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Revisiting an idea of Brézis and Nirenberg

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Abstract

Let $n \geqslant 3$ and Ω be a C^1 bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$. Suppose $\partial\Omega$ is C^2 at 0 and the mean curvature of $\partial\Omega$ at 0 is negative, we prove the existence of positive solutions for the equation:

$$\begin{cases} \Delta u + \lambda u^{\frac{n+2}{n-2}} + \dfrac{u^{2^*(s)-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\lambda > 0$, $0 < s < 2$, $2^*(s) = \frac{2(n-s)}{n-2}$ and $n \geqslant 4$. For $n = 3$, the existence result holds for $0 < s < 1$. Under the same assumption of the domain Ω, for $p \leqslant 2^*(s) - 1$, we also prove the existence of a positive solution for the following equation:

$$\begin{cases} \Delta u - \lambda u^p + \dfrac{u^{2^*(s)-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

where

$$\lambda > 0 \quad \text{and} \quad 1 \leqslant p < \frac{n}{n-2}.$$

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1. Introduction

The Caffarelli–Kohn–Nirenberg (CKN) inequalities assert that for all $u \in C_0^\infty(\mathbb{R}^n)$, there is a constant $C > 0$ such that

$$C\left(\int_{\mathbb{R}^n} |x|^{-bq}|u|^q\,dx\right)^{\frac{2}{q}} \leqslant \int_{\mathbb{R}^n} |x|^{-2a}|\nabla u|^2\,dx, \tag{1.1}$$

where $n \geqslant 3$,

$$-\infty < a < \frac{n-2}{2}, \qquad 0 \leqslant b-a \leqslant 1 \quad \text{and} \quad q = \frac{2n}{n-2+2(b-a)}. \tag{1.2}$$

See [2] and their generalization [12]. Suppose $\Omega \subset \mathbb{R}^n$ and $D_a^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_a^2 := \int_\Omega |x|^{-2a}|\nabla u|^2\,dx,$$

then inequality (1.1) holds for all functions in $D_a^{1,2}(\Omega)$. The corresponding best constant is defined as

$$S(a,b;\Omega) := \inf_{u \in D_a^{1,2}(\Omega)\setminus\{0\}} E_{a,b}(u), \tag{1.3}$$

where

$$E_{a,b}(u) := \frac{\int_\Omega |x|^{-2a}|\nabla u|^2\,dx}{(\int_\Omega |x|^{-bq}|u|^q\,dx)^{\frac{2}{q}}}.$$

It is easy to see that the existence of a minimizer of (1.3) is equivalent to finding a least-energy solution to the following equation:

$$\begin{cases} -\operatorname{div}\big(|x|^{-2a}\nabla u\big) = |x|^{-bq}u^{q-1}, \quad u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

To study Eq. (1.4), one could let $w(x) = |x|^{-a}u(x)$. A direct computation shows

$$\int_\Omega |x|^{-2a}|\nabla u|^2\,dx = \int_\Omega |\nabla w|^2\,dx - \gamma \int_\Omega \frac{w^2}{|x|^2}\,dx, \tag{1.5}$$

where

$$\gamma = a(n-2-a). \tag{1.6}$$

Thus, a corollary of (1.5) is that for $a < \frac{n-2}{2}$, $u \in D_a^{1,2}(\Omega)$ if and only if $w \in H_0^1(\Omega)$. Indeed, $u(x)$ is a solution of (1.4) if and only if $w(x)$ satisfies

$$\begin{cases} \Delta w + \gamma \dfrac{w}{|x|^2} + \dfrac{w^{2^*(s)-1}}{|x|^s} = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $s = (b - a)q$ and $2^*(s) = \frac{2(n-s)}{n-2}$. By (1.2), we see that $0 \leqslant s \leqslant 2$. For the problem finding minimizers of CKN inequalities and related subjects, we refer the readers to [3,7–11,13,14,16,17].

Let $L^{2^*(s)}(\Omega, |x|^{-s}\,dx)$ denote the space of $L^{2^*(s)}$-integrable functions with respect to the measure $|x|^{-s}\,dx$. By the CKN inequality, the embedding of $H_0^1(\Omega)$ in $L^{2^*(s)}(\Omega, |x|^{-s}\,dx)$ is a family of non-compact embeddings for $s \in [0, 2)$. In [9], among other things, Ghoussoub and Kang considered positive solutions of the following equation:

$$\begin{cases} \Delta u + \lambda u^p + \dfrac{u^{2^*(s)-1}}{|x|^s} = 0, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\lambda > 0$, $1 < p < \frac{n+2}{n-2}$, and Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. They proved the following result:

Theorem A. *Suppose* $0 \in \partial\Omega$ *and* $\partial\Omega$ *is* C^2 *at* 0. *Then for* $n \geqslant 4$, $\lambda > 0$ *and* $0 < s < 2$, *Eq.* (1.8) *has positive solutions if one of the following conditions is satisfied*:

(i) $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.
(ii) $1 < p < \frac{n+2}{n-2}$ *and* $\partial\Omega$ *has non-positive principal curvatures in a neighborhood of* 0.

The proof of Theorem A in [9] is based on the idea of Brézis and Nirenberg [1] where they considered the equation:

$$\begin{cases} \Delta u + \lambda\, u^p + u^{\frac{n+2}{n-2}} = 0 & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, & u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $1 < p < \frac{n+2}{n-2}$. To solve Eq. (1.9), one may consider the corresponding nonlinear functional Φ defined in $H_0^1(\Omega)$:

$$\Phi(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{p+1}\int_\Omega \left(u^+\right)^{p+1} dx - \frac{n-2}{2n}\int_\Omega \left(u^+\right)^{\frac{2n}{n-2}} dx. \tag{1.10}$$

Set

$$c_* := \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi(w), \tag{1.11}$$

where $\mathcal{P}$ denotes the class of continuous paths P connecting 0 and v, where $\Phi(v) < 0$. It is known that the functional Φ of (1.10) does not satisfy the Palais–Smale condition in $H_0^1(\Omega)$ due to the non-compactness of $H_0^1(\Omega)$ in $L^{\frac{2n}{n-2}}(\Omega)$. However, Brézis and Nirenberg [1] observed that Φ still satisfies $(P-S)_c$, provided that $c < \frac{1}{n}S_n^{\frac{n}{2}}$, where S_n is the Sobolev best constant. Thus, if $c_* < \frac{1}{n}S_n^{\frac{n}{2}}$ then c_* is a critical value. Recall that a functional Φ is said to satisfy the Palais–Smale condition $(P-S)_c$ at level c, if any sequence $u_k \in H_0^1(\Omega)$ that satisfies $\Phi(u_k) \to c$ and $\Phi'(u_k) \to 0$ in $H^{-1}(\Omega)$ as $k \to +\infty$, is necessarily relatively compact in $H_0^1(\Omega)$.

To solve (1.8), Ghoussoub and Kang [9] considered the functional:

$$\Phi_s(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{p+1}\int_\Omega \big(u^+\big)^{p+1}\,dx - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}\,dx,$$

and let c_* be the constant as defined in (1.11) by the minimax method. Similar to (1.9), Φ_s satisfies $(P-S)_{c_*}$ if

$$c_* < \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_s(\Omega)^{\frac{2^*(s)}{2^*(s)-2}}, \tag{1.12}$$

where $\mu_s(\Omega)$ is the best constant of the CKN inequality:

$$\mu_s(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2\,dx \;\middle|\; u \in H_0^1(\Omega) \text{ and } \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx = 1\right\}.$$

To illustrate how to apply the idea of Brézis and Nirenberg to Eq. (1.8), we would like to give a very brief account of the proof. Let $u_k \in H_0^1(\Omega)$ be a sequence such that $\Phi(u_k) \to c_*$ and $\Phi'(u_k) \to 0$ in $H^{-1}(\Omega)$ as $k \to +\infty$. From the assumption, it is easy to find $\|u_k\|_{H_0^1(\Omega)} \leqslant C$. Thus by passing to a subsequence of u_k, $u_k \rightharpoonup u$ in $H_0^1(\Omega)$. We have to show $u \neq 0$. Suppose $u \equiv 0$, then

$$c_* = \frac{1}{2}\lim_{k\to\infty}\int_\Omega |\nabla u_k|^2\,dx - \frac{1}{2^*(s)}\lim_{k\to\infty}\int_\Omega \frac{(u_k^+)^{2^*(s)}}{|x|^s}\,dx = \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\lim_{k\to\infty}\int_\Omega |\nabla u_k|^2\,dx.$$

On the other hand, by the definition of $\mu_s(\Omega)$, we see

$$\begin{aligned}\int_\Omega |\nabla u_k|^2\,dx &\geqslant \mu_s(\Omega)\left(\int_\Omega \frac{(u_k^+)^{2^*(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*(s)}} \\ &= \mu_s(\Omega)\left(\int_\Omega |\nabla u_k|^2\,dx\right)^{\frac{2}{2^*(s)}}\big(1+o(1)\big).\end{aligned} \tag{1.13}$$

Thus we deduce that

$$c_* = \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\lim_{k\to\infty}\int_\Omega |\nabla u_k|^2\,dx \geqslant \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_s(\Omega)^{\frac{2^*(s)}{2^*(s)-2}},$$

which violates (1.12).

This concludes that u is a nontrivial solution to (1.8). Indeed, Ghoussoub and Kang obtained positive solutions to (1.8) by showing c_* satisfies (1.12) if one of the conditions of Theorem A holds. The only difference between (1.8) and (1.9) is that the quantity $\mu_s(\Omega)$ might depend on the domain Ω, but the Sobolev best constant S_n does not.

However, in case the coefficient of u^p in (1.9) is negative, the equation reads

$$\begin{cases} \Delta u - \lambda u^p + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \text{ and } u|_{\partial\Omega} = 0, \end{cases} \tag{1.14}$$

where $\lambda > 0$. Let Φ^- be the corresponding functional:

$$\Phi^-(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{p+1}\int_\Omega \left(u^+\right)^{p+1} dx - \frac{n-2}{2n}\int_\Omega \left(u^+\right)^{\frac{2n}{n-2}} dx,$$

$u \in H_0^1(\Omega)$ and c_* be the constant of (1.11) defined by the minimax method. Then the inequality $c_* < \frac{1}{n}S_n^{n/2}$ does not hold any more. Thus the idea of Brézis and Nirenberg cannot be applied to Eq. (1.14). This is the reason why there are very few existence results for (1.14).

In this paper, we investigate the case that the coefficient of u^p in Eq. (1.8) is negative. Namely, we consider

$$\begin{cases} \Delta u - \lambda u^p + \dfrac{u^{2^*(s)-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \text{ and } u|_{\partial\Omega} = 0, \end{cases} \tag{1.15}$$

and

$$\Psi_s(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{p+1}\int_\Omega \left(u^+\right)^{p+1} dx - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s}\,dx,$$

where $\lambda > 0$ and p satisfies

$$1 \leqslant p \leqslant 2^*(s) - 1. \tag{1.16}$$

We will prove in Section 2 that Ψ_s satisfies $(P-S)_c$ for any c such that

$$0 < c < \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_s\left(\mathbb{R}^n_+\right)^{\frac{2^*(s)}{2^*(s)-2}},$$

where $\mu_s(\mathbb{R}^n_+)$ is the best constant of the CKN inequalities on the half-space $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \mid x_n > 0\}$. Note that if $0 \in \partial\Omega$ and the mean curvature $H(0) < 0$, then it was proved in [9] and [13] that

$$\mu_s(\Omega) < \mu_s\left(\mathbb{R}^n_+\right).$$

Hence, our result is a sharp improvement of the original result of Brézis and Nirenberg for such a domain. Our first result is the following theorem.

Theorem 1.1. *Let $\lambda > 0$ and Ω be a C^1 bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$. Suppose the mean curvature of $\partial\Omega$ at 0 is negative, $1 \leqslant p < \frac{n}{n-2}$ and* (1.16) *holds. Then the equation*:

$$\begin{cases} \Delta u - \lambda u^p + \dfrac{u^{2^*(s)-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.17}$$

has a positive solution.

