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Relative centers of motion, implicit bars and dead-center positions for planar mechanisms

Rudi Penne

Department of Industrial Sciences and Technology, Karel de Grote-Hogeschool, Antwerp, Belgium

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ABSTRACT

This paper characterizes the dead-center positions of a planar mechanism in terms of implicit bars, that can be described in their turn in terms of relative motion centers. Consequently, a graphical procedure for finding motion centers leads to a geometric description for dead-point positions. We give a survey of existing geometric constructions for motion centers, and we illustrate a new technique that makes use of the "Baracs construction".

1. Introduction

This paper is a contribution to the kinematic study of *planar mechanisms*. Our work space will be always assumed to be the plane. Most authors define a *linkage* as a connected construction consisting of rigid components (*links*), some of which are attached by means of *joints*, revolute (turning) or prismatic (sliding). For most applications we can assume that the links of such a linkage are either bars or consist of bars (e.g. a triangle), using only revolute joints; under this specification a linkage is sometimes called a *bar framework* [2,16]. A linkage might or might not be pinned down to avoid global displacements. If so, the "ground" will be considered as a link of the linkage, the *ground link*. Even if a linkage is not pinned down ("free") then it is convenient to factor out global rotations and translations in its configuration space, because these Euclidean motions are irrelevant for the mechanical behaviour of the linkage. Also, in counting the *degrees of freedom* of a free linkage it is common to subtract 3 (= dimension of the Euclidean group) from the dimension of the configuration space, yielding the "internal" degrees of freedom.

If a linkage or bar framework does not allow any deformation then it is called a (rigid) *structure*, otherwise a *mechanism*. However, we follow the convention to preserve this name for linkages with 1 dof, allowing but not requiring the presence of a ground link.

E-mail address: Rudi.Penne@kdg.be.

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The first order motions of such mechanisms are completely determined by the so-called "relative centers of motion" C_{ij} for each pair of links $\{B_i, B_j\}$ [2]. These relative centers are 3-vectors and can be visualized as "centers of rotation" in the projective plane if we consider them as homogeneous coordinates, while their magnitude stands for the "angular velocity", thus describing how B_i moves infinitesimally relative to B_j. This motion might be an infinitesimal translation, in which case the geometric center lies "at infinity" w.r.t. the physical work plane (canonically embedded in the projective plane). The geometric position of a relative center is commonly referred to as an *instant center* by mechanical engineers [11,14].

Of course, if B_i is connected to B_j by means of a joint P then for any relative motion the center C_{ij} has its position at P. Because a mechanism has only one internal degree of freedom, each C_{ij} corresponds to a fixed point in the projective plane, depending on the actual position of the mechanism, but independent from the applied velocities, also if B_i and B_j are not directly linked together. This observation has motivated many authors to find the instant centers of a given planar mechanism by a graphical procedure rather than by analytical computations [17,6,9,10,7,4,13], avoiding the use or knowledge of the involved velocities. Furthermore, a graphical algorithm leads to a synthetic coordinate-free expression for the instant centers, yielding a better qualitative understanding of the kinematics of the mechanism.

This paper has two purposes. First we attempt to synthesize the existing graphical techniques for constructing the instant centers of a given mechanism (Section 3). Roughly speaking, if a center is not already available as mechanical joint then one tries to constraint its location by a line.

Often, this line is a consequence of the *Aronhold–Kennedy property*: the three instant centers corresponding to three rigid bodies in relative motion are collinear. If two of these three centers are known (or previously constructed) then the AK-property provides a linear constraint for the third center. The most commonly used technique for finding instant centers is the intersection of two different AK-lines for the same center, called the *rule of four*.

Sometimes, when the number of available AK-lines is not sufficient, another type of linear constraint for the relative center C_{ij} is used, the *relative center line*. By deleting a bar (or link) we can obtain a 2-underbraced framework (a 2-dof linkage) such that the relative center for B_i and B_j must lie on a fixed line l_{ij}. This relative center line can be constructed by introducing two separate bracings for the 2-underbraced framework (different from the given mechanism) in which the center for B_i and B_j can be found easily, yielding two points of l_{ij}. This technique for constructing a relative center line is called the *swap principle*.

One can hope that for a given mechanism all instant centers can be constructed from the mechanical joints by intersecting lines belonging to one of these two types (AK-lines or relative center lines). But at the point of writing this article no justification for this hope has been published. On the other hand, in [7] it has been proven that AK-lines and relative center lines do suffice if the procedure of line intersection is generalized by a more general construction, the *Baracs construction*. The use of this construction for finding instant centers is illustrated in Section 4.

The second purpose of this paper is to deduce geometric descriptions for the *dead-center positions* of a given mechanism. In such a position the "driving link" of the mechanism is "locked", such that the mechanism loses its mobility. First we introduce in Section 5 the notion of *implicit bars* in a bar framework. We show how implicit bars can be characterized in terms of a collinearity condition on relative centers. Finally, in Section 6 we characterize a dead-center position by the existence of an implicit bar between a "free end" of the driving link and a "pinned down vertex" of the ground link. We conclude by illustrating how a graphical procedure for locating instant centers provides a geometric description of all dead-center positions of a given mechanism.

