and Corollary 25. Part 3 of the theorem is an immediate consequence of part 1.

In a sequel to this paper we apply Theorem 1 to the study of the Hilbert schemes $S^{[n]}$, $n \geqslant 2$, of length n zero-dimensional subschemes of a K3 surface S [30]. The Hilbert scheme $S^{[n]}$ has complex deformations, which are not Hilbert schemes on any K3 surface. We determine the subgroup Mon^2 of $\mathrm{Aut}[H^2(S^{[n]}, \mathbb{Z})]$, generated by monodromy operators of families of deformations of $S^{[n]}$, consisting of smooth hyperkähler varieties. We find an arithmetic obstruction for elements of $\mathrm{Aut}[H^2(S^{[n]}, \mathbb{Z})]$ to extend to monodromy operators of the full integral cohomology ring $H^*(S^{[n]}, \mathbb{Z})$. Mon^2 turns out to be smaller than expected, when $n-1$ is not a prime power. As a consequence, we get that the weight 2 Hodge structure, of hyperkähler deformations X of $S^{[n]}$, does not determine the bimeromorphic class of X, when $n-1$ is not a prime power.

Denote by $A^*(\mathcal{M}_H(v))$ the Chow ring of $\mathcal{M}_H(v)$. The following theorem is proven in Section 2.4.

Theorem 2. *Let S be a rational Poisson surface, and v, H as in Theorem* 1. *The class of the diagonal in $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$ is in the image of the exterior-product homomorphism*

$$A^*\big(\mathcal{M}_H(v)\big) \otimes A^*\big(\mathcal{M}_H(v)\big) \to A^*\big(\mathcal{M}_H(v) \times \mathcal{M}_H(v)\big).$$

Furthermore, the natural homomorphism $A^*(\mathcal{M}_H(v)) \to H^*(\mathcal{M}_H(v), \mathbb{Z})$ is an isomorphism.

1.1. Higgs bundles

Let Σ be a smooth compact and connected Riemann surface of genus $g \geqslant 2$, and D an effective divisor on Σ. The important special case $D = 0$ is included. A *Higgs bundle on Σ, with possible poles along D*, is a pair (E, φ), consisting of a vector bundle E on Σ and a 1-form valued endomorphism $\varphi : E \to E \otimes K_\Sigma(D)$. The Higgs bundle is *stable*, if any non-zero proper

φ-invariant subbundle $F \subset E$ satisfies the inequality $\deg(F)/\operatorname{rank}(F) < \deg(E)/\operatorname{rank}(E)$. The moduli space $\mathcal{H}_\Sigma(r,d,D)$, of stable rank r Higgs bundles of degree d, with possible poles along D, is a smooth quasi-projective variety [35,37]. Denote by f_i, $i = 1,2$, the projections from $\Sigma \times \mathcal{H}_\Sigma(r,d,D)$. Assume that r and d are relatively prime. Then there exists over $\Sigma \times \mathcal{H}_\Sigma(r,d,D)$ a universal vector bundle $\mathcal{E}$ and a universal Higgs field $\Phi : \mathcal{E} \to \mathcal{E} \otimes f_1^* K_\Sigma(D)$. Choose a basis $\{x_1, x_2, \ldots, x_{2g}\}$ of $K^1_{\mathrm{top}}(\Sigma)$, and $\{x_{2g+1}, x_{2g+2}\}$ of $K^0_{\mathrm{top}}(\Sigma)$. Define the Künneth factors $e_i \in K^*_{\mathrm{top}}(\mathcal{H}_\Sigma(r,d,D))$, $1 \leqslant i \leqslant 2g+2$, of the universal bundle $\mathcal{E}$, as in Eq. (2).

Theorem 3. *The cohomology ring $H^*(\mathcal{H}_\Sigma(r,d,D),\mathbb{Z})$ is generated by the Chern classes $c_j(e_i)$.*

The theorem is proven in Section 4. It sharpens results of Hausel–Thaddeus and the author, where the cohomology was considered with rational coefficients [20,29].

1.2. Related works

Ellingsrud and Strømme proved Theorems 1 and 2 when the surface S is the projective plane [16]. Beauville found generators, for the cohomology ring with *rational* coefficients of moduli spaces as above, when X is a non-symplectic Poisson surface [8]. When S is a K3 or abelian surface, generators for the cohomology ring $H^*(\mathcal{M},\mathbb{Q})$, with *rational* coefficients, where found in [29]. In case S is an arbitrary projective surface and $\mathcal{M} = S^{[n]}$ is the Hilbert scheme, parametrizing ideals sheaves of length n subscheme, generators for the ring $H^*(S^{[n]},\mathbb{Q})$ were found in [25,26]. The ring structure of $H^*(S^{[n]},\mathbb{Q})$ was calculated by Lehn and Sorger for a K3 or abelian surface S [24]. The cohomology ring was shown to be isomorphic to the orbifold cohomology of the symmetric product [17]. The ring structure of $H^*(S^{[n]},\mathbb{Q})$, for any smooth projective surface S, was determined in [12].

Theorem 1, about integral generators for the cohomology of moduli spaces, follows easily from the Main Theorems of [8] and [29], stated as Theorem 8 below. Theorem 8 expresses the class of the diagonal in $\mathcal{M} \times \mathcal{M}$, in terms of the universal sheaf $\mathcal{E}$ over $S \times \mathcal{M}$. The Künneth decomposition (2) of the class of $\mathcal{E}$, in the topological K-group of $S \times \mathcal{M}$, leads to a decomposition of the class of the diagonal in $\mathcal{M} \times \mathcal{M}$. More precisely, the class of the diagonal is Poincaré dual to the sum of exterior products of Chern classes of the classes $e_i \in K^*_{\mathrm{top}}(\mathcal{M})$ given in (2). Theorem 1 follows easily from the latter decomposition. Our decomposition of the diagonal is the precise topological analogue of that of [16], replacing $\mathbb{P}^2$ by a symplectic or Poisson surface S, and the algebraic K-group by the topological one.

The paper is organized as follows. In Section 2 we prove Theorem 1 for moduli spaces of sheaves over Poisson surfaces, assuming that a universal sheaf exists. The case of Poisson surfaces with vanishing odd cohomology is particularly simple, and is treated first. In Section 3 we find integral generators for the cohomology of such moduli spaces, when a universal sheaf does not exist. In Section 4 generators are found for the integral cohomology ring of the moduli spaces of Higgs bundles.

1.3. Notation

Given a smooth complex algebraic variety X, we denote by $K^0_{\mathrm{top}}X$ the topological K-ring of vector bundles on X. We let $K^0_{\mathrm{alg}}X$ be its algebraic analogue and

$$\alpha : K^0_{\mathrm{alg}}X \to K^0_{\mathrm{top}}X$$

the natural homomorphism. Given a morphism $f : X \to Y$, we denote by $f^! : K^0_{\mathrm{top}} Y \to K^0_{\mathrm{top}} X$ the pullback. When f is a proper morphism, we denote by $f_! : K^0_{\mathrm{top}} X \to K^0_{\mathrm{top}} Y$ the topological Gysin map ([9], [23, Proposition IV.5.24]). We used above the assumption, that X and Y are smooth, which enables us to identify K^0_{top} with the K-homology groups K_0^{top}, for both spaces. Similarly, we identify the Grothendieck group $K_0^{\mathrm{alg}} X$, of coherent sheaves, with $K^0_{\mathrm{alg}} X$, replacing the class of a sheaf by that of any of its locally free resolutions. The algebraic push-forward $f_! : K^0_{\mathrm{alg}} X \to K^0_{\mathrm{alg}} Y$ takes the class of a coherent sheaf E on X, to the alternating sum $\sum_{i \geqslant 0} (-1)^i R^i_{f_*} E$ of the classes of the higher direct image sheaves on Y. When Y is a point $\{pt\}$, we identify $K^0_{\mathrm{top}}(\{pt\})$ with $\mathbb{Z}$ and $f_!$ is the *Euler characteristic* $\chi : K^0_{\mathrm{top}}(X) \to \mathbb{Z}$. The algebraic and topological Gysin maps are compatible via the equality $f_! \circ \alpha = \alpha \circ f_!$ (see [9]).

We denote by $x \cup y$ the product in K^0_{top}. The Gysin homomorphisms satisfy the projection formula

$$f_!\big(x \cup f^!(y)\big) = f_!(x) \cup y \tag{3}$$

both for algebraic and for topological K-groups. The topological Gysin homomorphism satisfies the following weak analogue, of the Cohomology and Base Change Theorem, applied to a Cartesian product instead of a fiber product

$$\begin{array}{ccc} X \times Z & \xrightarrow{f \times \iota} & Y \times Z \\ {\scriptstyle \pi_X} \downarrow & & \downarrow {\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y. \end{array}$$

The equivalence

$$(f \times \iota)_!\big(\pi_X^! x\big) = \pi_Y^! f_!(x) \tag{4}$$

holds, where $x \in K^0_{\mathrm{top}}(X)$, $\iota : Z \to Z$ is the identity, $f \times \iota : X \times Z \to Y \times Z$ is the product map, and π_X, π_Y are the projections [5].

S will denote a surface and $\mathcal{M}$ a moduli space of stable sheaves on S. The morphism f_i is the projection from $S \times \mathcal{M}$ on the ith factor, $i = 1, 2$. The morphism p_i is the projection from $\mathcal{M} \times \mathcal{M}$ on the ith factor.

2. Integral generators via a universal sheaf

In Section 2.1 the necessary background on moduli spaces of stable sheaves on Poisson surfaces is reviewed. We recall in Section 2.2, that the class of the diagonal in $\mathcal{M} \times \mathcal{M}$ can be expressed in terms of the universal sheaf over $S \times \mathcal{M}$ (Theorem 8). In Section 2.3 we prove Theorem 1, finding generators for $H^*(\mathcal{M}, \mathbb{Z})$, when S is K3 or a rational Poisson surface. These are the Poisson surfaces with vanishing odd cohomology groups. We treat these special cases first, because the argument is simple, yet it illustrates the general idea. Everything follows from the formula for the class of the diagonal in $\mathcal{M} \times \mathcal{M}$ and the Künneth decomposition of the universal sheaf in $K^0_{\mathrm{top}}(S \times \mathcal{M})$. In Section 2.4 we establish the isomorphism $A^*(\mathcal{M}) \cong H^*(\mathcal{M}, \mathbb{Z})$, between the Chow ring and the integral cohomology ring, when S is a rational Poisson surface

(Theorem 2). The results in Sections 2.3 and 2.4, and their proofs, are natural extensions of those of Ellingsrud and Strømme in the case $S = \mathbb{P}^2$ [16]. Available to us is the formula for the class of the diagonal in $\mathcal{M} \times \mathcal{M}$, provided by Theorem 8, which was proven in [16] in the special case of $\mathbb{P}^2$. In Section 2.5 we treat Poisson surfaces with non-vanishing odd cohomology groups.