To see why the condition (1.16) appears, we may consider a sequence $u_j \in H_0^1(\Omega)$ such that $\Psi_s(u_j) \to c_*$ and $\Psi_s'(u_j) \to 0$ in $H^{-1}(\Omega)$, i.e., u_j satisfies

$$\frac{1}{2}\int_\Omega |\nabla u_j|^2\,dx + \frac{\lambda}{p+1}\int_\Omega \left(u_j^+\right)^{p+1} dx - \frac{1}{2^*(s)}\int_\Omega \frac{(u_j^+)^{2^*(s)}}{|x|^s}\,dx = c_* + o(1), \quad \text{and}$$

$$\int_\Omega |\nabla u_j|^2\,dx + \lambda\int_\Omega \left(u_j^+\right)^{p+1} dx - \int_\Omega \frac{(u_j^+)^{2^*(s)}}{|x|^s}\,dx = o(1)\|\nabla u_j\|_{L^2(\Omega)}.$$

From the above, it can be deduced that

$$\left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\int_\Omega |\nabla u_j|^2\,dx + \left(\frac{\lambda}{p+1} - \frac{\lambda}{2^*(s)}\right)\int_\Omega \left(u_j^+\right)^{p+1} dx = c_* + o(1)\left(\|\nabla u_j\|_{L^2(\Omega)} + 1\right).$$

If $p \leqslant 2^*(s) - 1$, then the above identity implies all quantities $\|\nabla u_j\|_{L^2(\Omega)}$, $\|u_j^+\|_{L^{p+1}(\Omega)}$ and $\|(u_j^+)^{2^*}|x|^{-s}\|_{L^1(\Omega)}$ are all uniformly bounded. If $p > 2^*(s) - 1$, then $\frac{1}{p+1} - \frac{1}{2^*(s)} < 0$ and it fails to prove the boundedness of those quantities. Thus, we are unable to show Ψ_s satisfies $(P-S)_{c_*}$ when (1.16) fails.

At the moment, condition (1.16) seems to be a technical assumption. However, by performing bubbling analysis, we will see that if (1.16) fails, there might occur a competition between these two nonlinear terms u^p and $u^{2^*(s)-1}|x|^{-s}$. Due to this competition, there might be a new bubbling phenomenon other than the one related to the entire solutions of

$$\Delta w + \frac{w^{2^*(s)-1}}{|x|^s} = 0 \quad \text{in } \mathbb{R}^n_+ \text{ or } \mathbb{R}^n.$$

This new phenomenon deserves a further study in a forthcoming paper.

Our next exploration is the case of Sobolev critical exponent of (1.8), i.e.,

$$\begin{cases} \Delta u + \lambda u^{\frac{n+2}{n-2}} + \dfrac{u^{2^*(s)-1}}{|x|^s} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega \text{ and } u|_{\partial\Omega} = 0. \end{cases} \tag{1.18}$$

In this case, there are two dominant terms, since none of $\int_\Omega u_j^{\frac{2n}{n-2}}\,dx$ and $\int_\Omega \frac{u_j^{2^*(s)}}{|x|^s}\,dx$ tends to 0 as $u_j \rightharpoonup 0$ in $H_0^1(\Omega)$. Hence, a natural question arises: is there a positive number c_0 such that the corresponding functional Φ_s of (1.18) satisfies $(P-S)_c$ for any $0<c<c_0$? If yes, we can guess this c_0 should be related to the least-energy solutions in $\mathbb{R}^n_+$:

$$\begin{cases} \Delta u+\lambda u^{\frac{n+2}{n-2}}+\dfrac{u^{2^*(s)-1}}{|x|^s}=0 \quad \text{in } \mathbb{R}^n_+, \\ u>0 \quad \text{in } \mathbb{R}^n_+ \text{ and } u|_{\partial\mathbb{R}^n_+}=0. \end{cases} \tag{1.19}$$

Theorem 1.2. *For $n\geqslant 3$ and $\lambda>0$, there exists a least-energy solution of Eq.* (1.19) *provided that*

$$\begin{cases} 0<s<1 & \text{if } n=3, \\ 0<s<2 & \text{if } n\geqslant 4. \end{cases}$$

Furthermore, the least-energy solution v satisfies

$$\int_{\mathbb{R}^n_+}\left(\frac{1}{2}|\nabla v|^2-\frac{(n-2)\lambda}{2n}v^{\frac{2n}{n-2}}-\frac{1}{2^*(s)}\frac{v^{2^*(s)}}{|x|^s}\right)dx<\frac{1}{n}\lambda^{\frac{2-n}{2}}S_n^{\frac{n}{2}}. \tag{1.20}$$

By applying Theorem 1.2, we can answer the above question regarding (1.18). Let u_0 be an entire least-energy solution of (1.19), and set

$$c_0=\frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla u_0|^2\,dx-\frac{(n-2)\lambda}{2n}\int_{\mathbb{R}^n_+}u_0^{\frac{2n}{n-2}}\,dx-\frac{1}{2^*(s)}\int_{\mathbb{R}^n_+}\frac{u_0^{2^*(s)}}{|x|^s}\,dx>0. \tag{1.21}$$

In Section 4, we will prove Φ_s satisfies $(P-S)_c$ whenever c satisfies

$$c<c_0. \tag{1.22}$$

As an application, we have the following result.

Theorem 1.3. *Suppose that Ω is a C^1 bounded domain in $\mathbb{R}^n$ with $0\in\partial\Omega$. Assume further that the mean curvature of $\partial\Omega$ at 0 is negative. Then for $\lambda>0$ Eq.* (1.18) *has a positive solution provided that one of the following cases holds*:

(i) $n=3$ *and* $0<s<1$.
(ii) $n\geqslant 4$ *and* $0<s<2$.

This paper is organized as follows. In Section 2, we will give a proof of Theorem 1.1. The proof is based on the original idea of Brézis and Nirenberg, together with a blowup analysis. In Section 3, we again apply the blowup argument to show existence of positive entire solutions of Eq. (1.19). The inequality (1.20) is interesting itself and is a byproduct of the blowup analysis. Finally, in Section 4, we deal with the Sobolev critical exponent case of (1.8) for any domain Ω in $\mathbb{R}^n$ satisfying the assumptions of Theorem 1.3.

2. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. For the proof of the main theorems, we will apply the mountain pass lemma of the following type:

Theorem B. *Let Φ be a C^1 functional on a Banach space E. Assume that there exist an open set $U \subset E$ and $\rho \in \mathbb{R}$ such that $0 \in U$ and*

$$\begin{cases} \Phi(u) \geqslant \rho \quad \text{for all } u \in \partial U, \\ \Phi(0) < \rho, \quad \Phi(v) < \rho \quad \text{for some } v \notin U. \end{cases} \tag{2.1}$$

Set

$$c_* := \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi(w),$$

where $\mathcal{P}$ denotes the class of continuous paths P joining 0 to v. Then $c_* \geqslant \rho$ and there exists a sequence $\{u_j\} \subset E$ such that

$$\begin{cases} \Phi(u_j) \to c_*, \\ \Phi'(u_j) \to 0 \quad \text{in } E^*. \end{cases}$$

Now set Ψ_s to be the functional corresponding to Eq. (1.17):

$$\Psi_s(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda}{p+1}\big(u^+\big)^{p+1} - \frac{1}{2^*(s)} \frac{(u^+)^{2^*(s)}}{|x|^s} \right) dx \quad \text{for } u \in H_0^1(\Omega).$$

As discussed in the introduction, we need the inequality:

$$c_* := \inf_{P \in \mathcal{P}} \max_{w \in P} \Psi_s(w) < \left(\frac{1}{2} - \frac{1}{2^*(s)} \right) \mu_s\big(\mathbb{R}^n_+\big)^{\frac{2^*(s)}{2^*(s)-2}}, \tag{2.2}$$

where $\mathcal{P}$ denotes the class of continuous paths joining 0 to some nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ with $\Psi_s(v_0) < 0$. To prove (2.2), we use the entire solution of the following equation:

$$\begin{cases} \Delta u + \frac{u^{2^*(s)-1}}{|y|^s} = 0 \quad \text{in } \mathbb{R}^n_+, \\ u(y) > 0 \quad \text{in } \mathbb{R}^n_+, \quad u = 0 \quad \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{2.3}$$

Lemma 2.1. *Let $u \in H_0^1(\mathbb{R}^n_+)$ be an entire solution of (2.3). Then the followings hold:*

(i) $$\begin{cases} u \in C^2\big(\overline{\mathbb{R}^n_+}\big) \quad \text{if } s < 1 + \frac{2}{n}, \\ u \in C^{1,\beta}\big(\overline{\mathbb{R}^n_+}\big) \quad \text{for all } 0 < \beta < 1 \quad \text{if } s = 1 + \frac{2}{n}, \\ u \in C^{1,\beta}\big(\overline{\mathbb{R}^n_+}\big) \quad \text{for all } 0 < \beta < \frac{n(2-s)}{n-2} \quad \text{if } s > 1 + \frac{2}{n}. \end{cases}$$

(ii) *There is a constant C such that* $|u(y)| \leqslant C(1+|y|)^{1-n}$ *and* $|\nabla u(y)| \leqslant C(1+|y|)^{-n}$.
(iii) $u(y', y_n)$ *is axially symmetric with respect to the* y_n*-axis, i.e.,* $u(y', y_n) = u(|y'|, y_n)$.

For the existence of least-energy solutions of (2.3), see Ghoussoub and Robert [10]. Concerning the proof of the assertions (i)–(iii), we refer to Ghoussoub and Robert [10] and Lin and Wadade [13]. By the same fashion as used in [7], we shall use Lemma 2.1 prove inequality (2.2).

Lemma 2.2. *Suppose that Ω is a C^1 bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$ and $\partial\Omega$ is C^2 at 0. If the mean curvature of $\partial\Omega$ at 0 is negative and $1 \leqslant p < \frac{n}{n-2}$, then there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Psi_s(v_0) < 0$ and*

$$\max_{0 \leqslant t \leqslant 1} \Psi_s(t v_0) < \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) \mu_s\left(\mathbb{R}_+^n\right)^{\frac{2^*(s)}{2^*(s)-2}}.$$

Proof. Without loss of generality, we may assume that in a neighborhood of 0, $\partial\Omega$ can be represented by $x_n = \varphi(x')$, where $x' = (x_1, \dots, x_{n-1})$, $\varphi(0) = 0$, $\nabla'\varphi(0) = 0$, $\nabla' = (\partial_1, \dots, \partial_{n-1})$, and the outer normal of $\partial\Omega$ at 0 is $-e_n = (0, 0, \dots, -1)$. Define

$$\phi(x) := \left(x', x_n - \varphi\left(x'\right)\right). \tag{2.4}$$

We choose a small positive number r_0 so that there exists neighborhoods of 0, U and $\tilde{U}$, such that $\phi(U) = B_{r_0}(0)$, $\phi(U \cap \Omega) = B_{r_0}^+(0)$, $\phi(\tilde{U}) = B_{\frac{r_0}{2}}(0)$ and $\phi(\tilde{U} \cap \Omega) = B_{\frac{r_0}{2}}^+(0)$. Here, we adopt the notation:

$$B_{r_0}^+(0) = B_{r_0} \cap \mathbb{R}_+^n \quad \text{for any } r_0 > 0.$$

Suppose $w \in H_0^1(\mathbb{R}_+^n)$ is a least-energy solution of (2.3) i.e.,

$$\mu_s\left(\mathbb{R}_+^n\right) = \frac{\int_{\mathbb{R}_+^n} |\nabla w|^2 \, dy}{\left(\int_{\mathbb{R}_+^n} \frac{w^{2^*(s)}}{|y|^s} \, dy\right)^{\frac{2}{2^*(s)}}} = \left(\int_{\mathbb{R}_+^n} \frac{w^{2^*(s)}}{|y|^s} \, dy\right)^{\frac{2^*(s)-2}{2^*(s)}},$$

then $u(y) := \mu_s(\mathbb{R}_+^n)^{\frac{1}{2-2^*(s)}} w(y)$ is a positive entire solution to the equation:

$$\begin{cases} \Delta u + \mu_s\left(\mathbb{R}_+^n\right) \dfrac{u^{2^*(s)-1}}{|y|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ u = 0 \quad \text{on } \partial\mathbb{R}_+^n & \text{with } \displaystyle\int_{\mathbb{R}_+^n} \frac{u^{2^*(s)}}{|y|^s} \, dy = 1. \end{cases}$$