2. Relative centers of motion

The work space of this paper will be the plane. Viewed as a Euclidean plane it will be modelled as $\mathbb{R}^2$. However, for a geometric treatment of kinematics it will be often more convenient to reason and calculate in the projective extension $\mathbb{RP}^2$. See e.g. [2,15,12] for other references that use projective geometry in kinematic applications.

We agree to embed the physical work plane in the projective plane as points with third homogeneous coordinate different from zero. We also introduce the vector space V of homogeneous coordinates coordinates, which is isomorphic to $\mathbb{R}^3$.

Let us consider an infinitesimal Euclidean motion in the plane, i.e. the first order approximation of a planar rotation or translation. Then we can always define a *(projective) center of motion* C, which is a vector,[1] in $\mathbb{R}^3$:

1. For a translation with constant velocity $v = (v_1, v_2)$ we put

$$C = (-v_2, v_1, 0).$$

2. For a rotation about $c = (a, b)$ with angular velocity α we put

$$C = (\alpha a, \alpha b, \alpha).$$

We also allow a translation with zero velocity, or a rotation with zero angular velocity, which means that nothing has been moved at all. For this (zero) motion we define the center C equal to (0, 0, 0).

If this center $C \neq (0, 0, 0)$ then it can be be regarded as a vector of homogeneous coordinates of a point $\pi(C)$ in the projective extension of the work plane. Thus, for a rotation with center $C = (\alpha a, \alpha b, \alpha)$ the corresponding point $\pi(C)$ is the Euclidean center of rotation (a, b). For a translation with center $C = (-v_2, v_1, 0)$, $\pi(C)$ is the point "at infinity" (in "direction" $(-v_2, v_1)$, perpendicular to the constant velocity vector). Note that the geometric point $\pi(C)$ contains less information than the algebraic center C of motion, since it does not determine the magnitude of velocity, only its direction.

If $C = (c_1, c_2, c_3)$ is a motion center and if $P = (p_1, p_2)$ is a point in the plane, then the velocity v_P of P can be recovered as the first two coordinates of the *cross product* (or *exterior product*):

$$(c_1, c_2, c_3) \times (p_1, p_2, 1) = (c_2 - p_2c_3, p_1c_3 - c_1, p_2c_1 - p_1c_2)$$
$$\Rightarrow v_P = (c_2 - p_2c_3, p_1c_3 - c_1).$$

Actually, this cross product gives line coordinates of the line CP, with v_P as normal, and with the third coordinate determined by the fact that this line contains P.

For understanding this projectively geometric treatment of kinematics it helps to compare with the "classical" Euclidean notations. In the latter setting an infinitesimal rotation about a center (a, b) assigns a velocity vector v_P to each point P:

$$v_P = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} p_1 - a \\ p_2 - b \end{bmatrix} = \begin{bmatrix} \alpha b \\ -\alpha a \end{bmatrix} + \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

with angular velocity α. Notice that this motion can be considered as an infinitesimal rotation about the origin plus a translation. Putting $(c_1, c_2, c_3) = (\alpha a, \alpha b, \alpha)$ shows the equivalence between both formulas for the velocity v_P. But the choice to work in the projective plane yields more elegance. Indeed, the algebraic computations for motions can be executed by a "central accountancy" in the projective center, rather than being forced to consider the velocities at point level. For example, the composition of two motions with centers C_1 and C_2 is a motion with center $C = C_1 + C_2$. It is a well known property of homogeneous coordinates that the geometric position $\pi(C)$ of this resulting center C is on the line that connects $\pi(C_1)$ and $\pi(C_2)$. Another advantage is the uniform treatment for rotations and translations.

If two rigid bodies in the plane, B_1 and B_2, are subject to a (separate) infinitesimal motion, with centers C_1 and C_2 respectively, then we define the *relative center of motion* of B_2 w.r.t. B_1 as:

$$C_{12} = C_2 - C_1 \in \mathbb{R}^3.$$

If $C_1 \neq C_2$ then B_1 and B_2 move relative to each other, and $C_{12} \neq (0, 0, 0)$. In this case, C_{12} can be considered as homogeneous coordinates of a point $\pi(C_{12})$ in the projective plane. Since $C_{12} = -C_{21}$, these relative centers "sit at" the same geometric point. If the bodies B_1 and B_2 are attached to each

1 For kinematics in 3 dimensions see [2,15] where centers of relative motion become vectors in $\mathbb{R}^6$.

other by a common point (*hinge* or *joint*) P, and if they move relative to each other, then $\pi(C_{12})$ coincides with this common joint. Indeed,

$$C_1 \times (P, 1) = C_2 \times (P, 1) \Rightarrow C_{12} \times (P, 1) = (0, 0, 0).$$

For a given joint P, this can be regarded as a system of three homogeneous linear equations with unknown coordinates of C_{12}. This system of equations has rank two, and will be referred to as the *joint conditions*.