2.1. *Stable sheaves and their moduli spaces*

Let S be a smooth and projective symplectic or Poisson surface and H an ample line bundle on S. The Hilbert polynomial of a coherent sheaf F on S is defined by

$$P_F(n) := \chi\big(F \otimes H^n\big) := h^0\big(F \otimes H^n\big) - h^1\big(F \otimes H^n\big) + h^2\big(F \otimes H^n\big).$$

Let r be the rank of F, $f_i := c_i(F)$, $i = 1, 2$, its Chern classes, $h := c_1(H)$, and $K := c_1(T^*S)$. Hirzebruch–Riemann–Roch yields the equality

$$P_F(n) = \big(rh^2/2\big)n^2 + \big[(h \cdot f_1) - r/2(h \cdot K)\big]n + \big[\big(f_1^2 - 2f_2\big) - f_1 \cdot K + 2r\chi(\mathcal{O}_S)\big]/2.$$

The degree d of $P_F(n)$ is equal to the dimension of the support of F. Let $l_0(F)/d!$ be the coefficient of n^d. Then $l_0(F)$ is a positive integer. Explicitly, if $r > 0$ then $l_0(F) := rh^2$. If $r = 0$ and $d = 1$ then $l_0(F) := h \cdot c_1(F)$. If $r = 0$ and $d = 0$, then $l_0(F) := -c_2(F)$. Given two polynomials p and q with real coefficients, we say that $p \succ q$ (respectively $p \succcurlyeq q$) if $p(n) > q(n)$ (respectively $p(n) \geqslant q(n)$) for all n sufficiently large.

Definition 4. A coherent sheaf F on S is called H*-semi-stable* (respectively H*-stable*) if it has support of pure dimension and any non-trivial subsheaf $F' \subset F$, $F' \neq (0)$, $F' \neq F$ satisfies

$$\frac{P_{F'}}{l_0(F')} \preccurlyeq \frac{P_F}{l_0(F)} \quad \text{(respectively } \prec\text{)}.$$

Let $v \in K^0_{\mathrm{top}}(S)$ be a class of rank $r \geqslant 0$ and first Chern class $c_1 \in H^2(S, \mathbb{Z})$ of Hodge type $(1, 1)$. The moduli space $\mathcal{M}_H(v)$, of isomorphism classes of H-stable sheaves of class v, is a quasi-projective scheme [19,37].

Definition 5. An ample line bundle H is said to be v*-generic*, if every H-semi-stable sheaf with class v is H-stable.

The class v is *primitive*, if v is not a multiple of another class in $K^0_{\mathrm{top}}S$, by an integer larger than 1. If v is primitive, then a v-generic polarization exists when $r > 0$, or when $r = 0$ and $\chi(v) \neq 0$ (see [39], when $r > 0$, [40, Lemma 1.2], when $r = 0$, and [28, Condition 3.1], for an existence criterion when $r = 0$ and $\chi(v) = 0$). If H is v-generic, then $\mathcal{M}_H(v)$ is projective [19,37]. *Caution*: The standard definition of the term v*-generic* is more general, and does not assume that v is primitive. We will assume, throughout the paper, the following:

Condition 6. The class v is primitive and H is v-generic.

When S is a K3 or abelian surface, the moduli space $\mathcal{M}_H(v)$ is smooth and holomorphic symplectic [33]. It is also connected ([40], or Corollary 10 below). When S is a non-symplectic Poisson surface, i.e., one with a non-trivial and effective anti-canonical line-bundle, then the moduli space $\mathcal{M}_H(v)$ is smooth, whenever the rank r is positive. When $r=0$, a moduli space $\mathcal{M}_H(v)$, of stable sheaves with pure one-dimensional support, may be singular (see [13, Example 8.6]). Smoothness and connectedness of $\mathcal{M}_H(v)$ are guaranteed, for sheaves with support of dimension 1 or 2, by the condition:

Condition 7. The sheaf $F\otimes K_S$ is isomorphic to a proper subsheaf of F, for every stable sheaf F parametrized by $\mathcal{M}_H(v)$.

The above condition and Serre's duality imply, that the extension group $\mathrm{Ext}^2(E,F)$ vanishes, for every two stable sheaves E, F in $\mathcal{M}_H(v)$. The condition is automatically satisfied when $r>0$ and S is a non-symplectic Poisson surface. The condition is also satisfied, for example, for sheaves with support of dimension 1 on a del-Pezzo surface S (with an ample anti-canonical line bundle K_S^{-1}). Smoothness follows from Condition 7, by a criterion of Artamkin [1]. Connectedness is proven below (Corollary 10).

A Poisson structure on S determines a Poisson structure on the smooth moduli space $\mathcal{M}_H(v)$ (see [38] for the definition of the tensor, [10] for the Jacobi identity when $r>0$, and [21] when $r=0$).

2.2. The class of the diagonal

Assume that S, v, and H satisfy Condition 6. The following result holds, when S is a K3 or abelian surface (by [29]), or if S is a non-symplectic Poisson surface and $\mathcal{M}_H(v)$ satisfies Condition 7 (by [8]). Let π_{ij} be the projection from $\mathcal{M}_H(v)\times S\times\mathcal{M}_H(v)$ onto the product of the ith and jth factors. Assume, that there exists a universal sheaf over $S\times\mathcal{M}_H(v)$.

Theorem 8. [8,29] *Let $\mathcal{E}'_v$, $\mathcal{E}''_v$ be any two universal families of sheaves over the m-dimensional moduli space $\mathcal{M}_H(v)$.*

1. *The class of the diagonal, in the Chow ring of $\mathcal{M}_H(v)\times\mathcal{M}_H(v)$, is identified by*

$$c_m\Big[-\pi_{13_!}\Big(\pi_{12}^*(\mathcal{E}'_v)^\vee\overset{L}{\otimes}\pi_{23}^*(\mathcal{E}''_v)\Big)\Big],\tag{5}$$

   where both the dual $(\mathcal{E}'_v)^\vee$ and the tensor product are taken in the derived category.
2. *When S is a K3 or abelian surface, the following vanishing holds*

$$c_{m-1}\Big[-\pi_{13_!}\Big(\pi_{12}^*(\mathcal{E}'_v)^\vee\overset{L}{\otimes}\pi_{23}^*(\mathcal{E}''_v)\Big)\Big]=0.\tag{6}$$

An analogue of Theorem 8 holds, even when a universal sheaf does not exist (see Proposition 24 below).

The class (5) is independent of the choice of the universal sheaves $\mathcal{E}'_v$ and $\mathcal{E}''_v$, as a consequence of part 1 of the theorem. This independence is proven by another method in Lemma 9, Eq. (8), in the non-symplectic Poisson case. In the symplectic case, Lemma 9, Eq. (9) relates the independence of the class (5) to the vanishing (6).

Lemma 9. *Let X be a topological space, x a class of rank $r\geqslant 0$ in $K^0_{\mathrm{top}}(X)$, and L a complex line-bundle on X. Then the Chern classes of $x\cup L$ satisfy the following equations*

$$c_{r+n}(x \cup L) = c_{r+n}(x) - (n-1)c_{r+n-1}(x)c_1(L) + \cdots + (-1)^d \binom{n-1}{d} c_{r+n-d}(x)c_1(L)^d + \cdots + (-1)^{n-1}c_{r+1}(x)c_1(L)^{n-1}, \quad (7)$$

for $n \geqslant 1$. *In particular,*

$$c_{r+1}(x \cup L) = c_{r+1}(x), \quad (8)$$

$$c_{r+2}(x \cup L) = c_{r+2}(x) - c_{r+1}(x)c_1(L). \quad (9)$$

Proof. Every element of $K^0_{\text{top}}(X)$ is of the form $[E] - [F]$, where E and F are vector bundles on X (see [3, Section 2.1]). Let e and f be the ranks of E and F, so that $r = e - f$. Using the *splitting principle*, we can write the Chern polynomials as products $c_t(E) = \prod_{i=1}^{e}(1 + \alpha_i t)$ and $c_t(F) = \prod_{j=1}^{f}(1 + \beta_j t)$, where the α_i and β_j are formal variables, and only the symmetric polynomials in the α_i or β_j are interpreted as cohomology classes. Set $\ell := c_1(L)$. The Chern polynomial of $x \cup L$ then satisfies:

$$c_t(x \cup L) = \prod_{i=1}^{e}\bigl(1 + [\alpha_i + \ell]t\bigr) \Big/ \prod_{j=1}^{f}\bigl(1 + [\beta_j + \ell]t\bigr) = (1 + \ell t)^r c_{t/(1+\ell t)}(x)$$
$$= \sum_q c_q(x)t^q(1 + \ell t)^{r-q}.$$

Thus, $c_{r+n}(x \cup L) = \sum_{i=0}^{r+n}\binom{i-n}{i}c_{r+n-i}(x)\ell^i$. The lemma follows from the vanishing of $\binom{i-n}{i}$, for $i \geqslant n$. □

Connectedness of $\mathcal{M}_H(v)$ is an immediate corollary of Theorem 8. This was observed by Mukai, for 2-dimensional moduli spaces over K3 surfaces [34], and by Kaledin, Lehn, and Sorger, for more general moduli spaces over K3 and abelian surfaces [22].

Corollary 10. $\mathcal{M}_H(v)$ *is connected.*

Proof. Let M be a connected component of $\mathcal{M}_H(v)$ and f_i, $i = 1, 2$, the projection from $S \times M$ onto the ith factor. Assume first, that a universal sheaf $\mathcal{E}_v$ exists over $S \times M$. Let F be a sheaf on S with class v and set

$$x := -f_{2_!}\Bigl[f_1^*(F)^\vee \overset{L}{\otimes} f_2^*(\mathcal{E}_v)\Bigr].$$

The class $c_m(x)$ depends only on the class v and is independent of the sheaf F. If F belongs to M, then $c_m(x)$ is Poincaré dual to the pullback of the class (5), via the embedding of M in $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$ sending E to (F, E). Thus $c_m(x)$ is Poincaré dual to a point.

Assume that there exists an H-stable sheaf F, which does not belong to M. Then the higher-direct images

$$R^i f_{2_*}\Bigl[f_1^*(F)^\vee \overset{L}{\otimes} f_2^*(\mathcal{E}_v)\Bigr]$$

vanish, for $i = 0, 2$. Consequently, the class x is represented by the locally free sheaf

$$R^1 f_{2_*}\left[f_1^*(F)^\vee \overset{L}{\otimes} f_2^*(\mathcal{E}_v)\right]$$

of rank $-\chi(v^\vee \otimes v)$. This rank is $m-2$, if S is symplectic, and $m-1$, if S is non-symplectic but Poisson. Thus $c_m(x)$ vanishes. This contradicts the non-vanishing of $c_m(x)$ proven above, so such F cannot exist.