Let $\varepsilon > 0$. We define

$$v_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}} u\left(\frac{\phi(x)}{\varepsilon}\right) \quad \text{for } x \in \Omega \cap U, \quad \text{and} \quad \hat{v}_\varepsilon := \eta v_\varepsilon \quad \text{in } \Omega, \tag{2.5}$$

where $\eta \in C_0^\infty(U)$ is a positive cut-off function with $\eta \equiv 1$ in $\tilde{U}$. For $t \geqslant 0$, we have

$$\Psi_s(t\hat{v}_\varepsilon) \leqslant \frac{t^2}{2}\int_\Omega |\nabla \hat{v}_\varepsilon|^2\,dx + \frac{\lambda t^{\,p+1}}{p+1}\int_\Omega \hat{v}_\varepsilon^{p+1}\,dx - \frac{t^{2^*(s)}}{2^*(s)}\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*(s)}}{|x|^s}\,dx. \tag{2.6}$$

In what follows, we estimate the order of each integral on the right-hand side of (2.6). By the change of the variable $\frac{\phi(x)}{\varepsilon} = y$, we get

$$\begin{aligned}
\int_\Omega |\nabla \hat{v}_\varepsilon|^2\,dx &= \int_{\Omega\cap U} \eta^2|\nabla v_\varepsilon|^2\,dx - \int_{\Omega\cap U} \eta(\Delta\eta)v_\varepsilon^2\,dx \\
&\leqslant \int_{\mathbb{R}^n_+} \big|\nabla u(y)\big|^2\,dy - 2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y)\nabla' u(y)\cdot\big(\nabla'\varphi\big)\big(\varepsilon y'\big)\,dy \\
&\quad + \int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2\big(\partial_n u(y)\big)^2\big|\big(\nabla'\varphi\big)\big(\varepsilon y'\big)\big|^2\,dy \\
&\quad - \varepsilon^2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)(\Delta\eta)\big(\phi^{-1}(\varepsilon y)\big)u(y)^2\,dy.
\end{aligned}$$

Note that, by using $|\nabla'\phi(y')| = O(|y'|)$ and the decay estimate of $|\nabla u|$ in Lemma 2.1, we see that

$$\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2\big(\partial_n u(y)\big)^2\big|\big(\nabla'\varphi\big)\big(\varepsilon y'\big)\big|^2\,dy \leqslant C\varepsilon^2\int_{\mathbb{R}^n}\big(1+|y|\big)^{-2n}|y|^2\,dy = O\big(\varepsilon^2\big).$$

Hence,

$$\int_\Omega |\nabla \hat{v}_\varepsilon|^2\,dx = \int_{\mathbb{R}^n_+}\big|\nabla u(y)\big|^2\,dy - 2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2\partial_n u(y)\nabla' u(y)\cdot\big(\nabla'\varphi\big)\big(\varepsilon y'\big)\,dy + O\big(\varepsilon^2\big).$$

Using integration by parts and Lemma 2.1, we obtain

$$\begin{aligned}
I &:= -2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2\partial_n u(y)\nabla' u(y)\cdot\big(\nabla'\varphi\big)\big(\varepsilon y'\big)\,dy \\
&= -\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2\partial_n u(y)\nabla' u(y)\cdot\nabla'\big[\varphi\big(\varepsilon y'\big)\big]\,dy
\end{aligned}$$

$$= \frac{4}{\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)\nabla'\big[\eta\big(\phi^{-1}(\varepsilon y)\big)\big] \cdot \partial_n u(y)\, \nabla' u(y)\varphi\big(\varepsilon y'\big)\, dy$$

$$+ \frac{2}{\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \nabla'\partial_n u(y) \cdot \nabla' u(y)\varphi\big(\varepsilon y'\big)\, dy$$

$$+ \frac{2}{\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y) \sum_{i=1}^{n-1} \partial_{ii} u(y)\varphi\big(\varepsilon y'\big)\, dy$$

$$= \frac{2}{\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y) \sum_{i=1}^{n-1} \partial_{ii} u(y)\varphi\big(\varepsilon y'\big)\, dy + O\big(\varepsilon^n\big).$$

Applying Eq. (2.3) and integration by parts, we have

$$I' := \frac{2}{\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_n u(y) \sum_{i=1}^{n-1} \partial_{ii} u(y)\varphi\big(\varepsilon y'\big)\, dy$$

$$= -\frac{2\mu_s(\mathbb{R}^n_+)}{2^*(s)\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{\partial_n [u(y)^{2^*(s)}]}{|y|^s} \varphi\big(\varepsilon y'\big)\, dy$$

$$- \frac{1}{\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_n\big[\big(\partial_n u(y)\big)^2\big]\varphi\big(\varepsilon y'\big)\, dy$$

$$= -\frac{2s\mu_s(\mathbb{R}^n_+)}{2^*(s)\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{u(y)^{2^*(s)} y_n}{|y|^{s+2}} \varphi\big(\varepsilon y'\big)\, dy$$

$$+ \frac{1}{\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}} \cap \partial\mathbb{R}^n_+} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \big(\partial_n u(y)\big)^2 \varphi\big(\varepsilon y'\big)\, dS_y + O\big(\varepsilon^n\big) =: J_1 + J_2 + O\big(\varepsilon^n\big).$$

Since $\partial\Omega$ is C^2 at 0, φ can be expanded as

$$\varphi\big(y'\big) = \sum_{i=1}^{n-1} \alpha_i y_i^2 + o(1)\big(\big|y'\big|^2\big). \tag{2.7}$$

Thus we see that

$$J_1 = -\frac{2s\mu_s(\mathbb{R}^n_+)}{2^*(s)\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{u(y)^{2^*(s)} y_n}{|y|^{s+2}} \varphi\big(\varepsilon y'\big)\,dy$$

$$= -\frac{2s\mu_s(\mathbb{R}^n_+)}{2^*(s)\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}} \setminus B^+_{\frac{r_0/2}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{u(y)^{2^*(s)} y_n}{|y|^{s+2}} \varphi\big(\varepsilon y'\big)\,dy$$

$$- \frac{2s\mu_s(\mathbb{R}^n_+)}{2^*(s)\varepsilon} \int\limits_{B^+_{\frac{r_0/2}{\varepsilon}}} \frac{u(y)^{2^*(s)} y_n}{|y|^{s+2}} \varphi\big(\varepsilon y'\big)\,dy =: J_{1,1} + J_{1,2}, \quad \text{and}$$

$$|J_{1,1}| \leqslant C\varepsilon \int\limits_{\{\frac{r_0}{2} \leqslant |\varepsilon y| < r_0\}} |y|^{2^*(s)(1-n)+1-s}\,dy = O\big(\varepsilon^{\frac{n(n-s)}{n-2}}\big).$$

Notice that

$$\varepsilon \int\limits_{\mathbb{R}^n_+ \setminus B^+_{\frac{r_0/2}{\varepsilon}}} u(y)^{2^*(s)} |y|^{1-s}\,dy = O\big(\varepsilon^{\frac{n(n-s)}{n-2}}\big). \tag{2.8}$$

Thus by using (2.7) and (2.8), we get

$$J_{1,2} = -\frac{2s\varepsilon\mu_s(\mathbb{R}^n_+)}{2^*(s)} \sum_{i=1}^{n-1} \alpha_i \int\limits_{\mathbb{R}^n_+} \frac{u(y)^{2^*(s)} y_i^2 y_n}{|y|^{2+s}}\,dy\big(1+o(1)\big) + O\big(\varepsilon^{\frac{n(n-s)}{n-2}}\big)$$

$$= -\frac{2s\varepsilon\mu_s(\mathbb{R}^n_+)}{2^*(s)(n-1)} \int\limits_{\mathbb{R}^n_+} \frac{u(y)^{2^*(s)} |y'|^2 y_n}{|y|^{2+s}}\,dy \left(\sum_{i=1}^{n-1} \alpha_i\right)\big(1+o(1)\big) + O\big(\varepsilon^{\frac{n(n-s)}{n-2}}\big)$$

$$= -K_1 H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^2\big),$$

where

$$H(0) := \frac{1}{n-1} \sum_{i=1}^{n-1} \alpha_i \quad \text{and} \quad K_1 := \frac{2s\mu_s(\mathbb{R}^n_+)}{2^*(s)} \int\limits_{\mathbb{R}^n_+} \frac{u(y)^{2^*(s)} |y'|^2 y_n}{|y|^{2+s}}\,dy. \tag{2.9}$$

In the above estimate, we used the fact $u(y', y_n) = u(|y'|, y_n)$. Next, we see that

$$J_2 = \frac{1}{\varepsilon} \int\limits_{B^+_{\frac{r_0}{\varepsilon}} \cap \partial\mathbb{R}^n_+} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \big(\partial_n u(y)\big)^2 \varphi\big(\varepsilon y'\big)\,dS_y$$

$$= \frac{1}{\varepsilon} \int\limits_{(B^+_{\frac{r_0}{\varepsilon}} \setminus B^+_{\frac{r_0/2}{\varepsilon}}) \cap \partial\mathbb{R}^n_+} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \big(\partial_n u(y)\big)^2 \varphi\big(\varepsilon y'\big)\,dS_y$$

$$+\frac{1}{\varepsilon}\int\limits_{B^+_{\frac{r_0/2}{\varepsilon}}\cap\partial\mathbb{R}^n_+}\big(\partial_n u(y)\big)^2\varphi\big(\varepsilon y'\big)\,dS_y =: J_{2,1}+J_{2,2},\quad \text{and}$$

$$\begin{aligned}|J_{2,1}| &\leqslant \frac{C}{\varepsilon}\int\limits_{\{\frac{r_0}{2}<|\varepsilon y'|\leqslant r_0\}}\big|(\partial_n u)\big(y',0\big)\big|^2\big|\varphi\big(\varepsilon y'\big)\big|\,dy'\\ &\leqslant C\varepsilon\int\limits_{\{\frac{r_0}{2}<|\varepsilon y'|\leqslant r_0\}}\big|y'\big|^{-2n+2}\,dy' = O\big(\varepsilon^n\big).\end{aligned}$$

Note that

$$\varepsilon\int\limits_{\{|\varepsilon y'|>\frac{r_0}{2}\}}\big|(\partial_n u)\big(y',0\big)\big|^2\big|y'\big|^2\,dy' = O\big(\varepsilon^n\big). \tag{2.10}$$

Thus by using (2.7) and (2.10), we get

$$\begin{aligned}J_{2,2} &= \varepsilon\sum_{i=1}^{n-1}\alpha_i\int\limits_{\mathbb{R}^{n-1}}\big((\partial_n u)\big(y',0\big)\big)^2 y_i^2\,dy'\big(1+o(1)\big)+O\big(\varepsilon^n\big)\\ &= \frac{\varepsilon}{n-1}\int\limits_{\mathbb{R}^{n-1}}\big|(\partial_n u)\big(y',0\big)\big|^2\big|y'\big|^2\,dy'\sum_{i=1}^{n-1}\alpha_i + O\big(\varepsilon^2\big) = K_2 H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^n\big),\end{aligned}$$

where

$$K_2 := \int\limits_{\mathbb{R}^{n-1}}\big|(\partial_n u)\big(y',0\big)\big|^2\big|y'\big|^2\,dy'. \tag{2.11}$$

After all, we get

$$\int\limits_{\Omega}|\nabla\hat{v}_\varepsilon|^2\,dx = \mu_s\big(\mathbb{R}^n_+\big) - K_1 H(0)\big(1+o(1)\big)\varepsilon + K_2\big(1+o(1)\big)H(0)\varepsilon + O\big(\varepsilon^2\big). \tag{2.12}$$

Next, by changing the variable $\frac{\phi(x)}{\varepsilon}=y$, we have

$$\int\limits_{\Omega}\hat{v}_\varepsilon^{p+1}\,dx = \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}}\int\limits_{\mathbb{R}^n_+}u^{p+1}\,dy + O\big(\varepsilon^{\frac{n(p+1)}{2}}\big). \tag{2.13}$$

Furthermore, the integral $\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*(s)}}{|x|^s}\,dx$ can be estimated as follows. By a change of the variable $\frac{\phi(x)}{\varepsilon} = y$, we have

$$\int\limits_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*(s)}}{|x|^s}\,dx = \int\limits_{B^+_{\frac{r_0/2}{\varepsilon}}} \frac{u^{2^*(s)}}{|\frac{\phi^{-1}(\varepsilon y)}{\varepsilon}|^s}\,dy. \tag{2.14}$$