A universal sheaf $\mathcal{E}_v$ exists always over $S \times \mathbb{P}$, where $\mathbb{P}$ is a projective bundle over $\mathcal{M}_H(v)$ given in Eq. (33). The above argument generalizes, as it proves that $c_m(x) = 0$, when $F \notin M$, and $c_m(x)$ is Poincaré dual to a fiber of $\mathbb{P}$ over a point of M, when $F \in M$. $\square$

2.3. Poisson surfaces with vanishing odd cohomology

Assume that a universal sheaf $\mathcal{E}$ exists over $S \times \mathcal{M}_H(v)$. The assumption is dropped in Section 3. Assume, in addition, the following:

Condition 11. The cohomology groups $H^i(S, \mathbb{Z})$ vanish, for odd i.

Equivalently, S is either a K3, or a smooth projective rational Poisson surface. The case of a general Poisson surface will be treated in Section 2.5. Condition 11 implies, that $K^1_{\mathrm{top}}(S)$ vanishes, the group $K^0_{\mathrm{top}}(S)$ is free of rank equal to $H^*(S, \mathbb{Z})$, and the Chern character

$$ch: K^0_{\mathrm{top}}(S) \to H^*(S, \mathbb{Q})$$

is an injective homomorphism [4, p. 19].

Given any cell complex M, the Künneth Theorem provides an isomorphism

$$K^0_{\mathrm{top}}(S) \otimes K^0_{\mathrm{top}}(M) \cong K^0_{\mathrm{top}}(S \times M),$$

given by the exterior product [3, Corollary 2.7.15]. We used here the vanishing of $K^1_{\mathrm{top}}S$. Use a basis $\{x_1, \ldots, x_n\}$ of $K^0_{\mathrm{top}}(S)$ to write the Künneth decomposition of the class of the universal sheaf:

$$\mathcal{E} \equiv \sum_{i=1}^{n} x_i \otimes e_i. \tag{10}$$

Proposition 12.

1. *The cohomology ring $H^*(\mathcal{M}_H(v), \mathbb{Z})$ is generated by the Chern classes $c_j(e_i)$ of the classes $e_i \in K^0_{\mathrm{top}}\mathcal{M}_H(v)$, which are given in Eq.* (10)*.*
2. *The cohomology group $H^i(\mathcal{M}_H(v), \mathbb{Z})$ vanishes, for odd i, and is torsion free, when i is even.*

Proof. 1. Set $\mathcal{M} := \mathcal{M}_H(v)$. The proof is similar to that of [16, Theorem 2.1]. The projection formula (3) yields the equivalence in $K^0_{\mathrm{top}}(\mathcal{M} \times \mathcal{M})$

$$-\pi_{13_!}\left(\pi_{12}^*(\mathcal{E})^\vee \overset{L}{\otimes} \pi_{23}^*(\mathcal{E})\right) \equiv -\sum_{i=1}^n \sum_{j=1}^n p_1^!\left(e_i^\vee\right) \cup p_2^!(e_j) \cup \pi_{13_!}\pi_2^!\left(x_i^\vee \cup x_j\right), \tag{11}$$

where p_i is the projection from $\mathcal{M} \times \mathcal{M}$ on the ith factor, $i = 1, 2$, and π_{ij} are the projections from $\mathcal{M} \times S \times \mathcal{M}$. Property (4) of the Gysin homomorphism implies the equality $\pi_{13_!}\pi_2^!(x_i^\vee \cup x_j) = \chi(x_i^\vee \cup x_j) \cdot 1$. We define the *Mukai pairing* on $K^0_{\mathrm{top}}(S)$ by

$$(x, y) := -\chi\left(x^\vee \cup y\right). \tag{12}$$

It is known to be a perfect pairing (see also Remark 18). The equivalence (11) becomes

$$-\pi_{13_!}\left(\pi_{12}^*(\mathcal{E})^\vee \overset{L}{\otimes} \pi_{23}^*(\mathcal{E})\right) \equiv \sum_{i=1}^n \sum_{j=1}^n (x_i, x_j) p_1^!\left(e_i^\vee\right) \cup p_2^!(e_j).$$

Theorem 8 translates to the equality

$$\delta = c_m\left(\sum_{i=1}^n \sum_{j=1}^n (x_i, x_j) p_1^!\left(e_i^\vee\right) \cup p_2^!(e_j)\right), \tag{13}$$

where δ is Poincaré dual to the class of the diagonal in $\mathcal{M} \times \mathcal{M}$. Clearly, the mth Chern class, on the right-hand side, can be written as a sum

$$\delta = \sum_{j \in J} p_1^*\alpha_j \cup p_2^*\beta_j, \tag{14}$$

where each α_j and β_j is a polynomial, with integral coefficients, in the Chern classes of e_i, for $1 \leqslant i \leqslant n$.

We have the well-known formula

$$x = p_{1,*}(\delta \cup p_2^* x), \tag{15}$$

where x is a class in $H^*(\mathcal{M}, \mathbb{Z})$. We recall the proof of that formula. Let $[\mathcal{M}] \in H_{2m}(\mathcal{M}, \mathbb{Z})$ and $[\mathcal{M} \times \mathcal{M}] \in H_{4m}(\mathcal{M} \times \mathcal{M}, \mathbb{Z})$ be the orientation classes and $[\Delta] \in H_{2m}(\mathcal{M} \times \mathcal{M}, \mathbb{Z})$ the class of the diagonal. The Poincaré Duality isomorphism, sending δ to $[\Delta]$, is induced by the cap product

$$(\bullet) \cap [\mathcal{M} \times \mathcal{M}] : H^*(\mathcal{M} \times \mathcal{M}, \mathbb{Z}) \xrightarrow{\cong} H_*(\mathcal{M} \times \mathcal{M}, \mathbb{Z}).$$

The Poincaré dual of Eq. (15) follows from the following equalities:

$$\begin{aligned} p_{1,*}\left\{(\delta \cup p_2^* x) \cap [\mathcal{M} \times \mathcal{M}]\right\} &= p_{1,*}\left\{p_2^* x \cap \left(\delta \cap [\mathcal{M} \times \mathcal{M}]\right)\right\} = p_{1,*}\left\{p_2^* x \cap [\Delta]\right\} \\ &= p_{1,*}\left\{p_1^* x \cap [\Delta]\right\} = x \cap [\mathcal{M}]. \end{aligned}$$

Using Eq. (15), the projection formula, and the decomposition (14), we express x as a linear combination of the α_j's:

$$x = \sum_{j \in J} \left(\int_{\mathcal{M}} x \cup \beta_j \right) \alpha_j. \tag{16}$$

2. Part 2 follows from the vanishing of the coefficient $\int_{\mathcal{M}} x \cup \beta_j$ in (16), whenever x is a torsion class or a class of odd degree. □

Example 13. Let $S = \mathbb{P}^2$ with the basis $\{y_1, y_2, y_3\}$ of $K^0_{\mathrm{alg}}\mathbb{P}^2$ given by $y_i := \mathcal{O}_{\mathbb{P}^2}(-i)$. The dual basis $\{x_1, x_2, x_3\}$, with respect to the pairing $\chi(x_i \cup y_j)$, is given by

$$x_1 := \mathcal{O}_{\mathbb{P}^2}(-2) - 3\mathcal{O}_{\mathbb{P}^2}(-1) - \mathcal{O}_{\mathbb{P}^2}, \quad x_2 := \mathcal{O}_{\mathbb{P}^2}(-1) - 3\mathcal{O}_{\mathbb{P}^2}, \quad \text{and} \quad x_3 := \mathcal{O}_{\mathbb{P}^2}.$$

It is well known, that $\alpha : K^0_{\mathrm{alg}}\mathbb{P}^2 \to K^0_{\mathrm{top}}\mathbb{P}^2$ is an isomorphism (see part 5 of Lemma 17). Let f_i be the projection from $\mathbb{P}^2 \times \mathcal{M}$ onto the ith factor, $i = 1, 2$. Then e_i in (10) is equivalent to $f_{2,!}(\mathcal{E} \cup f_1^! y_i)$. We conclude, that the classes e_i in Proposition 12 are given by

$$e_i = f_{2,!}\big(\mathcal{E} \otimes f_1^! \mathcal{O}_{\mathbb{P}^2}(-i)\big), \quad \text{for } 1 \leqslant i \leqslant 3.$$

These are precisely the classes in $K^0_{\mathrm{alg}}\mathcal{M}$ chosen by Ellingsrud and Strømme in their version of the statement of part 1 of Proposition 12 above (they worked with K^0_{alg} and the Chow-ring, instead of K^0_{top} and integral cohomology). The similarity is not a coincidence; the proof of Proposition 12 is modeled after that of [16, Theorem 2.1].

2.4. A decomposition of the diagonal in the Chow ring

We prove Theorem 2 in this section. Assume, that S is a smooth, projective, rational and Poisson surface. Let $v \in H^*(S, \mathbb{Z})$ be a vector satisfying Condition 7. The cohomological decomposition (14), of the diagonal class, has an analogue in the Chow ring $A^*(\mathcal{M} \times \mathcal{M})$ of $\mathcal{M} \times \mathcal{M}$ (Theorem 2).

Definition 14. Let X be an algebraic variety and $\Delta \subset X \times X$ the diagonal. We say, that X *admits a decomposition of the diagonal in* $K^0_{\mathrm{alg}}(X \times X)$, if the class of $\mathcal{O}_\Delta$ is in the image of $K^0_{\mathrm{alg}}X \otimes K^0_{\mathrm{alg}}X \to K^0_{\mathrm{alg}}(X \times X)$.

The decomposition in the definition exists, when X is a projective space $\mathbb{P}^n$ or the projectivization $\mathbb{P}E$ of a vector bundle E over $\mathbb{P}^n$ [18, Example 15.1.1]. Special cases include the minimal rational surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and the Hirzebruch surfaces.

Lemma 15. *Every smooth and projective rational surface S admits a decomposition of the diagonal in $K^0_{\mathrm{alg}}(S \times S)$.*

Lemma 15 follows from Lemma 16. Let X be a smooth complex algebraic surface, $\beta : \widehat{X} \to X$ the blow-up of X at a closed point, $E \subset \widehat{X}$ the exceptional divisor, $\delta : \Delta \to X \times X$, $\hat{\delta} : \widehat{\Delta} \to \widehat{X} \times \widehat{X}$, the diagonals, and $\pi_i : \widehat{X} \times \widehat{X} \to \widehat{X}$, $i = 1, 2$, the projections.

Lemma 16. *The following equivalence holds in $K^0_{\mathrm{alg}}(\widehat{X} \times \widehat{X})$:*

$$\hat{\delta}_! \mathcal{O}_{\widehat{\Delta}} \equiv (\beta \times \beta)^! [\delta_! \mathcal{O}_\Delta] - \bigl\{\pi_1^! \bigl[\mathcal{O}_{\widehat{X}}(E) - \mathcal{O}_{\widehat{X}}\bigr] \cup \pi_2^! \bigl[\mathcal{O}_{\widehat{X}}(E) - \mathcal{O}_{\widehat{X}}\bigr]\bigr\}. \tag{17}$$

Consequently, if X admits a decomposition of the diagonal in $K^0_{\mathrm{alg}}(X \times X)$, then the same is true when we replace X by its blow-up $\widehat{X}$.