Since $\phi^{-1}(y) = (y', y_n + \varphi(y'))$, it holds $|\phi^{-1}(y)|^2 = |y|^2 + 2y_n\varphi(y') + (\varphi(y'))^2$, and then

$$\begin{aligned}
\frac{1}{|\frac{\phi^{-1}(\varepsilon y)}{\varepsilon}|^s} &= \frac{1}{|y|^s}\cdot\frac{1}{(1+\frac{2y_n\varphi(\varepsilon y')}{\varepsilon|y|^2}+\frac{(\varphi(\varepsilon y'))^2}{\varepsilon^2|y|^2})^{\frac{s}{2}}}\\
&= \frac{1}{|y|^s}\left(1 - \frac{sy_n\varphi(\varepsilon y')}{\varepsilon|y|^2} - \frac{s(\varphi(\varepsilon y'))^2}{2\varepsilon^2|y|^2}\right)\\
&\quad + \frac{1}{|y|^s}O\left(\left(\frac{2y_n\varphi(\varepsilon y')}{\varepsilon|y|^2} + \frac{(\varphi(\varepsilon y'))^2}{\varepsilon^2|y|^2}\right)^2\right).
\end{aligned} \tag{2.15}$$

Thus from (2.14) and (2.15), we obtain

$$\begin{aligned}
\int\limits_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*(s)}}{|x|^s}\,dx &= \int\limits_{\mathbb{R}^n_+} \frac{u^{2^*(s)}}{|y|^s}\,dy - \frac{s}{\varepsilon}\int\limits_{B^+_{\frac{r_0/2}{\varepsilon}}} \frac{u(y)^{2^*(s)}\,y_n\varphi(\varepsilon y')}{|y|^{2+s}}\,dy + O\big(\varepsilon^2\big)\\
&= 1 - s\varepsilon\sum_{i=1}^{n-1}\alpha_i\int\limits_{\mathbb{R}^n_+}\frac{u(y)^{2^*(s)}y_i^2y_n}{|y|^{2+s}}\,dy\big(1+o(1)\big) + O\big(\varepsilon^2\big)\\
&= 1 - \frac{s\varepsilon}{n-1}\int\limits_{\mathbb{R}^n_+}\frac{u(y)^{2^*(s)}|y'|^2y_n}{|y|^{2+s}}\,dy\left(\sum_{i=1}^{n-1}\alpha_i\right)\big(1+o(1)\big) + O\big(\varepsilon^2\big)\\
&= 1 - \frac{2^*(s)K_1}{2\mu_s(\mathbb{R}^n_+)}H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^2\big),
\end{aligned}$$

where K_1 is the same positive constant as in (2.9).

After all, each integral on the right-hand side of (2.6) can be expressed by

$$\begin{cases} \int\limits_{\Omega} |\nabla \hat{v}_\varepsilon|^2\,dx = \mu_s\big(\mathbb{R}^n_+\big) - K_1 H(0)\big(1+o(1)\big)\varepsilon + K_2 H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^2\big), \\ \int\limits_{\Omega} \hat{v}_\varepsilon^{\,p+1}\,dx = \varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} \int\limits_{\mathbb{R}^n_+} u^{p+1}\,dy + O\big(\varepsilon^{\frac{n(p+1)}{2}}\big), \\ \int\limits_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*(s)}}{|x|^s}\,dx = 1 - \frac{2^*(s)K_1}{2\mu_s(\mathbb{R}^n_+)} H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^2\big). \end{cases} \tag{2.16}$$

By (2.6) and (2.16), we have for $t \geqslant 0$,

$$\begin{aligned} \Psi_s(t\hat{v}_\varepsilon) \leqslant{}& \frac{t^2}{2}\Big(\mu_s\big(\mathbb{R}^n_+\big) - K_1 H(0)\big(1+o(1)\big)\varepsilon + K_2 H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^2\big)\Big) \\ &+ \frac{\lambda t^{p+1}}{p+1}\bigg(\varepsilon^{\frac{n+2}{2}-\frac{(n-2)p}{2}} \int\limits_{\mathbb{R}^n_+} u^{p+1}\,dy + O\big(\varepsilon^{\frac{n(p+1)}{2}}\big)\bigg) \\ &- \frac{t^{2^*(s)}}{2^*(s)}\bigg(1 - \frac{2^*(s)K_1}{2\mu_s(\mathbb{R}^n_+)} H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^2\big)\bigg). \end{aligned} \tag{2.17}$$

Since $2^*(s) > 2$ and $\frac{n+2}{2} - \frac{n-2}{2}p > 1$, there exists $T > (\frac{2^*(s)}{2}\mu_s(\mathbb{R}^n_+))^{\frac{1}{2^*(s)-2}}$ and $\varepsilon_0 > 0$, such that, for all ε with $\varepsilon_0 > \varepsilon > 0$,

$$\Psi_s(T\hat{v}_\varepsilon) < 0. \tag{2.18}$$

By (2.17), we have

$$\begin{aligned} \Psi_s(t\hat{v}_\varepsilon) \leqslant{}& \bigg(\frac{\mu_s(\mathbb{R}^n_+)}{2}t^2 - \frac{t^{2^*(s)}}{2^*(s)}\bigg) \\ &+ \bigg(\frac{(K_2 - K_1 + o(1))}{2}t^2 + \frac{(K_1 + o(1))}{2\mu_s(\mathbb{R}^n_+)}t^{2^*(s)}\bigg)H(0)\varepsilon \\ &+ \frac{\lambda t^{p+1}}{p+1}\bigg(\int\limits_{\mathbb{R}^n_+} u^{p+1}\,dy\bigg)\varepsilon^{\frac{n+2}{2}-\frac{n-2}{2}p} + O\big(\varepsilon^2\big) \\ =:{}& g_1(t) + g_2(t)H(0)\varepsilon + \frac{\lambda t^{p+1}}{p+1}\bigg(\int\limits_{\mathbb{R}^n_+} u^{p+1}\,dy\bigg)\varepsilon^{\frac{n+2}{2}-\frac{n-2}{2}p} + O\big(\varepsilon^2\big). \end{aligned} \tag{2.19}$$

Since $2^*(s) > 2$, we see that $g_1(t)$ has only one maximum value

$$g_1\big(t_1^*\big) = \bigg(\frac{1}{2} - \frac{1}{2^*(s)}\bigg)\mu_s\big(\mathbb{R}^n_+\big)^{\frac{2^*(s)}{2^*(s)-2}},$$

where at $t_1^* = \mu_s(\mathbb{R}^n_+)^{\frac{1}{2^*(s)-2}}$. Also, we see that, for ε small,

$$g_2(t_1^*) = \left(\frac{k_2 + o(1)}{2}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2}{2^*(s)-2}} > 0.$$

Hence, if $\frac{n+2}{2} - \frac{n-2}{2}p > 1$, i.e., $p < \frac{n}{n-2}$, we have $\Psi_s(t\hat{v}_\varepsilon) < g_1(t_1^*)$ for all $0 \leqslant t \leqslant T$ and for any small $\varepsilon > 0$. Therefore, by taking ε small and $v_0 = T\hat{v}_\varepsilon$, we obtain

$$\max_{0\leqslant t\leqslant 1} \Psi_s(tv_0) < \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*(s)}{2^*(s)-2}}.$$

This completes the proof of Lemma 2.2. □

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1.
Step 1. The subcritical cases
For any fixed small $\varepsilon > 0$, we let

$$\Psi_s^\varepsilon(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda}{p-\varepsilon+1}(u^+)^{p-\varepsilon+1} - \frac{1}{2^*(s)-\varepsilon}\frac{(u^+)^{2^*(s)-\varepsilon}}{|x|^s}\right)dx \quad \text{for } u \in H_0^1(\Omega),$$

where $\Psi_s^0 = \Psi_s$. It is easy to see, by Lemma 2.2, there exists $v_0 \in H_0^1(\Omega)$ such that $\Psi_s^\varepsilon(v_0) < 0$ for $0 < \varepsilon \leqslant \varepsilon_0$, and there is a constant $\rho > 0$ such that

$$\rho \leqslant c_*^\varepsilon := \inf_{P\in\mathcal{P}} \max_{w\in P} \Psi_s^\varepsilon(w) \leqslant \max_{0\leqslant t\leqslant 1} \Psi_s^\varepsilon(tv_0) < \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_s(\mathbb{R}^n_+)^{\frac{2^*(s)}{2^*(s)-2}}, \tag{2.20}$$

provided ε_0 is small. By applying the mountain pass lemma, there exists a sequence $\{u_{\varepsilon,k}\}_{k\in\mathbb{N}} \subset H_0^1(\Omega)$ such that

$$\Psi_s^\varepsilon(u_{\varepsilon,k}) \to c_*^\varepsilon \quad \text{and} \quad \left(\Psi_s^\varepsilon\right)'(u_{\varepsilon,k}) \to 0 \quad \text{in } H^{-1}(\Omega) \quad \text{as } k \to \infty, \quad \text{i.e.,}$$

$$\int_\Omega \left(\frac{1}{2}|\nabla u_{\varepsilon,k}|^2 + \frac{\lambda}{p-\varepsilon+1}(u_{\varepsilon,k}^+)^{p-\varepsilon+1} - \frac{1}{2^*(s)-\varepsilon}\frac{(u_{\varepsilon,k}^+)^{2^*(s)-\varepsilon}}{|x|^s}\right)dx$$
$$= c_*^\varepsilon + o(1) \quad \text{as } k \to \infty, \tag{2.21}$$

and

$$-\Delta u_{\varepsilon,k} + \lambda\left(u_{\varepsilon,k}^+\right)^{p-\varepsilon} - \frac{(u_{\varepsilon,k}^+)^{2^*(s)-1-\varepsilon}}{|x|^s} =: \zeta_{\varepsilon,k} \quad \text{with } \zeta_{\varepsilon,k} \to 0 \quad \text{in } H^{-1}(\Omega)$$
$$\text{as } k \to \infty. \tag{2.22}$$

Multiplying (2.22) by $u_{\varepsilon,k}$, we obtain

$$\int_\Omega \left(|\nabla u_{\varepsilon,k}|^2 + \lambda\big(u_{\varepsilon,k}^+\big)^{p-\varepsilon+1} - \frac{(u_{\varepsilon,k}^+)^{2^*(s)-\varepsilon}}{|x|^s}\right) dx = \langle \zeta_{\varepsilon,k}, u_{\varepsilon,k}\rangle. \tag{2.23}$$

From (2.21) and (2.23), we derive

$$\begin{aligned}&\left(\frac{1}{2} - \frac{1}{2^*(s)-\varepsilon}\right)\int_\Omega |\nabla u_{\varepsilon,k}|^2\,dx + \left(\frac{\lambda}{p-\varepsilon+1} - \frac{\lambda}{2^*(s)-\varepsilon}\right)\int_\Omega \big(u_{\varepsilon,k}^+\big)^{p-\varepsilon+1}\,dx \\ &\quad = c_*^\varepsilon + o(1)\big(\|u_{\varepsilon,k}\|_{H_0^1(\Omega)} + 1\big).\end{aligned} \tag{2.24}$$

Since $p \leqslant 2^*(s) - 1$, we see that $\frac{1}{p-\varepsilon+1} - \frac{1}{2^*(s)-\varepsilon} \geqslant 0$. Hence, (2.24) implies

$$\|u_{\varepsilon,k}\|_{H_0^1(\Omega)} \leqslant C, \tag{2.25}$$

where $C > 0$ is independent of $k \in \mathbb{N}$ and small $\varepsilon > 0$.

Thus, extracting a subsequence, still denoted by $u_{\varepsilon,k}$, we see that

$$\begin{cases} u_{\varepsilon,k} \rightharpoonup u_\varepsilon & \text{weakly in } H_0^1(\Omega), \\ u_{\varepsilon,k}^+ \to u_\varepsilon^+ & \text{strongly in } L^{p-\varepsilon+1}(\Omega), \\ u_{\varepsilon,k}^+ \to u_\varepsilon^+ & \text{strongly in } L^{2^*(s)-\varepsilon}\big(\Omega, |x|^{-s}\,dx\big), \end{cases} \tag{2.26}$$

as $k \to \infty$. It is easy to see that $u_\varepsilon \not\equiv 0$ in $H_0^1(\Omega)$. Indeed, if $u_\varepsilon \equiv 0$ in Ω, then

$$0 < c_*^\varepsilon = \liminf_{k\to+\infty} \frac{1}{2}\int_\Omega |\nabla u_{\varepsilon,k}|^2\,dx, \quad \text{and} \quad \lim_{k\to+\infty}\int_\Omega |\nabla u_{\varepsilon,k}|^2\,dx = 0,$$

where the identities follow from (2.21) and (2.23) respectively. Obviously, it yields a contradiction. Passing to the limit $k \to \infty$ in (2.22), u_ε satisfies

$$-\Delta u_\varepsilon + \lambda\big(u_\varepsilon^+\big)^{p-\varepsilon} - \frac{(u_\varepsilon^+)^{2^*(s)-1-\varepsilon}}{|x|^s} = 0.$$

By the maximum principle, we obtain $u_\varepsilon > 0$ in Ω. As a consequence, for any small $\varepsilon > 0$, we get that c_*^ε is a critical value for Ψ_s^ε and $u_\varepsilon \in H_0^1(\Omega)$ is a positive solution to

$$\Delta u_\varepsilon - \lambda u_\varepsilon^{p-\varepsilon} + \frac{u_\varepsilon^{2^*(s)-1-\varepsilon}}{|x|^s} = 0. \tag{2.27}$$

Thus,

$$\begin{cases} \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\,dx + \frac{\lambda}{p-\varepsilon+1}\int_\Omega u_\varepsilon^{p-\varepsilon+1}\,dx - \frac{1}{2^*(s)-\varepsilon}\int_\Omega \frac{u_\varepsilon^{2^*(s)-\varepsilon}}{|x|^s}\,dx = c_\varepsilon^*, \\ \int_\Omega |\nabla u_\varepsilon|^2\,dx + \lambda\int_\Omega u_\varepsilon^{p-\varepsilon+1}\,dx - \int_\Omega \frac{u_\varepsilon^{2^*(s)-\varepsilon}}{|x|^s}\,dx = 0. \end{cases} \tag{2.28}$$

Moreover, by (2.25), we have

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leqslant C,$$

where $C > 0$ is independent of $\varepsilon > 0$.

Next, as $\varepsilon \to 0$, by extracting a subsequence $\{u_{\varepsilon_j}\}$, we get

$$\begin{cases} u_{\varepsilon_j} \rightharpoonup u_0 & \text{weakly in } H_0^1(\Omega), \\ u_{\varepsilon_j} \to u_0 & \text{strongly in } L^{p+1}(\Omega), \\ u_{\varepsilon_j} \rightharpoonup u_0 & \text{weakly in } L^{2^*(s)}(\Omega, |x|^{-s}\,dx). \end{cases}$$

Then (2.27) yields

$$\Delta u_0 - u_0^p + \frac{u_0^{2^*(s)-1}}{|x|^s} = 0.$$

We shall prove $u_0 \neq 0$ in $H_0^1(\Omega)$. Suppose $u_0 \equiv 0$ in Ω. Up to a subsequence, we may assume

$$C_1 = \lim_{j\to\infty}\int_\Omega |\nabla u_{\varepsilon_j}|^2\,dx \quad \text{and} \quad C_2 = \lim_{j\to\infty}\int_\Omega \frac{u_{\varepsilon_j}^{2^*(s)-\varepsilon_j}}{|x|^s}\,dx. \tag{2.29}$$

Inferring from (2.28), we get

$$\frac{C_1}{2} - \frac{C_2}{2^*(s)} = c_* \quad \text{and} \quad C_1 - C_2 = 0, \quad \text{i.e.} \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)C_1 = c_*. \tag{2.30}$$

By (2.29) and (2.30), we derive

$$C_1 = C_2 \geqslant \mu_s(\Omega)^{\frac{2^*(s)}{2^*(s)-2}} \quad \text{and} \quad c_* \geqslant \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_s(\Omega)^{\frac{2^*(s)}{2^*(s)-2}}.$$

However, this lower bound of c_* is too weak to obtain a contradiction. In the followings, we shall apply the bubbling analysis to obtain a sharp lower bound for c_*. This bubbling analysis has been used for the curvature equations, see [4–6,15]. However, we have to go through for all the details because $|x|^{-s}$ has singularity at 0.

We first note $u_{\varepsilon_j}(x_{\varepsilon_j}) = \max_\Omega u_{\varepsilon_j} \to \infty$ as $j \to \infty$. Otherwise, $u_{\varepsilon_j} \to 0$ strongly in $H_0^1(\Omega)$, and we get

$$0 = \frac{1}{2} \lim_{j\to\infty} \int_\Omega |\nabla u_{\varepsilon_j}|\, dx = c_*,$$

which is a contradiction. Let

$$\kappa_j := u_{\varepsilon_j}(x_{\varepsilon_j})^{-\frac{2^*(s)-2-\varepsilon_j}{2-s}}. \tag{2.31}$$

Step 2. We claim $|x_{\varepsilon_j}| = O(\kappa_j)$ as $j \to \infty$. Suppose that, up to a subsequence, $\lim_{j\to\infty} \frac{|x_{\varepsilon_j}|}{\kappa_j} = \infty$. By scaling, set

$$v_{\varepsilon_j}(y) := \frac{u_{\varepsilon_j}(x_{\varepsilon_j} + \kappa_j y)}{u_{\varepsilon_j}(x_{\varepsilon_j})} \quad \text{for } y \in \Omega_j, \tag{2.32}$$

where

$$\Omega_j := \left\{ y \in \mathbb{R}^n \;\middle|\; x_{\varepsilon_j} + \kappa_j y \in \Omega \right\}. \tag{2.33}$$

By (2.27) and (2.31), v_{ε_j} satisfies

$$\begin{cases} \Delta v_{\varepsilon_j} - \lambda \kappa_j^2 u_{\varepsilon_j}(x_{\varepsilon_j})^{p-\varepsilon_j-1} v_{\varepsilon_j}^{p-\varepsilon_j} + \left(\dfrac{\kappa_j}{|x_{\varepsilon_j}|}\right)^s \dfrac{v_{\varepsilon_j}^{2^*(s)-1-\varepsilon_j}}{|\frac{x_{\varepsilon_j}}{|x_{\varepsilon_j}|} + \frac{\kappa_j}{|x_{\varepsilon_j}|} y|^s} = 0 \quad \text{in } \Omega_j, \\ v_{\varepsilon_j} = 0 \quad \text{on } \partial\Omega_j. \end{cases}$$

Furthermore, we have

$$\kappa_j^2 u_{\varepsilon_j}(x_{\varepsilon_j})^{p-\varepsilon_j-1} = \kappa_j^{2-\frac{(2-s)(p-\varepsilon_j-1)}{2^*(s)-2-\varepsilon_j}} \to 0 \quad \text{as } j \to \infty, \tag{2.34}$$

since $\kappa_j \to 0$ and $2 - \frac{(2-s)(p-\varepsilon_j-1)}{2^*(s)-2-\varepsilon_j} > 0$, i.e., $p < \frac{n+2}{n-2}$. Thus v_{ε_j} converges to some v smoothly in any compact set, and v satisfies $v(0) = 1$ and

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathbb{R}^n, \\ 0 \leqslant v \leqslant v(0) = 1, \end{cases} \tag{2.35}$$

provided that $\Omega_j \to \mathbb{R}^n$, or

$$\begin{cases} \Delta v = 0 \quad \text{in some half space } H, \\ v \leqslant v(0) = 1, \text{ and } v = 0 \quad \text{on } \partial H \end{cases} \tag{2.36}$$

provided that up to an affine transformation $\Omega_j \to H := \{y \in \mathbb{R}^n \mid y_n > -a\}$ for some $a > 0$.

On the other hand, we have

$$\int_{\Omega_j} v_{\varepsilon_j}^{\frac{2n}{n-2}}\,dy = \kappa_j^{\frac{n\varepsilon_j}{2^*(s)-2-\varepsilon_j}} \int_{\Omega} u_{\varepsilon_j}^{\frac{2n}{n-2}}\,dx \leqslant C,$$

and then $\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}}\,dy$ is finite. However, in the former case, by the Liouville's theorem, $v(x) \equiv 1$ for $x \in \mathbb{R}^n$. This contradicts to that $\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}}\,dy$ is finite. In the latter case, by the boundary condition on ∂H and the maximum principle, $v(x) \equiv 0$ for $x \in H$ which violates to $v(0) = 1$. Therefore we conclude $|x_{\varepsilon_j}| = O(\kappa_j)$ as $j \to \infty$.

Note that Step 2 implies that the origin is the only blow up point.

Step 3. We claim that, up to a subsequence, $\frac{x_{\varepsilon_j}}{\kappa_j} \to y_0 \neq 0$ as $j \to \infty$.

Suppose that $\frac{x_{\varepsilon_j}}{\kappa_j} \to 0$ as $j \to \infty$. As in the proof of Step 2, we define v_{ε_j} and Ω_j by (2.32) and (2.33), respectively. Then by (2.27), v_{ε_j} satisfies

$$\begin{cases} \Delta v_{\varepsilon_j} - \lambda\kappa_j^2 u_{\varepsilon_j}(x_{\varepsilon_j})^{p-\varepsilon_j-1} v_{\varepsilon_j}^{p-\varepsilon_j} + \dfrac{v_{\varepsilon_j}^{2^*(s)-1-\varepsilon_j}}{|\frac{x_{\varepsilon_j}}{\kappa_j} + y|^s} = 0 & \text{in } \Omega_j, \\ v_{\varepsilon_j} = 0 & \text{on } \partial\Omega_j. \end{cases}$$

Since by (2.34) $\kappa_j^2 u_{\varepsilon_j}(x_{\varepsilon_j})^{p-\varepsilon_j-1} \to 0$, v_{ε_j} converges to some v smoothly in any compact set in $\overline{\mathbb{R}^n_+}$, and v satisfies

$$\begin{cases} \Delta v + \dfrac{v^{2^*(s)-1}}{|y|^s} = 0 & \text{in } \mathbb{R}^n_+, \\ v = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Because $0 \in \partial\mathbb{R}^n_+$, we have $v(0) = 0$ which is a contradiction to $v(0) = 1$. Thus Step 3 is proved.

Step 4. We complete the proof of Theorem 1.1 in this step. We note after an affine transformation, v_{ε_j} converges to some v smoothly in any compact set in $\overline{\mathbb{R}^n_+}$, and v satisfies

$$\begin{cases} \Delta v + \dfrac{v^{2^*(s)-1}}{|y|^s} = 0 & \text{in } \mathbb{R}^n_+, \\ v(x) > 0 \quad \text{in } \mathbb{R}^n_+, \text{ and } v = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{2.37}$$

By (2.37), we have

$$\mu_s(\mathbb{R}^n_+) \leqslant \frac{\int_{\mathbb{R}^n_+} |\nabla v|^2\,dy}{(\int_{\mathbb{R}^n_+} \frac{v^{2^*(s)}}{|y|^s}\,dy)^{\frac{2}{2^*(s)}}} = \left(\int_{\mathbb{R}^n_+} \frac{v^{2^*(s)}}{|y|^s}\,dy\right)^{\frac{2^*(s)-2}{2^*(s)}},$$

and then

$$\int_{\mathbb{R}^n_+} |\nabla v|^2\,dy = \int_{\mathbb{R}^n_+} \frac{v^{2^*(s)}}{|y|^s}\,dy \geqslant \mu_s\big(\mathbb{R}^n_+\big)^{\frac{2^*(s)}{2^*(s)-2}}. \tag{2.38}$$

Furthermore, note that

$$\begin{aligned} C_1 &= \lim_{j\to\infty} \int_{\Omega} |\nabla u_{\varepsilon_j}|^2\,dx = \lim_{j\to\infty} \kappa_j^{-\frac{(n-2)\varepsilon_j}{2^*(s)-2-\varepsilon_j}} \int_{\Omega_j} |\nabla v_{\varepsilon_j}|^2\,dy \\ &\geqslant \lim_{j\to\infty} \int_{\Omega_j} |\nabla v_{\varepsilon_j}|^2\,dy \geqslant \int_{\mathbb{R}^n_+} |\nabla v|^2\,dy. \end{aligned} \tag{2.39}$$

Then by (2.30), (2.38) and (2.39), we derive

$$c_* \geqslant \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)C_1 \geqslant \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_s\big(\mathbb{R}^n_+\big)^{\frac{2^*(s)}{2^*(s)-2}},$$

which yields a contradiction to (2.20). Thus $u_0 \neq 0$ in $H_0^1(\Omega)$, and Theorem 1.1 is proved. □

3. Existence of entire solution in $\mathbb{R}^n_+$

In this section, we shall prove Theorem 1.2. We note that for some parameter $\lambda > 0$, the existence of least-energy solutions of (1.19) was proved in [13]. In this section, we want to prove the same result for all $\lambda > 0$. To this end, we prepare the following lemma in which the condition $0 \in \partial\Omega$ is not necessary. This lemma is also needed for the proof of Theorem 1.3 in the next section.