Note, that $(\beta \times \beta)_! \circ (\beta \times \beta)^! = id$, $\eta := (\beta \times \beta)^! \circ (\beta \times \beta)_!$ is an idempotent, and the right-hand side of (17) is compatible with the decomposition $\mathrm{Im}(\eta) \oplus \ker(\eta)$ of $K^0_{\mathrm{alg}}(\widehat{X} \times \widehat{X})$.

Proof. Let $\Delta_E \subset E \times E$ be the diagonal. Set $Z := \widehat{X} \cup [E \times E]$. The inclusions of $E \times E$, Z, and Δ_E into $\widehat{X} \times \widehat{X}$ are all denoted by ι. The desired decomposition (17) follows from the following equivalences:

$$\iota_! \mathcal{O}_Z \equiv (\beta \times \beta)^! [\delta_! \mathcal{O}_\Delta], \tag{18}$$

$$\iota_! \mathcal{O}_Z \equiv \hat{\delta}_! \mathcal{O}_{\widehat{\Delta}} + \iota_! \mathcal{O}_{E \times E} - \iota_! \mathcal{O}_{\Delta_E}, \tag{19}$$

$$\iota_! \mathcal{O}_{E \times E} \equiv \pi_1^! \bigl[1 - \mathcal{O}_{\widehat{X}}(-E)\bigr] \cup \pi_2^! \bigl[1 - \mathcal{O}_{\widehat{X}}(-E)\bigr], \tag{20}$$

$$\iota_! \mathcal{O}_{\Delta_E} \equiv \bigl\{1 - \bigl[\pi_1^! \mathcal{O}_{\widehat{X}}(E) \cup \pi_2^! \mathcal{O}_{\widehat{X}}(E)\bigr]\bigr\} \cup \iota_! \mathcal{O}_{E \times E}. \tag{21}$$

We prove (18) first. Z is the total transform of Δ in $\widehat{X} \times \widehat{X}$ and

$$\iota_* \mathcal{O}_Z = (\beta \times \beta)^* (\delta_* \mathcal{O}_\Delta).$$

We need to prove the vanishing of the torsion sheaves $\mathcal{T}or_i^{\mathcal{O}_{X \times X}}(\mathcal{O}_{\widehat{X} \times \widehat{X}}, \delta_* \mathcal{O}_\Delta)$, $i \geqslant 1$. Both Δ and Z have codimension 2. Δ is smooth and is hence a local complete intersection. Hence, so is Z. Furthermore, local equations g_1, g_2 of Δ in $X \times X$ pull back to local equations of Z, which form a regular sequence [31, Chapter 6, Section 16, Theorem 31]. Thus, the Koszul complex, locally resolving the sheaf $\delta_* \mathcal{O}_\Delta$, pulls back via $\beta \times \beta$ to a Koszul complex locally resolving the sheaf $\iota_* \mathcal{O}_Z$ [31, Chapter 7, Section 18, Theorem 43]. The vanishing of the higher torsion sheaves follows.

Let I_{Δ_E} be the ideal sheaf of Δ_E in $E \times E$. The embedding $E \times E \hookrightarrow Z$ pushes forward I_{Δ_E} to the ideal sheaf of $\widehat{\Delta}$ in Z. Equivalence (19) follows. Equivalence (20) is clear. Note, that $\mathcal{O}_{\widehat{X}}(E)$ restricts to E as $\mathcal{O}_E(-1)$. Hence, $1 - [\pi_1^! \mathcal{O}_{\widehat{X}}(E) \cup \pi_2^! \mathcal{O}_{\widehat{X}}(E)]$ is sent to the class of $\mathcal{O}_{\Delta_E}$ via the composition

$$K_{\mathrm{alg}}(\widehat{X}) \otimes K_{\mathrm{alg}}(\widehat{X}) \to K_{\mathrm{alg}}(E) \otimes K_{\mathrm{alg}}(E) \to K_{\mathrm{alg}}(E \times E).$$

Equivalence (21) now follows from the projection formula (3). □

Lemma 17. *Let X be a projective algebraic variety, which admits a decomposition*

$$\mathcal{O}_{\Delta_X} \equiv \sum_{i \in I} p_1^! x_i \cup p_2^! y_i \tag{22}$$

of the diagonal in $K^0_{\mathrm{alg}}(X \times X)$. Then the following statements hold.

1. *The x_i generate $K^0_{\mathrm{alg}}X$.*
2. *$K^0_{\mathrm{alg}}X$ is a finitely generated free $\mathbb{Z}$-module.*
3. *Suppose that the set $\{x_i\}_{i\in I}$ is minimal. Then $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are dual bases, with respect to the pairing $\chi(x_i \cup y_j)$.*
4. *Exterior product*

$$\mu : K^0_{\mathrm{alg}}(X) \otimes K^0_{\mathrm{alg}}(M) \to K^0_{\mathrm{alg}}(X \times M)$$

is an isomorphism, for every algebraic variety M.
5. *$\alpha : K^0_{\mathrm{alg}}X \to K^0_{\mathrm{top}}X$ is an isomorphism.*
6. *If X is smooth and projective, and we replace assumption* (22) *by its Chow ring analogue*

$$[\Delta] \equiv \sum_{i\in I} p_1^*\alpha_i \cup p_2^*\beta_i \tag{23}$$

in $A^*(X \times X)$, then the statements above hold, after replacing K^0_{alg} by the Chow ring, $K^0_{\mathrm{top}}X$ by $H^*(X,\mathbb{Z})$, and the pairing $\chi(x_i \cup y_j)$ by the intersection pairing $\int_X \alpha_i \cup \beta_j$.

Proof. The proof is again similar to that of [16, Theorem 2.1]. Let p_i be the projection from $X \times X$, f_i the projection from $X \times M$, and π_{ij} the projection from $X \times X \times M$ onto the product of the ith and jth factors. Everything follows from the evident formula

$$x \equiv p_{1,!}\big(\mathcal{O}_{\Delta_X} \cup p_2^! x\big),$$

for any x in $K^0_{\mathrm{alg}}X$. Using the projection formula (3) and the decomposition (22), we get

$$x \equiv \sum_{i\in I} \chi(x \cup y_i)x_i. \tag{24}$$

1. Part 1 follows from Eq. (24).

2. For part 2 observe, that if $x \in K^0_{\mathrm{alg}}X$ is a torsion class, then the coefficients $\chi(x \cup y_i)$ in Eq. (24) vanish for all i.

3. The minimality assumption implies, that the x_i are linearly independent. The statement follows by setting $x = x_j$ in Eq. (24).

4. Let $\mathcal{E}$ be a class in $K^0_{\mathrm{alg}}(X \times M)$. The projection formula (3) yields the decomposition

$$\mathcal{E} \equiv \pi_{13_!}\big(\pi_{12}^!\mathcal{O}_{\Delta_X} \cup \pi_{23}^!\mathcal{E}\big) \equiv \sum_{i\in I} f_1^! x_i \cup \pi_{13_!}\big(\pi_2^! y_i \cup \pi_{23}^!\mathcal{E}\big).$$

The Cohomology and Base Change Theorem, applied to the fiber product

$$\begin{tikzcd} X \times X \times M \arrow[r, "\pi_{23}"] \arrow[d, "\pi_{13}"'] & X \times M \arrow[d, "f_2"] \\ X \times M \arrow[r, "f_2"] & M, \end{tikzcd}$$

implies the second equality below

$$\pi_{13_!}\big(\pi_2^! y_i \cup \pi_{23}^! \mathcal{E}\big) \equiv \pi_{13_!}\pi_{23}^!\big(f_1^! y_i \cup \mathcal{E}\big) \equiv f_2^! f_{2_!}\big(f_1^! y_i \cup \mathcal{E}\big).$$

Consequently, $\mathcal{E} \equiv \sum_{i\in I} f_1^! x_i \cup f_2^! f_{2_!}(f_1^! y_i \cup \mathcal{E})$ and the exterior product is surjective.

We prove injectivity next. Choose dual bases as in part 3. Suppose $\mathcal{E} := \sum_{i\in I} x_i \otimes e_i$ is in the kernel of μ. The projection formula (3) yields

$$0 = f_{2_!}\big(f_1^! y_j \cup \mu(\mathcal{E})\big) = f_{2_!}\left(f_1^! y_j \cup \left[\sum_{i\in I} f_1^! x_i \cup f_2^! e_i\right]\right) = \sum_{i\in I} \chi(y_j \cup x_i) e_i = e_j.$$

Hence, all the e_j vanish.

5. Surjectivity of $\alpha : K^0_{\mathrm{alg}} X \to K^0_{\mathrm{top}} X$ follows from Eq. (24), interpreted in $K^0_{\mathrm{top}} X$. Injectivity follows from the vanishing of $\chi(x \cup y_i)$ in Eq. (24), for a class x in the kernel of α.

6. The proof of part 6 is a straightforward translation of the above proofs. We include only the translation of the proof of part 5. Once we replace Eq. (14) by Eq. (23), then Eq. (16) expresses every cohomology class x in $H^*(X,\mathbb{Z})$ as a linear combination of classes coming from $A^*(X)$. The surjectivity of $A^*(X) \to H^*(X,\mathbb{Z})$ follows. When the class x belongs to $A^*(X)$, Eq. (16) holds in $A^*(X)$. The injectivity follows from the vanishing of the coefficients $\int_X x \cup \beta_j$ in Eq. (16), when x is a class in the kernel of $A^*(X) \to H^*(X,\mathbb{Z})$. $\square$

Remark 18. The topological Mukai pairing (12) is a perfect pairing. This follows from the argument in the proof of part 3 of Lemma 17, provided Eq. (22) is taken in $K^0_{\mathrm{top}}[S \times S]$.

Proof of Theorem 2. There exists a class x in $K^0_{\mathrm{alg}}(S)$, such that $\chi(x \cup v) = 1$, by part 3 of Lemma 17. Hence, a universal sheaf exists [34]. The decomposition (10), of the universal sheaf, can be taken in $K^0_{\mathrm{alg}}(S \times \mathcal{M})$, by part 4 of Lemma 17. Consequently, the decomposition (14), of the diagonal in $\mathcal{M} \times \mathcal{M}$, is given in the Chow ring $A^*(\mathcal{M} \times \mathcal{M})$. The fact, that $A^*(\mathcal{M}) \to H^*(\mathcal{M},\mathbb{Z})$ is an isomorphism, follows from parts 5 and 6 of Lemma 17. $\square$

2.5. Non-simply connected Poisson surfaces

We omit Condition 11 and consider any smooth, projective, symplectic or Poisson surface S.