Lemma 3.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$. For $\lambda > 0$, if* (i) *$n = 3$ and $0 < s < 1$ or* (ii) *$n \geqslant 4$ and $0 < s < 2$, there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Phi_s(v_0) < 0$ and*

$$\max_{t\geqslant 0} \Phi_s(tv_0) < \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

where

$$\Phi_s(u) := \int_{\Omega} \left(\frac{1}{2}|\nabla u|^2 - \frac{(n-2)\lambda}{2n}\big(u^+\big)^{\frac{2n}{n-2}} - \frac{1}{2^*(s)}\frac{(u^+)^{2^*(s)}}{|x|^s}\right)dx \quad \textit{for } u \in H_0^1(\Omega)$$

and

$$S_n := \inf\left\{ \int_{\Omega} |\nabla u|^2\,dx \;\Big|\; u \in H_0^1(\Omega) \textit{ and } \int_{\Omega} |u|^{\frac{2n}{n-2}}\,dx = 1 \right\}.$$

Remark 3.2. It is well known that S_n is independent of Ω and when $\Omega = \mathbb{R}^n$, S_n is achieved by the function

$$g(x) = C\big(1+|x|^2\big)^{\frac{-(n-2)}{2}},$$

where C is a normalization constant.

Proof of Lemma 3.1. The following calculation was basically done in [1]. We include it here for the sake of completeness. Let x_0 be an interior point of Ω such that $B_{3r}(x_0) \subset \Omega$. Take $\phi(x) \in C_0^\infty(B_{2r}(x_0))$ be a cut off function with $\phi|_{B_r(x_0)} \equiv 1$. Consider $g_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}} g(\frac{x-x_0}{\varepsilon})\phi(x) \in H_0^1(\Omega)$.

For $t \geqslant 0$, we have

$$\Phi_s(tg_\varepsilon) = \frac{t^2}{2}\int_\Omega |\nabla g_\varepsilon|^2\,dx - \frac{(n-2)\lambda t^{\frac{2n}{n-2}}}{2n}\int_\Omega g_\varepsilon^{\frac{2n}{n-2}}\,dx - \frac{t^{2^*(s)}}{2^*(s)}\int_\Omega \frac{g_\varepsilon^{2^*(s)}}{|x|^s}\,dx. \tag{3.1}$$

Using the integration by parts and a change of the variable $y = \frac{x-x_0}{\varepsilon}$, we get

$$\begin{aligned}
&\int_\Omega |\nabla g_\varepsilon(x)|^2\,dx \\
&= \varepsilon^{-n}\int_{B_{2r}(x_0)} \left|(\nabla g)\left(\frac{x-x_0}{\varepsilon}\right)\right|^2 \phi^2(x)\,dx - \varepsilon^{2-n}\int_{B_{2r}(x_0)\setminus B_r(x_0)} g^2\left(\frac{x-x_0}{\varepsilon}\right)\phi\Delta\phi\,dx \\
&= \int_{B_{\frac{2r}{\varepsilon}}(0)} |\nabla g(y)|^2\phi^2(x_0+\varepsilon y)\,dy - \varepsilon^2\int_{B_{\frac{2r}{\varepsilon}}(0)\setminus B_{\frac{r}{\varepsilon}}(0)} g^2(y)\phi(x_0+\varepsilon y)\Delta\phi\,dy.
\end{aligned}$$

Direct calculation gives

$$\int_{B_{\frac{2r}{\varepsilon}}(0)} \big|\nabla g(y)\big|^2\,dy = \int_{\mathbb{R}^n} \big|\nabla g(y)\big|^2\,dy + O\big(\varepsilon^{n-2}\big),$$

$$\int_{B_{\frac{2r}{\varepsilon}}(0)\setminus B_{\frac{r}{\varepsilon}}(0)} g^2(y)\,dy = O\big(\varepsilon^{n-4}\big).$$

Hence,

$$\int_\Omega \big|\nabla g_\varepsilon(x)\big|^2\,dx = \int_{\mathbb{R}^n} \big|\nabla g(y)\big|^2\,dy + O\big(\varepsilon^{n-2}\big) \quad \text{as } \varepsilon \to 0^+.$$

Next, we have

$$\int_{\Omega} g_\varepsilon(x)^{\frac{2n}{n-2}}\,dx = \int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}} + O\big(\varepsilon^n\big),$$

and

$$\int_{\Omega} \frac{g_\varepsilon(x)^{2^*(s)}}{|x|^s}\,dx = C^{2^*(s)}\varepsilon^s \int_{B_{\frac{2r}{\varepsilon}}(0)} \frac{(1+|y|^2)^{-n+s}}{|x_0+\varepsilon y|^s}\phi(x_0+\varepsilon y)^{2^*(s)}\,dy = O\big(\varepsilon^s\big).$$

Therefore, as $T>0$ is fixed, there exists $\varepsilon_0>0$ such that

$$\begin{aligned}
\Phi_s(tg_\varepsilon) &= \frac{t^2}{2}\int_{\mathbb{R}^n}\big|\nabla g(y)\big|^2\,dy - \frac{(n-2)\lambda t^{\frac{2n}{n-2}}}{2n}\int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}}\,dy \\
&\quad - t^{2^*(s)}\varepsilon^s \int_{B_{\frac{2r}{\varepsilon}}(0)} C^{2^*(s)}\frac{(1+|y|^2)^{-n+s}}{|x_0+\varepsilon y|^s}\phi(x_0+\varepsilon y)^{2^*(s)}\,dy + O\big(\varepsilon^{n-2}\big)t^2 + O\big(\varepsilon^n\big)t^{\frac{2n}{n-2}} \\
&< \frac{t^2}{2}\int_{\mathbb{R}^n}\big|\nabla g(y)\big|^2\,dy - \frac{(n-2)\lambda t^{\frac{2n}{n-2}}}{2n}\int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}}\,dy,
\end{aligned}$$

for $0\leqslant t<T$ and all positive $\varepsilon\leqslant\varepsilon_0$, provided $0<s<n-2$, i.e., $0<s<1$ as $n=3$ and $0<s<2$ as $n\geqslant 4$.

Now, by choosing T large, we have $\Phi_s(Tg_\varepsilon)<0$ for all $\varepsilon\leqslant\varepsilon_0$. Inferring from (3.1), we see that $\Phi_s(tg_\varepsilon)<0$ for $t\geqslant T$ and all positive $\varepsilon\leqslant\varepsilon_0$. Hence, for $0<\varepsilon\leqslant\varepsilon_0$, elementary calculus gives

$$\max_{t\geqslant 0}\Phi_s(tg_\varepsilon) < \max_{t\geqslant 0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^n}\big|\nabla g(y)\big|^2\,dy - \frac{(n-2)\lambda t^{\frac{2n}{n-2}}}{2n}\int_{\mathbb{R}^n} g(y)^{\frac{2n}{n-2}}\,dy\right\} = \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.$$

By choosing $v_0 = Tg_{\varepsilon_0}$, we obtain

$$\max_{t\geqslant 0}\Phi_s(tv_0) < \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}} \quad\text{and}\quad \Phi_s(v_0)<0,$$

provided either (i) $n=3$ and $0<s<1$ or (ii) $n\geqslant 4$ and $0<s<2$. This completes the proof. □

Next, we explore the following lemma:

Lemma 3.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $n\geqslant 3$, $\lambda>0$, $0<s<2$ and $2^*(s)=\frac{2(n-s)}{n-2}$. If Ω is star-shaped about the origin, then Eq.* (1.18) *has no positive solution.*

Proof. As usual, Lemma 3.3 is a consequence of Pohozaev's identity. Multiplying (1.18) by $x \cdot \nabla u$ and ∇u respectively and taking integrations, we obtain

$$\begin{cases} -\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x + \frac{2-n}{2}\int_{\Omega}|\nabla u|^2\,dx = \frac{2-n}{2}\left(\lambda\int_{\Omega}u^{\frac{2n}{n-2}}\,dx + \int_{\Omega}\frac{u^{2^*(s)}}{|x|^s}\,dx\right), \\ \int_{\Omega}|\nabla u|^2\,dx = \lambda\int_{\Omega}u^{\frac{2n}{n-2}}\,dx + \int_{\Omega}\frac{u^{2^*(s)}}{|x|^s}\,dx, \end{cases}$$

where ν denotes the outward normal to $\partial\Omega$. Here, we derive the following Pohozaev's identity

$$\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 dS_x = 0.$$

Since Ω is star-shaped about the origin, we deduce that $\frac{\partial u}{\partial\nu} = 0$ a.e. on $\partial\Omega$. Hence,

$$\lambda\int_{\Omega}u^{\frac{n+2}{n-2}}\,dx + \int_{\Omega}\frac{u^{2^*(s)-1}}{|x|^s}\,dx = -\int_{\Omega}\Delta u\,dx = 0,$$

which implies $u \equiv 0$. $\square$

By using Lemma 3.1 and Lemma 3.3, we shall prove Theorem 1.2 in the following.

Proof of Theorem 1.2. We shall apply the blowing up argument to show the existence of positive solutions of Eq. (1.19). To do it, we let Ω be a star-shaped domain with respect to 0 and $0 \in \partial\Omega$. For any $\varepsilon > 0$, by applying Theorem B and Lemma 3.1, we can find a positive solution of

$$\begin{cases} \Delta u_\varepsilon + \lambda u_\varepsilon^{p_\varepsilon - 1} + \frac{u_\varepsilon^{2^*(s)-1-\varepsilon}}{|x|^s} = 0 \quad \text{in } \Omega, \\ u_\varepsilon > 0 \quad \text{in } \Omega \text{ and } u_\varepsilon|_{\partial\Omega} = 0, \end{cases} \tag{3.2}$$

which satisfies

$$\int_{\Omega}\left(\frac{1}{2}|\nabla u_\varepsilon|^2 - \frac{\lambda}{p_\varepsilon}u_\varepsilon^{p_\varepsilon} - \frac{1}{2^*(s)-\varepsilon}\frac{u_\varepsilon^{2^*(s)-\varepsilon}}{|x|^s}\right)dx = c_*^\varepsilon < \frac{1}{n}\lambda^{\frac{2-n}{2}}S_n^{\frac{n}{2}}, \tag{3.3}$$

where

$$p_\varepsilon := \frac{2n}{n-2} - \frac{2\varepsilon}{2-s}.$$

By applying Lemma 3.1, the proof of the above statement is the same as in [1]. Since it is standard now, the details of the proof is omitted.

Multiplying (3.2) by u_ε, we obtain

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx - \lambda\int_\Omega u_\varepsilon^{p_\varepsilon}\,dx - \int_\Omega \frac{u_\varepsilon^{2^*(s)-\varepsilon}}{|x|^s}\,dx = 0. \tag{3.4}$$

Note that by (3.3) and (3.4), one can readily derive that

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leqslant C,$$

where C is some constant independent of small $\varepsilon > 0$. Thus, by extracting a subsequence $\{u_j := u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ with $\varepsilon_j \to 0$ as $j \to \infty$, there exists a function $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } H_0^1(\Omega),\\ u_j \rightharpoonup u & \text{weakly in } L^{\frac{2n}{n-2}}(\Omega),\\ u_j \rightharpoonup u & \text{weakly in } L^{2^*(s)}\left(\Omega, \dfrac{dx}{|x|^s}\right). \end{cases}$$

Now, passing to the limit $j \to \infty$ yields that

$$\Delta u + \lambda u^{\frac{n+2}{n-2}} + \frac{u^{2^*(s)-1}}{|x|^s} = 0 \quad \text{in } \Omega.$$

However, by Lemma 3.3, the above equation has no positive solution provided that Ω is a star-shaped domain. Thus we conclude that $u \equiv 0$ in Ω. But by (3.3),

$$\frac{1}{2}\int_\Omega |\nabla u_j|\,dx > c_* > 0, \quad \text{for } j \text{ large}.$$

Therefore, u_j must blow up somewhere in $\overline{\Omega}$.

Let

$$m_j := u_j(x_j) = \max_{\overline{\Omega}} u_j(x) \to \infty \quad \text{as } j \to \infty.$$

We consider the scaling:

$$v_j(y) := m_j^{-1} u_j(x_j + k_j y) \quad \text{for } y \in \Omega_j := \left\{ z \in \mathbb{R}^n \;\middle|\; x_j + k_j z \in \Omega \right\},$$

where $k_j = m_j^{-\frac{p_j-2}{2}}$ and $p_j = \frac{2n}{n-2} - \frac{2\varepsilon_j}{2-s}$. By (3.2), v_j satisfies

$$\begin{cases} \Delta v_j + \lambda v_j^{p_j-1} + \dfrac{v_j^{2^*(s)-1-\varepsilon_j}}{|\frac{x_j}{k_j} + y|^s} = 0 & \text{in } \Omega_j,\\ v_j = 0 \quad \text{on } \partial\Omega_j. \end{cases} \tag{3.5}$$

We claim

$$0 < \liminf_{j\to\infty} \frac{|x_j|}{k_j} \leqslant \limsup_{j\to\infty} \frac{|x_j|}{k_j} < +\infty. \tag{3.6}$$

Suppose that, up to a subsequence, $\frac{|x_j|}{k_j} \to \infty$. Then $\Omega_j \to \mathbb{R}^n$ as $j \to +\infty$ and $v_j(y)$ converges to some $v(y)$ uniformly in every compact subset of $\mathbb{R}^n$. It is easy to see $v(0) = 1$ and v is the solution of the equation:

$$\Delta v + \lambda v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n.$$

It is readily checked that

$$C_1 := \lim_{j\to\infty} \int_\Omega |\nabla u_j|^2\,dx = \lim_{j\to\infty} \left(m_j^{(\frac{n-2}{2-s})\varepsilon_j} \int_{\Omega_j} |\nabla v_j|^2\,dy \right) \geqslant \int_{\mathbb{R}^n} |\nabla v|^2\,dy =: A_1, \tag{3.7}$$

$$C_2 := \lim_{j\to\infty} \int_\Omega u_j^{p_j}\,dx = \lim_{j\to\infty} \left(m_j^{(\frac{n-2}{2-s})\varepsilon_j} \int_{\Omega_j} v_j^{p_j}\,dy \right) \geqslant \int_{\mathbb{R}^n} v^{\frac{2n}{n-2}}\,dy =: A_2, \tag{3.8}$$

$$C_3 := \lim_{j\to\infty} \int_\Omega \frac{u_j^{2^*(s)-\varepsilon_j}}{|x|^s}\,dx = \lim_{j\to\infty} \left(m_j^{(\frac{n-2}{2-s})\varepsilon_j} \int_{\Omega_j} \frac{v_j^{2^*(s)-\varepsilon_j}}{|\frac{x_j}{k_j} + y|^s}\,dy \right). \tag{3.9}$$

Note that

$$\frac{C_1}{2} - \frac{(n-2)\lambda}{2n} C_2 - \frac{C_3}{2^*(s)} = c_*, \tag{3.10}$$

$$C_1 - \lambda C_2 - C_3 = 0, \tag{3.11}$$

$$A_1 = \lambda A_2. \tag{3.12}$$

By (3.7)–(3.12), we have

$$\begin{aligned} c_* &= \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) C_1 + \lambda \left(\frac{1}{2^*(s)} - \frac{n-2}{2n}\right) C_2 \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) A_1 + \lambda \left(\frac{1}{2^*(s)} - \frac{n-2}{2n}\right) A_2 \\ &= \frac{\lambda}{n} A_2. \end{aligned}$$

On the other hand, by the Sobolev inequality, we see that

$$S_n A_2^{\frac{n-2}{n}} \leqslant A_1.$$

This leads to

$$A_2 \geqslant \lambda^{-\frac{n}{2}} S_n^{\frac{n}{2}}.$$

Hence,

$$c_* \geqslant \frac{\lambda}{n} A_2 \geqslant \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

which contradicts to

$$c_* \leqslant \max_{0 \leqslant t \leqslant 1} \Phi_s(t v_0) < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.$$

Hence, we have proved

$$\limsup_{j \to \infty} \frac{|x_j|}{k_j} < +\infty.$$

Next, we want to prove

$$\liminf_{j \to \infty} \frac{|x_j|}{k_j} > 0.$$

Suppose not. Then, up to a subsequence, $\lim_{j\to\infty} \frac{|x_j|}{k_j} = 0$. In this case, up to a rotation, $\Omega_j \to \mathbb{R}^n_+$ and v_j converges to some v uniformly in any compact subset of $\overline{\mathbb{R}^n_+}$, where v is the solution of the equation:

$$\begin{cases} \Delta v + \lambda v^{\frac{n+2}{n-2}} + \dfrac{v^{2^*(s)-1}}{|y|^s} = 0 & \text{in } \mathbb{R}^n_+, \\ v = 0 \quad \text{on } \partial \mathbb{R}^n_+, \end{cases} \tag{3.13}$$

with $v(0) = 1$, which is a contradiction. Hence (3.6) holds true.

Now we may pick up a subsequence $\frac{x_j}{k_j} \to y_0 \neq 0$, then up to an affine transformation $\Omega_j \to \mathbb{R}^n_+$. Therefore, v_j converges to some v uniformly in any compact subset of $\overline{\mathbb{R}^n_+}$, where v is the solution of Eq. (3.13) with $v(y_1) = 1$ for some $y_1 \in \mathbb{R}^n_+$.

By (3.3) and (3.4), we have

$$c_*^\varepsilon = \lambda\left(\frac{1}{2} - \frac{1}{p_\varepsilon}\right) \int_\Omega u_\varepsilon^{p_\varepsilon}\, dx + \left(\frac{1}{2} - \frac{1}{2^*(s) - \varepsilon}\right) \int_\Omega u_\varepsilon^{2^*(s)-\varepsilon}\, dx.$$

Notice that

$$\int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}}\, dy \leqslant \lim_{j \to +\infty} \int_\Omega u_{\varepsilon_j}^{p_{\varepsilon_j}}\, dx, \quad \text{and} \quad \int_{\mathbb{R}^n_+} \frac{v^{2^*(s)}}{|y|^s}\, dy \leqslant \lim_{j \to +\infty} \int_\Omega \frac{u_{\varepsilon_j}^{2^*(s)-\varepsilon_j}}{|x|^s}\, dx.$$

Thus, by (3.13) and the above observation, we have

$$\begin{aligned}
\Phi_s(v) &= \int_{\mathbb{R}^n_+} \left(\frac{1}{2}|\nabla v|^2 - \frac{(n-2)\lambda}{2n} v^{\frac{2n}{n-2}} - \frac{1}{2^*(s)} \frac{v^{2^*(s)}}{|x|^s} \right) dx \\
&= \lambda \left(\frac{1}{2} - \frac{n-2}{2n} \right) \int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}}\, dx + \left(\frac{1}{2} - \frac{1}{2^*(s)} \right) \int_{\mathbb{R}^n_+} \frac{v^{2^*(s)}}{|x|^s}\, dx \\
&\leqslant \lim_{\varepsilon_j \to 0} c_*^{\varepsilon_j} < \frac{1}{n} \lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}.
\end{aligned}$$

This proves (1.20).

To complete the proof of Theorem 1.2, we need the following lemma:

Lemma 3.4. *Let $\lambda > 0$ and $u \in H_0^1(\mathbb{R}^n_+)$ be an entire positive solution of Eq.* (1.19)*. Then the followings hold*:

(i) $$\begin{cases} u \in C^2\left(\overline{\mathbb{R}^n_+}\right) & \text{if } s < 1 + \frac{2}{n}, \\ u \in C^{1,\beta}\left(\overline{\mathbb{R}^n_+}\right) \quad \text{for all } 0 < \beta < 1 & \text{if } s = 1 + \frac{2}{n}, \\ u \in C^{1,\beta}\left(\overline{\mathbb{R}^n_+}\right) \quad \text{for all } 0 < \beta < \frac{n(2-s)}{n-2} & \text{if } s > 1 + \frac{2}{n}. \end{cases}$$

(ii) *There is a constant C such that $|u(y)| \leqslant C(1+|y|)^{1-n}$ and $|\nabla u(y)| \leqslant C(1+|y|)^{-n}$.*

(iii) *$u(y', y_n)$ is axially symmetric with respect to the y_n-axis, i.e., $u(y', y_n) = u(|y'|, y_n)$.*

(iv) *There exists a $\xi > 0$ such that*

$$u(x) = \left(\frac{\xi}{|x|} \right)^{n-2} u\left(\frac{\xi^2 x}{|x|^2} \right).$$

For some $\lambda \in \mathbb{R}^+$, Lemma 3.4 has been proved in [13]. The proof there can work for Eq. (1.19) for all $\lambda > 0$. So we omit the proof here. We refer Lin and Wadade [13] for the details of the proof.

Now we come back to the proof of Theorem 1.2. Let

$$c_0 = \inf\left\{ \Phi_s(v) \;\middle|\; v \text{ is a positive solution of (1.19) and } \Phi_s(v) > 0 \right\}.$$

It is easy to see $c_0 > 0$. Now suppose v_j is a sequence of solutions of (1.19) with $\Phi_s(v_j) \to c_0$. By scaling and Lemma 3.4, we may assume v_j satisfies

$$v_j(x) = \left(\frac{1}{|x|} \right)^{n-2} v_j\left(\frac{x}{|x|^2} \right), \quad x \in \mathbb{R}^n_+. \tag{3.14}$$

If v_j is uniformly bounded, then $v_j \to v$ where v is a positive solution to (1.19) with $\Phi_s(v) = c_0$, and the proof is done. So we may assume $\max v_j = v_j(x_j) \to +\infty$. By (3.14), we may assume $|x_j| \leqslant 1$. By using the inequality (1.20),

$$\Phi_s(v_j) < \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},$$

and applying the same argument as before, we can prove the rescaling $\hat{v}_j$,

$$\hat{v}_j(y) = \frac{v_j(x_j + k_j y)}{v_j(x_j)},$$

where $k_j = v_j(x_j)^{-\frac{2}{n-2}}$, converges to $\hat{v}$ uniformly in any compact set of $\overline{\mathbb{R}^n_+}$ where $\hat{v}$ is a positive solution of

$$\Delta\hat{v} + \lambda\hat{v}^{\frac{n+2}{n-2}} + \frac{\hat{v}^{2^*(s)-1}}{|y|^s} \quad \text{in } \mathbb{R}^n_+, \qquad \hat{v}|_{\partial\mathbb{R}^n_+} = 0,$$

and $\Phi_s(\hat{v}) \leqslant c_0$. Then $\Phi_s(\hat{v}) = c_0$ and Theorem 1.2 is completely proved. □

For the case $n = 3$ and $1 \leqslant s < 2$, we can obtain the entire solution to (1.19) if λ is small. To see this, we can take the positive function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ which achieves the best constant $\mu_s(\Omega)$, i.e.,

$$\mu_s(\Omega) = \frac{\int_\Omega |\nabla v_0|^2\,dx}{(\int_\Omega \frac{v_0^{2^*(s)}}{|x|^s}\,dx)^{\frac{2}{2^*(s)}}},$$

so that for $t \geqslant 0$,

$$\begin{aligned}
\max_{t\geqslant 0} \Phi_s(tv_0) &\leqslant \max_{t\geqslant 0}\left\{\frac{t^2}{2}\int_\Omega |\nabla v_0|^2\,dx - \frac{t^{2^*(s)}}{2^*(s)}\int_\Omega \frac{v_0^{2^*(s)}}{|x|^s}\,dx\right\} \\
&= \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\left(\int_\Omega |\nabla v_0|^2\,dx\right)^{\frac{2^*(s)}{2^*(s)-2}}\left(\int_\Omega \frac{v_0^{2^*(s)}}{|x|^s}\,dx\right)^{\frac{-2}{2^*(s)-2}} \\
&= \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_s(\Omega)^{\frac{2^*(s)}{2^*(s)-2}} \\
&< \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}},
\end{aligned} \tag{3.15}$$

if $\lambda < (\frac{n}{2} - \frac{n}{2^*(s)})^{\frac{2}{2-n}} S_n^{\frac{n}{n-2}} \mu_s(\Omega)^{\frac{2^*(s)2}{(n-2)(2-2^*(s))}}$. Once the inequality (3.15) holds, then by the same proof of Theorem 1.2, we can establish the following result.

Theorem 3.5. *For $n=3$ and $1 \leqslant s < 2$, there exists a least-energy solution of Eq.* (1.19) *provided that*

$$0 < \lambda < \left(\frac{n}{2} - \frac{n}{2^*(s)}\right)^{\frac{2}{2-n}} S_n^{\frac{n}{n-2}} \mu_s(\Omega)^{\frac{2^*(s)2}{(n-2)(2-2^*(s))}}.$$

4. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. To this end, we need the following lemma.