We need to review first some background from topological K-theory. Let X be a connected topological space. The reduced K-group $\widetilde{K}^0_{\mathrm{top}}(X)$ is the kernel of the restriction homomorphism $K^0_{\mathrm{top}}(X) \to K^0_{\mathrm{top}}(x_0)$, where x_0 is a point of X. Given two topological spaces X and Y, with points $x_0 \in X$ and $y_0 \in Y$, we set $X \vee Y := X \times \{y_0\} \cup \{x_0\} \times Y$ and $X \wedge Y := X \times Y / X \vee Y$. Denote by SX the reduced suspension of X, $SX := S^1 \wedge X$, where s_0 is a fixed point of the circle S^1. Let X^+ be the disjoint union of X and a point x_0. The notation SX^+ stands for $S(X^+)$. Note, that $S(X^+) = S^1 \times X / \{s_0\} \times X$. The associativity of the operation $\wedge$ yields $S(S(X^+)) = S^2 \times X/\{s_0\} \times X$ and similarly for the nth iterate $S^n(X^+) = S^n \times X/\{s_0\} \times X$. Recall, that the odd K-group $K^1_{\mathrm{top}}(X)$ is defined to be $\widetilde{K}^0_{\mathrm{top}}(SX^+)$ (see [3]). The latter group is naturally isomorphic to the kernel of

$$K^0_{\mathrm{top}}\big(S^1 \times X\big) \xrightarrow{\iota^*} K^0_{\mathrm{top}}\big(\{s_0\} \times X\big),$$

where s_0 is a fixed point in S^1 and $\iota : \{s_0\} \times X \hookrightarrow S^1 \times X$ is the natural embedding [3, Corollary 2.4.7]. Set

$$K^*_{\mathrm{top}}(X) := K^0_{\mathrm{top}}(X) \oplus K^1_{\mathrm{top}}(X).$$

Let X and Y be smooth complex algebraic varieties. A proper morphisms $f : X \to Y$ extends to a proper continuous map $\tilde{f} : SX \to SY$. Gysin maps are defined, more generally, for proper morphism between differentiable manifolds, with even-dimensional fibers, satisfying an additional condition; existence and choice of a certain relative ^{c}spinorial structure (see [23, Proposition IV.5.24 and Remark IV.5.27]). When f is a proper morphism between complex manifolds X and Y, a natural ^{c}spinorial structure exists for f, as well as for the suspension $\tilde{f}$ (use [23, Theorem II.4.8] for the latter). Consequently, we get a Gysin map $f_! := \tilde{f}_! : K^1_{\mathrm{top}}(X) \to K^1_{\mathrm{top}}(Y)$. The projection formula (3) and property (4) extend for classes in K^1_{top}.

The exterior product $K^0_{\mathrm{top}}(X) \otimes K^1_{\mathrm{top}}(Y) \to K^1(X \times Y)$ is defined as follows. There is a natural map $q : X \times SY^+ \to S(X \times Y)^+$. The image of the exterior product $K^0_{\mathrm{top}}(X) \otimes \widetilde{K}^0_{\mathrm{top}}(SY^+) \to \widetilde{K}^0(X \times SY^+)$ is contained in the image of $\widetilde{K}^0[S(X \times Y)^+]$ via $q^!$. Since $q^!$ is injective, the exterior product has values in $K^1(X \times Y)$. The exterior product

$$K^1_{\mathrm{top}}(X) \otimes K^1_{\mathrm{top}}(Y) \to K^0(X \times Y)$$

is defined, using the natural map $q^2 : SX^+ \times SY^+ \to S^2(X \times Y)^+$ and Bott's Periodicity Theorem, which implies the isomorphism

$$\beta : K^0[X \times Y] \xrightarrow{\cong} \widetilde{K}^0\big[S^2(X \times Y)^+\big]$$

(see Theorem 2.4.9 and Section 2.6 in [3]). Under the inclusion of $\widetilde{K}^0[S^2(X \times Y)^+]$ in $K^0[S^2 \times X \times Y]$, the isomorphism β sends a class α in $K^0[X \times Y]$ to the exterior product of α with the generator of $\widetilde{K}^0(S^2)$.

Definition 19. Let x be a class in $K^1_{\mathrm{top}}(X)$, corresponding to a class $\tilde{x}$ in $\widetilde{K}^0_{\mathrm{top}}(SX^+)$, and $i \geqslant 1/2$ a half-integer. The Chern class $c_i(x)$ of x is defined as the image in $H^{2i}(X, \mathbb{Z})$ of $c_{i+1/2}(\tilde{x})$, via the isomorphism

$$H^{2i}(X, \mathbb{Z}) \cong H^{2i}(X^+, \mathbb{Z}) \cong H^{2i+1}(SX^+, \mathbb{Z}).$$

One extends, similarly, the Chern character to $K^*_{\mathrm{top}}X$. The Chern character is a linear homomorphism, which preserves products and commutes with pullbacks. It maps $K^1_{\mathrm{top}}X$ into $H^{\mathrm{odd}}(X, \mathbb{Q})$. When $H^*(X, \mathbb{Z})$ is torsion free, then so is $K^*_{\mathrm{top}}X$. Furthermore,

$$ch : K^*_{\mathrm{top}}X \otimes \mathbb{Q} \to H^*(X, \mathbb{Q})$$

is an isomorphism ([4] and [23, V.3.26]). The Künneth Theorem holds as well [3, Theorem 2.7.15]. When $H^*(X, \mathbb{Z})$ is torsion free, it states that the exterior product

$$\big[K^0_{\mathrm{top}}X \otimes K^0_{\mathrm{top}}M\big] \oplus \big[K^1_{\mathrm{top}}X \otimes K^1_{\mathrm{top}}M\big] \xrightarrow{\cong} K^0_{\mathrm{top}}[X \times M]$$

is an isomorphism, for any cell complex M.

The cohomology group $H^*(S,\mathbb{Z})$ is torsion free, for a smooth projective symplectic or Poisson surface, by the classification of such surfaces [7]. Consequently, the exterior product (1) is an isomorphism.

Proposition 20. *Theorem* 1 *holds, under the additional assumption that a universal sheaf $\mathcal{E}$ exists over $S \times \mathcal{M}_H(v)$.*

Proof. The proof of Proposition 12 extends to the more general setup. In formula (11) we implicitly used the Künneth decomposition of $\mathcal{E}^\vee = \sum_{i=1}^n x_i^\vee \otimes e_i^\vee$. When x_i and e_i are classes in K^1_{top}, the exterior product $f_1^! x_i^\vee \cup f_2^! e_i^\vee$ need not be equal to $(f_1^! x_i \cup f_2^! e_i)^\vee$, if we interpret ${}^\vee : K^1_{\mathrm{top}} \to K^1_{\mathrm{top}}$ as the duality operator on the suspension. Instead, we avoid relating the Künneth decompositions of $\mathcal{E}$ and $\mathcal{E}^\vee$, write $[\mathcal{E}^\vee] = \sum_{i=1}^n x_i \otimes e_i'$, replace the Mukai pairing by $(x,y) := -\chi(x \cup y)$, and replace $e_i^\vee$ by e_i' in (13).

Somewhat delicate is the analogue of the statement, that the classes β_j, in the decomposition (14), are polynomials, with integral coefficients, in the Chern classes of the e_i, for $1 \leqslant i \leqslant n$. The analogous statement follows from Lemma 21. □

Lemma 21. *Let $x, y \in K^1_{\mathrm{top}} X$ and d an integer $\geqslant 1$. The class $c_d(x \cup y)$ in $H^{2d}(X,\mathbb{Z})$ can be written as a polynomial, with integral coefficients, in the even-dimensional classes $c_{i+1/2}(x) \cup c_{k-i-1/2}(y)$, for $0 \leqslant i \leqslant k-1$ and $1 \leqslant k \leqslant d$.*

The proof of Lemma 21 will depend on Lemmas 22 and 23. Let $\mathcal{P}_n$, $n \geqslant 1$, be the set of descending partitions $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \cdots$, $\lambda_i \geqslant 0$, $\sum_{i=1}^\infty \lambda_i = n$. The length $\ell(\lambda)$ is $\max\{i\colon \lambda_i \neq 0\}$. The multiplicity $m_i := m_i(\lambda)$ of i in λ is the number of j with $\lambda_j = i$.

Lemma 22. *Let X be a topological space and x a class in $K^*_{\mathrm{top}}(X)$. Write $x = y + z$, where $y \in K^0_{\mathrm{top}}(X)$ and $z \in K^1_{\mathrm{top}}(X)$. Let $ch_i(x)$ be the degree $2i$ summand of $ch(x)$ in $H^{2i}(X,\mathbb{Q})$. Let $k \geqslant 1$ be an integer.*

1. $(-1)^k (k-1)! ch_k(x) = \displaystyle\sum_{i_1+2i_2+\cdots+ki_k=k} (-1)^{i_1+\cdots+i_k} \frac{(i_1+\cdots+i_k-1)!}{i_1!\cdots i_k!} c_1(y)^{i_1}\cdots c_k(y)^{i_k}.$
2. $ch_{k-1/2}(x) = \dfrac{(-1)^{k-1}}{(k-1)!} c_{k-1/2}(z).$
3. $c_k(y) = \displaystyle\sum_{\lambda\in\mathcal{P}_k} (-1)^{k-\ell(\lambda)} \prod_{i\geqslant 1} \frac{[(i-1)! ch_i(y)]^{m_i}}{m_i!}.$

Proof. 1. The equality $ch_k(x) = ch_k(y)$ follows from the linearity of the Chern character. $ch_k(y)$ is a polynomial in $c_1(y), \ldots, c_{i-1}(y)$. It suffices to calculate its coefficients when the class y is represented by a vector bundle. For such y, the equality is Girard's formula [32, Chapter 16, Problem 16-A].

2. Let $\tilde{z} \in \widetilde{K}^0_{\mathrm{top}}(SX^+)$ be the class corresponding to z. The Chern classes $c_i(\tilde{z})$ belong to the reduced cohomology $\widetilde{H}^*(SX^+)$, with integral coefficients. Let S^1 be the 1-sphere and $u \in H^1(S^1,\mathbb{Z})$ a generator. Choose a base point $s_0 \in S^1$. Recall the short exact sequence:

$$0 \to \widetilde{H}^*(SX^+) \to \widetilde{H}^*(S^1 \times X) \to \widetilde{H}^*(\{s_0\} \times X) \to 0$$

[3, Corollary 2.4.7]. In particular, the image of $\widetilde{H}^*(SX^+)$ is the principal ideal generated in $\widetilde{H}^*(S^1 \times X)$ by $u \otimes 1$. The cup product $\cup: \widetilde{H}^*(SX^+) \otimes \widetilde{H}^*(SX^+) \to \widetilde{H}^*(SX^+)$ vanishes, since $u \cup u = 0$ in $H^*(S^1)$. Consequently, $c_i(\tilde{z}) \cup c_j(\tilde{z}) = 0$, for any two positive integers i, j. Part 1 of the lemma implies the equality

$$ch_i(\tilde{z}) = \frac{(-1)^{i-1}}{(i-1)!} c_i(\tilde{z}). \tag{25}$$

Let π_i be the projection from $S^1 \times X$ to the ith factor, $i = 1, 2$. Let $v \in K^1_{\mathrm{top}}(S^1)$ be the generator with $c_{1/2}(v) = u$. Then

$$\tilde{z} = \pi_1^!(v) \cup \pi_2^!(z), \quad ch(\tilde{z}) = \pi_1^* ch(v) \cup \pi_2^* ch(z), \quad \text{and} \quad ch(v) = u.$$

We get the equality $ch_i(\tilde{z}) = \pi_1^*(u) \cup \pi_2^* ch_{i-1/2}(z)$. Part 2 of the lemma follows from the latter equality and Eq. (25).

3. See [27, Chapter I, Section 2, Eq. (2.14′)]. □

Lemma 23. [2, Lemma 2] *Let X be a finite CW-complex. Then X can be embedded as a subcomplex of a finite CW-complex A, so that both $H^*(A, \mathbb{Z})$ and $K^*(A)$ are free abelian groups and $K^*(A) \to K^*(X)$ is surjective.*

Proof of Lemma 21. If $f: X \to A$ is continuous, $x = f^!(x')$, $y = f^!(y')$ and Lemma 21 holds for the class $c_d(x' \cup y')$, then the lemma holds also for the class $c_d(x \cup y)$. We may thus assume that $H^*(X, \mathbb{Z})$ is free, by Lemma 23.

Part 2 of Lemma 22 and the multiplicative property of the Chern character yield:

$$(d-1)! ch_d(x \cup y) = (-1)^{d-1} \sum_{i=0}^{d-1} \binom{d-1}{i} c_{i+1/2}(x) \cup c_{d-i-1/2}(y). \tag{26}$$

Hence $(d-1)! ch_d(x \cup y)$ belongs to the subring of $H^*(X, \mathbb{Z})$ generated by $c_{i-1/2}(x) \cup c_{j-1/2}(y)$, $1 \leqslant i, j \leqslant d$. The powers $[c_{i-1/2}(x) \cup c_{j-1/2}(y)]^k$ vanish for $k \geqslant 2$. We get the identity

$$\begin{aligned} &\bigl[(d-1)! ch_d(x \cup y)\bigr]^m \\ &\quad = m!(-1)^{m(d-1)} \sum_{0 \leqslant k_1 < k_2 < \cdots < k_m \leqslant d-1} \prod_{i=1}^{m} \left[\binom{d-1}{k_i} c_{k_i+1/2}(x) \cup c_{d-k_i-1/2}(y)\right]. \end{aligned}$$

Lemma 21 follows from the above equality and part 3 of Lemma 22. □

3. Generators in the absence of a universal sheaf

We complete the proof of Theorem 1 in this section, dropping the assumption that a universal sheaf exists. We construct a universal class e in $K^0_{\mathrm{top}}(S \times \mathcal{M})$, regardless of the existence of an algebraic universal sheaf (Definition 26). The details are worked out in Section 3.1. In Section 3.2 we prove the following proposition. Let S, v, H satisfy the assumptions of Theorem 8, with one exception: a universal sheaf need not exist.

Proposition 24.

1. *Using the notation of Theorem* 8, *the class*

$$c_m\big[-\pi_{13_!}\big(\pi_{12}^!(e)^\vee \cup \pi_{23}^!(e)\big)\big] \tag{27}$$

is Poincaré-dual to the class of the diagonal, where e is the class in Definition 26.

2. *When S is a K3 or abelian surface, the following vanishing holds*

$$c_{m-1}\big[-\pi_{13_!}\big(\pi_{12}^!(e)^\vee \cup \pi_{23}^!(e)\big)\big] = 0. \tag{28}$$

An immediate consequence is:

Corollary 25. *Theorem* 1 *holds also in the absence of a universal sheaf, once we replace in* (2) *the class $\mathcal{E}$, of the universal sheaf, by the class e in Definition* 26.

Following is a summary of the construction of the universal class e. Let X be a topological space, a complex analytic space, or a scheme over $\mathbb{C}$ endowed with the étale topology. Denote by $\mathcal{O}_X^*$ the sheaf of invertible complex valued continuous, holomorphic, or algebraic functions. Let θ be a Čech 2-cocycle with coefficients in $\mathcal{O}_X^*$. There is a notion of θ-twisted vector bundles over X (Definition 27). One can define the K-group $K^0(X)_\theta$ of θ-twisted vector bundles. The topological version $K^0_{\mathrm{top}}(X)_\theta$ is defined in [6,15]. For the analytic $K^0_{\mathrm{hol}}(X)_\theta$ or algebraic $K^0_{\mathrm{alg}}(X)_\theta$ see [11,14]. $K^0(X)_\theta$ is a $K^0(X)$-module. $K^0(X)_\theta$ depends only on the Čech cohomology class of θ, canonically up to tensorization by the class in $K^0(X)$ of a line-bundle. Note, that in the topological category $H^2(X, \mathcal{O}_X^*)$ is isomorphic to $H^3(X, \mathbb{Z})$, via the connecting homomorphism of the exponential sequence.

Let S be a K3, abelian, or a smooth projective Poisson surface, and $\mathcal{M}_H(v)$ a moduli space as in Theorem 1. Over $S \times \mathcal{M}_H(v)$ there always is a twisted universal sheaf $\mathcal{E}_v$. The twisting cocycle is the pullback $f_2^*\theta$ of some Čech 2-cocycle θ, with coefficient in the sheaf $\mathcal{O}^*_{\mathcal{M}_H(v)}$, in the étale or classical topology of $\mathcal{M}_H(v)$ (see Appendix in [34] or Section 3.1 below). The twisted universal sheaf $\mathcal{E}_v$ determines a class in $K^0_{\mathrm{hol}}(\mathcal{M}_H(v) \times S)_{f_2^*\theta}$. Let $\bar{\theta}$ be the image of θ, as a Čech 2-cocycle with coefficient in the sheaf of invertible continuous functions. We prove that $\bar{\theta}$ is a coboundary (Lemma 28). It follows that $K_{\mathrm{top}}(\mathcal{M}_H(v) \times S)_{f_2^*\bar{\theta}}$ is isomorphic to the untwisted group $K_{\mathrm{top}}(\mathcal{M}_H(v) \times S)$, canonically up to tensorization of the latter with a topological line-bundle on $\mathcal{M}_H(v)$.

Definition 26. The universal class e is the image of $\mathcal{E}_v$ under the composition

$$K_{\mathrm{hol}}\big(\mathcal{M}_H(v) \times S\big)_{f_2^*\theta} \to K_{\mathrm{top}}\big(\mathcal{M}_H(v) \times S\big)_{f_2^*\bar{\theta}} \to K_{\mathrm{top}}\big(\mathcal{M}_H(v) \times S\big).$$

The details are worked out in Section 3.1.

3.1. A universal class in $K^0_{\mathrm{top}}(S \times \mathcal{M})$

Let S be a K3, abelian, or a smooth projective Poisson surface. The moduli space $\mathcal{M}_H(v)$, with a v-generic polarization H, always admits a twisted universal sheaf (see Appendix in [34]).

There exists a covering $\mathcal{U} := \{U_\alpha\}_{\alpha \in I}$ of $\mathcal{M}_H(v)$, in the étale or classical topology, universal sheaves $\mathcal{E}_\alpha$ over $S \times U_\alpha$, isomorphisms

$$g_{\alpha\beta} : (\mathcal{E}_\beta)|_{[S \times U_{\alpha\beta}]} \xrightarrow{\cong} (\mathcal{E}_\alpha)|_{[S \times U_{\alpha\beta}]},$$

such that the cocycle

$$(\delta g)_{\alpha\beta\gamma} := g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha}$$

comes from a 2-cocycle θ in $Z^2(\mathcal{U}, \mathcal{O}^*_{\mathcal{M}_H(v)})$.

Given a class in $K^0_{\mathrm{alg}}(S)$, represented by a complex F, we get the line-bundle $L_\alpha := \det f_{2_!}(\mathcal{E}_\alpha \otimes f_1^* F)$ on U_α. The isomorphisms $g_{\alpha\beta}$ induce isomorphisms

$$g^F_{\alpha\beta} : (L_\beta)|_{U_{\alpha\beta}} \xrightarrow{\cong} (L_\alpha)|_{U_{\alpha\beta}}, \tag{29}$$

satisfying $(\delta g^F)_{\alpha\beta\gamma} = \theta^{\chi(v \cup F)}_{\alpha\beta\gamma}$. Consequently, $\theta^{\chi(v \cup F)}$ is a coboundary. We conclude, that θ^n is a coboundary, where the natural number n is given by

$$n := \gcd\bigl\{\chi(v \cup w) \colon\ w \in K^0_{\mathrm{alg}}(S)\bigr\}. \tag{30}$$

If $n = 1$, then θ is a coboundary $\theta = \delta\eta$, for some one-cochain $\eta \in C^1(\mathcal{U}, \mathcal{O}^*_{\mathcal{M}_H(v)})$. Then the new transition functions

$$\varphi_{\alpha\beta} := g_{\alpha\beta} \cdot \eta^{-1}_{\alpha\beta} \tag{31}$$

glue the local universal sheaves $\mathcal{E}_\alpha$ to a global universal sheaf. In general, we get only a $f_2^*\theta$-twisted sheaf $\mathcal{E}_v$, in the following sense.

Definition 27. Let X be a scheme or a complex analytic space, $\mathcal{U} := \{U_\alpha\}_{\alpha \in I}$ a covering, open in the complex or étale topology, and $\theta \in Z^2(\mathcal{U}, \mathcal{O}^*_X)$ a Čech 2-cocycle. A *θ-twisted sheaf* consists of sheaves $\mathcal{E}_\alpha$ of $\mathcal{O}_{U_\alpha}$-modules over U_α, for all $\alpha \in I$, and isomorphisms $g_{\alpha\beta} : (\mathcal{E}_\beta)|_{U_{\alpha\beta}} \to (\mathcal{E}_\alpha)|_{U_{\alpha\beta}}$ satisfying the conditions:

(1) $g_{\alpha\alpha} = id$,
(2) $g_{\alpha\beta} = g^{-1}_{\beta\alpha}$,
(3) $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \theta_{\alpha\beta\gamma} \cdot id$.

The θ-twisted sheaf is *coherent*, if the $\mathcal{E}_i$ are.