Lemma 4.1. *Suppose that Ω is a C^1 bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$ and the mean curvature of $\partial\Omega$ at 0 is negative. Then, for $\lambda > 0$, there exists a nonnegative function $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\Phi_s(v_0) < 0$ and*

$$\max_{t \geqslant 0} \Phi_s(t v_0) < c_0 := \int_{\mathbb{R}^n_+} \left(\frac{1}{2}|\nabla v|^2 - \frac{(n-2)\lambda}{2n} v^{\frac{2n}{n-2}} - \frac{1}{2^*(s)} \frac{v^{2^*(s)}}{|x|^s}\right) dx,$$

where

$$\Phi_s(u) := \int_{\Omega} \left(\frac{1}{2}|\nabla u|^2 - \frac{(n-2)\lambda}{2n}\big(u^+\big)^{\frac{2n}{n-2}} - \frac{1}{2^*(s)} \frac{(u^+)^{2^*(s)}}{|x|^s}\right) dx \quad \text{for } u \in H_0^1(\Omega), \tag{4.1}$$

and $v \in H_0^1(\mathbb{R}^n_+)$ is a least-energy solution of Eq. (1.19).

Proof. As in the proof of Lemma 2.2, we take $\hat{v}_\varepsilon(x) := \varepsilon^{-\frac{n-2}{2}} \eta(x) v(\frac{\phi(x)}{\varepsilon})$, where $\phi(x)$ is as defined in (2.4) and $\eta(x)$ is a cut-off function. From the estimates (2.16) with $p = \frac{n+2}{n-2}$, we get for $t \geqslant 0$,

$$\begin{aligned}
\Phi_s(t\hat{v}_\varepsilon) &\leqslant \frac{t^2}{2} \int_{\Omega} |\nabla \hat{v}_\varepsilon|^2\, dx - \frac{(n-2)\lambda\, t^{\frac{2n}{n-2}}}{2n} \int_{\Omega \cap \tilde{U}} v_\varepsilon^{\frac{2n}{n-2}}\, dx - \frac{t^{2^*(s)}}{2^*(s)} \int_{\Omega \cap \tilde{U}} \frac{v_\varepsilon^{2^*(s)}}{|x|^s}\, dx \\
&\leqslant \frac{t^2}{2}\left(\int_{\mathbb{R}^n_+} |\nabla v|^2\, dy - K_1' H(0)\big(1+o(1)\big)\varepsilon + K_2' H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^2\big)\right) \\
&\quad - \frac{(n-2)\lambda\, t^{\frac{2n}{n-2}}}{2n}\left(\int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}}\, dy + O\big(\varepsilon^{\frac{n^2}{n-2}}\big)\right) \\
&\quad - \frac{t^{2^*(s)}}{2^*(s)}\left(\int_{\mathbb{R}^n_+} \frac{v^{2^*(s)}}{|y|^s}\, dy - \frac{2^*(s) K_1'}{2} H(0)\big(1+o(1)\big)\varepsilon + O\big(\varepsilon^2\big)\right) \\
&= \frac{t^2}{2} \int_{\mathbb{R}^n_+} |\nabla v|^2\, dy - \frac{(n-2)\lambda t^{\frac{2n}{n-2}}}{2n} \int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}}\, dy - \frac{t^{2^*(s)}}{2^*(s)} \int_{\mathbb{R}^n_+} \frac{v^{2^*(s)}}{|y|^s}\, dy
\end{aligned}$$

$$+\frac{H(0)}{2}\big(\big(K_2'-K_1'+o(1)\big)t^2+\big(K_1'+o(1)\big)t^{2^*(s)}\big)\varepsilon+O\big(\varepsilon^2\big)$$
$$:=f_1(t)+\frac{H(0)\varepsilon}{2}f_2(t)+O\big(\varepsilon^2\big), \tag{4.2}$$

where

$$K_1':=\frac{2s}{2^*(s)}\int\limits_{\mathbb{R}^n_+}\frac{v^{2^*(s)}|y'|^2y_n}{|y|^{2+s}}\,dy,$$
$$K_2':=\int\limits_{\mathbb{R}^{n-1}}\big|(\partial_n v)\big(y',0\big)\big|^2\big|y'\big|^2\,dy'+\frac{(n-2)\lambda}{n}\int\limits_{\mathbb{R}^{n-1}}v\big(y',0\big)^{\frac{2n}{n-2}}\big|y'\big|^2\,dy'.$$

Since $2^*(s)>2$, $\frac{2n}{n-2}>2$ and

$$\int\limits_{\mathbb{R}^n_+}|\nabla v|^2\,dy=\lambda\int\limits_{\mathbb{R}^n_+}v^{p+1}\,dy+\int\limits_{\mathbb{R}^n_+}\frac{v^{2^*(s)}}{|y|^s}\,dy,$$

we find

$$\max_{t\geqslant 0}f_1(t)=f_1(1)=c_0,$$

and

$$f_2(1)=K_2'+o(1)>0,$$

provided ε is small.

Hence, in case $H(0)<0$ and ε small, we conclude for all $t\geqslant 0$,

$$\Phi_s(t\hat{v}_\varepsilon)<f_1(1)=c_0.$$

Finally, we take $v_0:=t_0\hat{v}_\varepsilon$ where t_0 is large enough so that $\Phi_s(v_0)<0$. The lemma is proved. □

Now we shall show Theorem 1.3.

Proof of Theorem 1.3. Under the assumption of Theorem 1.3, for small $\varepsilon>0$, by applying Theorem B, Theorem 1.2 and Lemma 4.1, we can find a positive solution of

$$\begin{cases}\Delta u_\varepsilon+\lambda u_\varepsilon^{p_\varepsilon-1}+\dfrac{u_\varepsilon^{2^*(s)-1-\varepsilon}}{|x|^s}=0 \quad \text{in } \Omega,\\ u_\varepsilon>0 \quad \text{in } \Omega \text{ and } u_\varepsilon|_{\partial\Omega}=0,\end{cases} \tag{4.3}$$

which satisfies

$$\int\limits_{\Omega}\left(\frac{1}{2}|\nabla u_\varepsilon|^2-\frac{\lambda}{p_\varepsilon}u_\varepsilon^{p_\varepsilon}-\frac{1}{2^*(s)-\varepsilon}\frac{u_\varepsilon^{2^*(s)-\varepsilon}}{|x|^s}\right)dx=c_*^\varepsilon<c_0, \tag{4.4}$$

where c_0 is as defined in Lemma 4.1 and

$$p_\varepsilon := \frac{2n}{n-2} - \frac{2\varepsilon}{2-s}.$$

Similar to the proof of Theorem 1.2, by extracting a subsequence, u_j has a weak limit $u \in H_0^1(\Omega)$ that satisfies

$$\Delta u + \lambda u^{\frac{n+2}{n-2}} + \frac{u^{2^*(s)-1}}{|x|^s} = 0 \quad \text{in } \Omega. \tag{4.5}$$

In what follows, we shall prove that u is a nontrivial solution. If $u \equiv 0$, the same as the proof of Theorem 1.2, we may assume

$$m_j := u_j(x_j) = \max_{\overline{\Omega}} u_j(x) \to \infty \quad \text{as } j \to \infty,$$

and consider the scaling:

$$v_j(y) := m_j^{-1} u_j(x_j + k_j y) \quad \text{for } y \in \Omega_j := \left\{ z \in \mathbb{R}^n \;\middle|\; x_j + k_j z \in \Omega \right\},$$

where $k_j = m_j^{-\frac{p_j-2}{2}}$ and $p_j = \frac{2n}{n-2} - \frac{2\varepsilon_j}{2-s}$. Then v_j satisfies

$$\begin{cases} \Delta v_j + \lambda v_j^{p_j-1} + \dfrac{v_j^{2^*(s)-1-\varepsilon_j}}{|\frac{x_j}{k_j} + y|^s} = 0 & \text{in } \Omega_j, \\ v_j = 0 & \text{on } \partial\Omega_j. \end{cases} \tag{4.6}$$

Noting that by the inequality (1.20), we see that $c_*^\varepsilon < c_0 < \frac{1}{n}\lambda^{\frac{2-n}{2}} S_n^{\frac{n}{2}}$. Hence, we may apply the blowing up argument in the proof of Theorem 1.2 to obtain

$$0 < \liminf_{j\to\infty} \frac{|x_j|}{k_j} \leqslant \limsup_{j\to\infty} \frac{|x_j|}{k_j} < +\infty. \tag{4.7}$$

Now we may pick up a subsequence $\frac{x_j}{k_j} \to y_0 \neq 0$ and assume, up to an affine transformation, $\Omega_j \to \mathbb{R}_+^n$. Therefore, v_j converges to some v uniformly in any compact set of $\overline{\mathbb{R}_+^n}$, where $v \in H_0^1(\mathbb{R}_+^n)$ is a nontrivial solution of the equation

$$\begin{cases} \Delta v + \lambda v^{\frac{n+2}{n-2}} + \dfrac{v^{2^*(s)-1}}{|z|^s} = 0 & \text{in } \mathbb{R}_+^n, \\ v = 0 & \text{on } \partial\mathbb{R}_+^n. \end{cases} \tag{4.8}$$

Direct calculation shows

$$C_1 := \lim_{j\to\infty}\int\limits_{\Omega} |\nabla u_j|^2\,dx = \lim_{j\to\infty}\left(m_j^{(\frac{n-2}{2-s})\varepsilon_j}\int\limits_{\Omega_j} |\nabla v_j|^2\,dy\right) \geqslant \int\limits_{\mathbb{R}^n_+} |\nabla v|^2\,dz =: B_1, \tag{4.9}$$

$$C_2 := \lim_{j\to\infty}\int\limits_{\Omega} u_j^{p_j}\,dx = \lim_{j\to\infty}\left(m_j^{(\frac{n-2}{2-s})\varepsilon_j}\int\limits_{\Omega_j} v_j^{p_j}\,dy\right) \geqslant \int\limits_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}}\,dz =: B_2, \tag{4.10}$$

$$C_3 := \lim_{j\to\infty}\int\limits_{\Omega} \frac{u_j^{2^*(s)-\varepsilon_j}}{|x|^s}\,dx = \lim_{j\to\infty}\left(m_j^{(\frac{n-2}{2-s})\varepsilon_j}\int\limits_{\Omega_j} \frac{v_j^{2^*(s)-\varepsilon_j}}{|\frac{x_j}{k_j}+y|^s}\,dy\right) \geqslant \int\limits_{\mathbb{R}^n_+} \frac{v^{2^*(s)}}{|z|^s}\,dz =: B_3. \tag{4.11}$$

Inferring from (4.4) and (4.8), we deduce

$$\begin{cases} C_1 - \lambda C_2 - C_3 = 0, \\ B_1 - \lambda B_2 - B_3 = 0, \\ c_* = \dfrac{C_1}{2} - \dfrac{(n-2)\lambda}{2n} C_2 - \dfrac{1}{2^*(s)} C_3. \end{cases}$$

By the definition of c_0, see (1.21), we have

$$\frac{B_1}{2} - \frac{(n-2)\lambda}{2n} B_2 - \frac{1}{2^*(s)} B_3 \geqslant c_0.$$

To sum up, we see

$$\begin{aligned} c_* &= \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) C_1 + \left(\frac{1}{2^*(s)} - \frac{n-2}{2n}\right)\lambda C_2 \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) B_1 + \left(\frac{1}{2^*(s)} - \frac{n-2}{2n}\right)\lambda B_2 \\ &= \frac{B_1}{2} - \frac{(n-2)\lambda}{2n} B_2 - \frac{1}{2^*(s)} B_3 \\ &\geqslant c_0. \end{aligned}$$

This contradicts to $c_* < c_0$. Hence, we have proved $u \neq 0$ in $H_0^1(\Omega)$ and the proof of Theorem 1.3 is complete. □

Notice that for the case $n = 3$ and $1 \leqslant s < 2$, the above arguments also work if λ is small, because by Theorem 3.5, Lemma 4.1 still holds for $n = 3$ and $1 \leqslant s < 2$ provided that λ is small. Therefore, we have the following theorem.

Theorem 4.2. *Suppose that Ω is a C^1 bounded domain in $\mathbb{R}^3$ with $0 \in \partial\Omega$. Assume further that the mean curvature of $\partial\Omega$ at 0 is negative. Then for $1 \leqslant s < 2$, Eq.* (1.18) *has a positive solution if*

$$0 < \lambda < \left(\frac{n}{2} - \frac{n}{2^*(s)}\right)^{\frac{2}{2-n}} S_n^{\frac{n}{n-2}} \mu_s(\Omega)^{\frac{2^*(s)2}{(n-2)(2-2^*(s))}}.$$

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