Locally free θ-twisted sheaves of finite rank form an abelian category, with the obvious notion of homomorphisms, and we let $K^0_{\mathrm{hol}}(X)_\theta$ be its K-group. Observe, that the determinant $\det(\mathcal{E})$, of a θ-twisted locally free sheaf $\mathcal{E}$ of rank r, is a θ^r-twisted line-bundle. Thus, θ^r is a coboundary. Consequently, the order of the class $[\theta]$, of θ in $H^2(X, \mathcal{O}^*_X)$, divides the rank of every θ-twisted locally free sheaf $\mathcal{E}$. $K^0_{\mathrm{hol}}(X)_\theta$ is thus trivial, if $[\theta]$ has infinite order. For a more general definition of twisted K-groups and derived categories, see [14] and references therein.

Let θ be a Čech 2-cocycle of continuous invertible complex valued functions. The twisted topological K-groups are defined for a class $[\theta]$ of arbitrary order [6]. When the order of $[\theta]$ is finite, the obvious analogue of the above definition yields a group $K^0_{\mathrm{top}}(X)_\theta$, which is naturally isomorphic to the one defined in [6].

We define the class $[\mathcal{E}_v]$ of the twisted universal sheaf $\mathcal{E}_v$ in $K^0_{\mathrm{hol}}(S \times \mathcal{M}_H(v))_{f_2^*\theta}$ using the following twisted locally free resolution of $\mathcal{E}_v$. Choose a sufficiently ample line-bundle H' on S and set

$$W_\alpha := f_{2_*}(\mathcal{E}_\alpha \otimes f_1^* H') \quad \text{and} \quad \mathcal{E}_{\alpha,0} := f_1^*(H')^{-1} \otimes f_2^* W_\alpha.$$

There is a natural evaluation homomorphism $ev_\alpha : \mathcal{E}_{\alpha,0} \to \mathcal{E}_\alpha$. We choose H' sufficiently ample, so that $R^i f_{2_*}(\mathcal{E}_\alpha \otimes f_1^* H')$ vanishes, for $i > 0$, each $\mathcal{E}_{\alpha,0}$ is locally free, and each ev_α is surjective. The gluing transformations $g_{\alpha\beta}$, of the $\mathcal{E}_\alpha$, induce gluing transformations $\psi_{\alpha\beta}$ of the vector bundles W_α, whose coboundary is diagonal $\theta_{\alpha\beta\gamma} \cdot id$. We denote the corresponding θ-twisted locally free sheaf by

$$W. \tag{32}$$

Similarly, the pullback $f_2^*(\psi_{\alpha\beta})$ is a 1-cochain of gluing transformations for the $\mathcal{E}_{\alpha,0}$, defining a $f_2^*\theta$-twisted vector bundle $\mathcal{E}_0$. Repeating the process once more, starting with $\mathcal{E}_0$, we get a surjective homomorphism $\mathcal{E}_1 \to \mathcal{E}_0$, with kernel $\mathcal{E}_2$, where $\mathcal{E}_i$, $0 \leqslant i \leqslant 2$, are locally free $f_2^*\theta$-twisted sheaves. The class $\sum (-1)^i [\mathcal{E}_i]$ is independent of the choices made and is denoted by $[\mathcal{E}_v]$.

When the cocycle θ is a coboundary, $\theta = \delta(\eta)$, the procedure described in Eq. (31) induces a well defined isomorphism $K^0_{\mathrm{hol}}(X)_\theta \to K^0_{\mathrm{hol}}(X)$.

Let $\mathcal{A}^*$ be the sheaf of continuous, complex valued, and invertible functions on $\mathcal{M}_H(v)$.

Lemma 28. *The homomorphism $Z^2(\mathcal{U}, \mathcal{O}^*_{\mathcal{M}_H(v)}) \to Z^2(\mathcal{U}, \mathcal{A}^*)$ maps the cocycle θ to a coboundary $\bar{\theta}$.*

The general idea for the proof of Lemma 28 is clear; the number n in Eq. (30) becomes 1, once we replace $K^0_{\mathrm{alg}} S$ by $K^0_{\mathrm{top}} S$. Assume the existence of the Gysin map

$$f_{2_!} : K^0_{\mathrm{top}}\big(S \times \mathcal{M}_H(v)\big)_{f_2^*\bar{\theta}} \to K^0_{\mathrm{top}}\big(\mathcal{M}_H(v)\big)_{\bar{\theta}}$$

in twisted topological K-theory, and choose $x \in K^0_{\mathrm{top}} S$ satisfying $\chi(x \cup v) = 1$. Then the class $f_{2_!}(f_1^!(x) \cup [\mathcal{E}_v])$ in $K^0_{\mathrm{top}}(\mathcal{M}_H(v))_{\bar{\theta}}$ has rank 1. But the order of the cohomology class of $\bar{\theta}$ divides the rank of any class in $K^0_{\mathrm{top}}(\mathcal{M}_H(v))_{\bar{\theta}}$. Hence $\bar{\theta}$ is a coboundary. We avoid using Gysin maps in a twisted version of topological K-theory, as we are unfamiliar with such a construction in the literature. The following elementary lemma provides an alternative proof of the triviality of $\bar{\theta}$. The lemma summarizes well-known facts about twisted sheaves, and will be needed also in the next section.

Lemma 29. *Let $E := (E_\alpha, g_{\alpha\beta})$ be a θ-twisted sheaf and $F := (F_\alpha, \psi_{\alpha\beta})$ a θ-twisted locally free sheaf over an analytic space X.*

1. *The projective bundles $\mathbb{P}F_\alpha$ glue to a global projective bundle $\phi : \mathbb{P}F \to X$.*

2. *The image of θ in $H^3(X,\mathbb{Z})$, under the connecting homomorphism of the exponential sequence, is the topological obstruction for $\mathbb{P}F$ to lift to a complex topological vector bundle.*
3. *There exists a line bundle L over $\mathbb{P}F \times_X \mathbb{P}F$, which restricts as $\mathcal{O}(1,-1)$ to each fiber $\mathbb{P}F_x \times \mathbb{P}F_x$, over a point $x \in X$.*
4. *There exists a sheaf $\widetilde{E}$ over $\mathbb{P}F$, which restricts to $\mathbb{P}F_\alpha$ as $\phi^* E_\alpha \otimes \mathcal{O}_{\mathbb{P}F_\alpha}(-1)$, satisfying*

$$p_1^*(\widetilde{E}) \otimes L = p_2^* \widetilde{E},$$

where $p_i : \mathbb{P}F \times_X \mathbb{P}F \to \mathbb{P}F$, $i = 1,2$, are the projections.
5. *The cocycle $\phi^*\theta$ is a coboundary, and the class of $\widetilde{E}$ in $K^0_{\mathrm{hol}}(\mathbb{P}F)$ is the image of E via the isomorphism $K^0_{\mathrm{hol}}(\mathbb{P}F)_{\phi^*\theta} \to K^0_{\mathrm{hol}}(\mathbb{P}F)$, determined by a suitable choice of a 1-cochain $\tilde{\psi}$ satisfying $\delta(\tilde{\psi}) = \phi^*\theta$.*

Proof. Part 1 is clear and part 2 is standard [11].

Proof of parts 3, 4, and 5. Let $\tau_\alpha := \mathcal{O}_{\mathbb{P}F_\alpha}(-1)$ be the tautological line sub-bundle over $\mathbb{P}F_\alpha$. The pullback $\phi^*\psi_{\alpha\beta}$ to $\mathbb{P}F$ restricts to an isomorphism

$$\tilde{\psi}_{\alpha\beta} : (\tau_\beta)|_{\mathbb{P}F_\alpha \cap \mathbb{P}F_\beta} \to (\tau_\alpha)|_{\mathbb{P}F_\alpha \cap \mathbb{P}F_\beta},$$

defining a 1-cochain $\tilde{\psi}$ satisfying $\delta(\tilde{\psi}) = \phi^*\theta$. The local sheaves $\phi^* E_\alpha \otimes \tau_\alpha^{-1}$, over $\mathbb{P}F_\alpha$, are glued to a global sheaf $\widetilde{E}$ over $\mathbb{P}F$ via the transformations $\phi^* g_{\alpha\beta} \otimes \tilde{\psi}_{\alpha\beta}^{-1}$. The cocycle $p_1^*\tilde{\psi}_{\alpha\beta} \otimes p_2^*\tilde{\psi}_{\alpha\beta}^{-1}$ over $\mathbb{P}F \times_X \mathbb{P}F$ glues $p_1^*\tau_\alpha \otimes p_2^*\tau_\alpha^{-1}$ to the global line-bundle L. □

Let

$$\phi : \mathbb{P} \to \mathcal{M}_H(v) \tag{33}$$

be a projective bundle, corresponding to a θ-twisted vector bundle, such as W given in (32). Let $\widetilde{\mathcal{E}}$ be the sheaf over $S \times \mathbb{P}$, corresponding to the twisted universal sheaf $\mathcal{E}_v$, via Lemma 29. Denote by $\tilde{f}_i$ the projection from $S \times \mathbb{P}$ on the ith factor, $i = 1,2$.

Proof of Lemma 28. It suffices to prove that $\mathbb{P}$ is the projectivization of a topological vector bundle, by Lemma 29, part 2. This, in turn, is equivalent to the surjectivity of the restriction homomorphism $H^2(\mathbb{P},\mathbb{Z}) \to H^2(\mathbb{P}_m,\mathbb{Z})$, where $\mathbb{P}_m$ is a fiber of $\mathbb{P}$ over a point m in $\mathcal{M}_H(v)$, by a well known criterion.[1]

A class $x \in K^0_{\mathrm{top}}(S)$, satisfying $\chi(x \cup v) = 1$, exists by Remark 18, since v is primitive. Set $y := \tilde{f}_{2_!}[\tilde{f}_1^!(x) \cup \widetilde{\mathcal{E}}]$. Then y is a class of rank 1 in $K^0_{\mathrm{top}}(\mathbb{P})$ satisfying the equality

$$p_1^!(y) \cup \ell = p_2^!(y),$$

[1] Details are included in the proof of Lemma 15, part 1 in the e-print version math.AG/0406016 v1 of this paper.

where $\ell \in K^0_{\mathrm{top}}(\mathbb{P} \times \mathbb{P})$ is the class of the line bundle L in Lemma 29, and p_i, $i = 1, 2$, are the projections from $\mathbb{P} \times \mathbb{P}$. The above equality is verified via the following sequence of simpler equalities. Let π_{ij} be the projections from $\mathbb{P} \times \mathbb{P} \times S$.

$$p_1^!\big(\tilde{f}_{2!}\big[\tilde{f}_1^!(x) \cup \widetilde{\mathcal{E}}\big]\big) \cup \ell = \pi_{12!}\big(\pi_3^!(x) \cup \pi_{13}^!(\widetilde{\mathcal{E}}) \cup \pi_{12}^!\ell\big) = \pi_{12!}\big(\pi_3^!(x) \cup \pi_{23}^!(\widetilde{\mathcal{E}})\big) = p_2^!(y).$$

The last equality follows from property (4) for Gysin maps and for the first use also the projection formula (3). The second equality follows from part 4 of Lemma 29. We conclude, that $c_1(y)$ restricts to each fiber $\mathbb{P}_m$ as a generator of $H^2(\mathbb{P}_m, \mathbb{Z})$. □

As a consequence of Lemma 28, we can choose a 1-cochain $\eta \in C^1(\mathcal{U}, \mathcal{A}^*)$, satisfying $\theta = \delta\eta$. Modifying the transition functions $g_{\alpha\beta}$, of each $f_2^*\bar{\theta}$-twisted vector bundle, as in Eq. (31), we get an isomorphism

$$K^0_{\mathrm{top}}\big(S \times \mathcal{M}_H(v)\big)_{f_2^*\bar{\theta}} \to K^0_{\mathrm{top}}\big(S \times \mathcal{M}_H(v)\big).$$

Use this isomorphism in Definition 26 to map the class of $\mathcal{E}_v$ in $K^0_{\mathrm{hol}}(S \times \mathcal{M}_H(v))_{f_2^*\theta}$ to a class

$$e \in K^0_{\mathrm{top}}\big[S \times \mathcal{M}_H(v)\big]. \tag{34}$$

3.2. Decomposition of the diagonal via a universal class

Proof of Proposition 24. 1. We will prove the equality of the class (27) and the class of the diagonal, by pulling back both via the homomorphism

$$(\phi \times \phi)^* : H^*\big(\mathcal{M}_H(v) \times \mathcal{M}_H(v), \mathbb{Z}\big) \to H^*(\mathbb{P} \times \mathbb{P}, \mathbb{Z}), \tag{35}$$

and comparing both pullbacks to a third class (36). $\mathbb{P}$ is the projectivization of a global topological vector bundle, by Lemma 28. The injectivity of $\phi^* : H^*(\mathcal{M}_H(v), \mathbb{Z}) \to H^*(\mathbb{P}, \mathbb{Z})$, and hence of (35), follows [23, Proposition V.3.12].

An extension of Theorem 8, carried out in [29, Section 3], implies that the pullback by $\phi \times \phi$, of the diagonal in $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$, is Poincaré dual to the class

$$c_m\big[-\tilde{\pi}_{13!}\big(\tilde{\pi}_{12}^!(\widetilde{\mathcal{E}})^\vee \cup \tilde{\pi}_{23}^!(\widetilde{\mathcal{E}})\big)\big], \tag{36}$$

where $\tilde{\pi}_{ij}$ is the projection from $\mathbb{P} \times S \times \mathbb{P}$ onto the product of the ith and jth factors. Furthermore, the vanishing

$$c_{m-1}\big[-\tilde{\pi}_{13!}\big(\tilde{\pi}_{12}^!(\widetilde{\mathcal{E}})^\vee \cup \tilde{\pi}_{23}^!(\widetilde{\mathcal{E}})\big)\big] = 0, \tag{37}$$

holds when S is a K3 or abelian surface (Eq. (6)).

The class $[\widetilde{\mathcal{E}}]$ of $\widetilde{\mathcal{E}}$ in $K^0_{\mathrm{top}}(S \times \mathbb{P})$ is the image of the class $(id_S \times \phi)^! \mathcal{E}_v$ via the composite homomorphism

$$K^0_{\mathrm{hol}}(S \times \mathbb{P})_{\phi^*(\theta)} \to K^0_{\mathrm{top}}(S \times \mathbb{P})_{\phi^*(\bar{\theta})} \to K^0_{\mathrm{top}}(S \times \mathbb{P}),$$

by Lemma 29, part 5. The same holds for the class $(id_S \times \phi)^! e$, by its definition (34). Hence, there exists a topological line bundle F over $\mathbb{P}$, satisfying

$$[\widetilde{\mathcal{E}}] = F \cup (id_S \times \phi)^! e. \tag{38}$$

We claim, that $\phi \times \phi$ pulls back the class (27) to the class (36). The proposition follows from this claim, since the homomorphism (35) is injective. The claim follows from the equivalence (38) and the invariance of the class (36), under replacement of $\widetilde{\mathcal{E}}$ by $\widetilde{\mathcal{E}} \otimes F^{-1}$. The invariance follows from Eq. (9) in Lemma 9, the vanishing (37) when the surface S is K3 or abelian, and the following rank calculation.

The class $-\tilde{\pi}_{13_!}(\tilde{\pi}_{12}^!(\widetilde{\mathcal{E}})^\vee \cup \tilde{\pi}_{23}^!(\widetilde{\mathcal{E}}))$ in $K^0_{\mathrm{alg}}(\mathbb{P} \times \mathbb{P})$ has rank $-\chi(v^\vee \cup v)$. The dimension m of $\mathcal{M}_H(v)$ is either $2 - \chi(v^\vee \cup v)$ or $1 - \chi(v^\vee \cup v)$, depending on S being symplectic, or non-symplectic Poisson.

2. Equation (28) pulls back to Eq. (37), as seen by the equivalence (38) and Eq. (8) in Lemma 9. □

4. Higgs bundles

We sketch the proof of Theorem 3 in this section. Let us first review the geometric set-up of the proof of Theorem 7 in [29]. Set $\mathcal{H} := \mathcal{H}_\Sigma(r, d, D)$ and otherwise keep the notation introduced in the paragraph preceding Theorem 3. Let S be the projectivization of the rank 2 vector bundle $K_\Sigma(D) \oplus \mathcal{O}_\Sigma$, and $b : S \to \Sigma$ the bundle map. S is a Poisson surface. $\mathcal{H}$ is a Zariski open subset of a compact moduli space $\mathcal{M}$, of stable sheaves of pure dimension 1 on S. The moduli space $\mathcal{M}$ may be singular outside $\mathcal{H}$, as points in the boundary correspond to sheaves on S, which may not satisfy Condition 7. There exists however a resolution $\nu : \widetilde{\mathcal{M}} \to \mathcal{M}$, which is an isomorphism over $\mathcal{H}$. Furthermore, $\widetilde{\mathcal{M}}$ is projective and the restriction homomorphism from $H^*(\widetilde{\mathcal{M}}, \mathbb{Z})$ to $H^*(\mathcal{H}, \mathbb{Z})$ is surjective. Let π_{ij} be the projection from $\mathcal{H} \times S \times \widetilde{\mathcal{M}}$ onto the product of the ith and jth factors. A universal sheaf $\mathcal{F}'$ exists over $S \times \mathcal{M}$ and we denote by $\mathcal{F}$ its restriction to $S \times \mathcal{H}$. The equality $(b \times id)_!(\mathcal{F}) = [\mathcal{E}]$, of classes in $K^0_{\mathrm{alg}}(\Sigma \times \mathcal{H})$, relates the universal sheaf to the universal bundle. The class, in Borel–Moore homology, of the graph of the embedding $\iota : \mathcal{H} \hookrightarrow \widetilde{\mathcal{M}}$ is given by $c_m[-\pi_{13_!}(\pi_{12}^!(\mathcal{F})^\vee \cup \pi_{23}^!(id \times \nu)^!(\mathcal{F}'))]$.

Step 1. We claim, that the Chern classes of the Künneth factors of $\mathcal{F}$ generate the integral cohomology ring of $\mathcal{H}$. This follows from the proof of Proposition 20, modulo minor modifications due to the non-compactness of $\mathcal{H}$. One needs to replace homology groups and classes by their Borel–Moore analogues, and the Poincaré Duality isomorphism, by the isomorphism between Borel–Moore homology and singular cohomology of the smooth varieties. In Eq. (13), for example, δ is the singular cohomology class, corresponding to the class, in Borel–Moore homology, of the graph of the embedding $\iota : \mathcal{H} \hookrightarrow \widetilde{\mathcal{M}}$. Equation (14) then holds, where p_i are the projections from $\mathcal{H} \times \widetilde{\mathcal{M}}$, and each α_j (respectively β_j) is a super-symmetric polynomial, with integral coefficients, in the Chern classes of the Künneth factors of $\mathcal{F}$ (respectively $(id \times \nu)^! \mathcal{F}'$). Here we use the fact that p_1 is smooth and proper, which is the reason for the construction of the auxiliary space $\widetilde{\mathcal{M}}$. Equations (15) and (16) are replaced by

$$\iota^*(x) = p_{1,*}\big(\delta \cup p_2^*(x)\big), \quad \text{and}$$

$$\iota^*(x) = \sum_{j \in J} \Bigg(\int_{\widetilde{\mathcal{M}}} x \cup \beta_j \Bigg) \alpha_j, \tag{39}$$

for $x \in H^*(\widetilde{\mathcal{M}}, \mathbb{Z})$. We conclude that the Chern classes of the Künneth factors of $\mathcal{F}$ generate $H^*(\mathcal{H}, \mathbb{Z})$, by Eq. (39) and the surjectivity of $\iota^* : H^*(\widetilde{\mathcal{M}}, \mathbb{Z}) \to H^*(\mathcal{H}, \mathbb{Z})$.

Step 2. It remains to relate the Künneth factors of $\mathcal{F}$ to those of $\mathcal{E}$. Let h be the class in $K^*_{\mathrm{top}}(S)$ of the line bundle $\mathcal{O}_S(-1)$. Then $K^*_{\mathrm{top}}(S)$ is a free $K^*_{\mathrm{top}}(\Sigma)$-module with basis $\{1, h\}$, by [23, Chapter IV, Theorem 2.16]. Let $\tilde{f}_i$, $i = 1, 2$, be the projections from $S \times \mathcal{H}$ and f_i, $i = 1, 2$, the projections from $\Sigma \times \mathcal{H}$. The following two equalities hold for all $x \in K^*_{\mathrm{top}}(\Sigma)$:

$$\tilde{f}_{2!}\big(\tilde{f}_1^!\big[(h-1)b^!(x)\big] \cup \mathcal{F}\big) = 0, \tag{40}$$

$$\tilde{f}_{2!}\big(\tilde{f}_1^! b^!(x) \cup \mathcal{F}\big) = f_{2!}\big(f_1^!(x) \cup \mathcal{E}\big). \tag{41}$$

Equality (40) follows from the fact, that the support of $\mathcal{F}$ is the universal spectral curve, which is contained in the open subset $K_\Sigma(D) \times \mathcal{H}$ of $S \times \mathcal{H}$, where $K_\Sigma(D)$ denotes also the total space of the line-bundle. Equality (41) follows from the relation $(b \times id)_!(\mathcal{F}) = \mathcal{E}$ and the projection formula (3). Equalities (40) and (41) imply, that the Künneth factors of $\mathcal{F}$ span the same subgroup of $K^*_{\mathrm{top}}(\mathcal{H})$ as those of $\mathcal{E}$.

5. Note

Zhenbo Qin and Weiqiang Wang have recently and independently obtained bases for the integral cohomology groups (modulo torsion) of Hilbert schemes of points on a projective surface X with vanishing $H^1(X, \mathcal{O}_X)$ and $H^2(X, \mathcal{O}_X)$ [36].

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