Relative centers are important for the study of rigidity or other kinematic properties of mechanisms or constructions. We refer e.g. to [2,7,16] for applications in the field of *bar frameworks*. Also in textbooks and articles by mechanical engineers relative centers have proven to be very useful, albeit under a different terminology (e.g. [14,11,4,9,10]). In mechanical engineering one often considers linkages with one degree of freedom (*mechanisms*) and the relative motion center for a pair of rigid components (*links*) B_1 and B_2. Here, the point $\pi(C_{12})$ is called the *instantaneous center of zero velocity*, or briefly the *instant center*. If the links B_1 and B_2 are connected to each other (by a joint) in the linkage then $\pi(C_{12})$ is a *primary* instant center, otherwise it is called *secondary*. Often, some vertices of a linkage are "pinned down", and consequently the "ground" can be considered as another link B_0 of the linkage. In this case, the points $\pi(C_{0i})$ are equal to $\pi(C_i)$, and are called *absolute* instant centers.

It is commonly known in the community of mechanical engineers that for a given position of a 1-dof linkage its instant centers are completely determined, regardless what particular velocities are used to move it. Of course, this is trivial for primary centers (= joints of the linkage), but it also holds for the secondary centers, which might be regarded as "virtual joints". For a proof of this property we refer to [6] (formulated in terms "almost-planar line diagrams") or [7] (formulated in terms of "1-underbraced planar bar frameworks").

Remark. In the sequel we will often confuse between the (relative) center C and its geometric position $\pi(C)$ for the ease of formulation and notation. However, formally C is a 3-vector, while $\pi(C)$ is obtained by intersecting the line through C with the work plane ($z = 1$).

An immediate but important consequence of the definition of relative centers is the following classical property (*Aronhold–Kennedy*):

If three rigid bodies in the plane are subject to an infinitesimal motion, and if none of the three relative centers is zero, then they correspondent to three collinear points.

Indeed,

$$C_{12} + C_{23} + C_{31} = 0.$$

If the position of two of these three relative centers is known (by earlier construction or as primary instant centers), this AK-property provides a linear constraint for the third center, called the *AK-line*.

Considering relative centers as 3-vectors, the AK-property merely states that C_{12}, C_{13}, C_{23} are coplanar.

3. Graphical methods for locating relative centers

We explained in the introduction that by a *mechanism* we will always mean a 1-dof linkage or a 1-underbraced bar framework. As already pointed out in Section 2, the geometric positions $\pi(C_{ij})$ of the relative centers are determined by a given configuration of the mechanism. So, in principle, they should be obtainable by graphical methods. In this section we distinguish two main classes of constructions that appear in the recent literature [14,11,4,6]. We will not describe the "joint-joining principle" of Dijksman [3], because it is a method that is only applicable for "ternary links" (links with three joints) and is not known to be extendable for general mechanisms. However, to our opinion this method (reducing ternary into binary links) deserves further research to explore the possibility for generalization.

We will use the notation $P \vee Q$ for the line joining the two points P and Q, and the notation $a \wedge b$ for the point in which the two lines a and b meet, and both are supposed to operate in the projective plane.

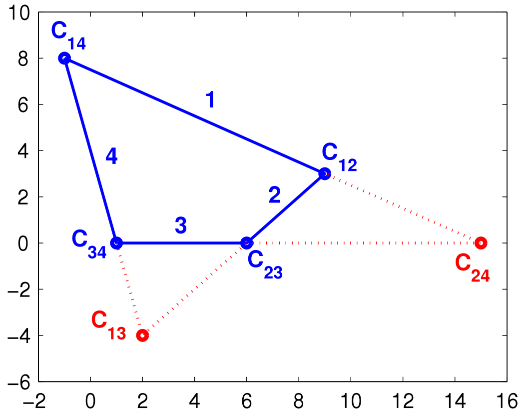

Fig. 1. Illustration of the rule of four. For a cycle of 4 bars, the "secondary" relative centers are located by the intersection of AK-lines.

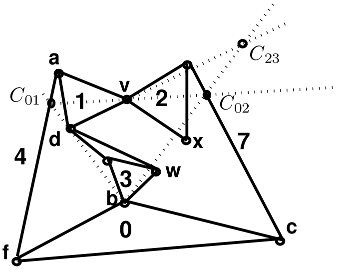

Fig. 2. By successively applying the rule of four we find first C_{01}, then C_{02} and finally C_{23}.

1. *The rule of four.*

We observed at the end of Section 2 that the Aronhold–Kennedy property may provide a line that contains the relative center that we try to locate. If two different AK-lines are available for the same relative center, the latter can be located as the intersection of these lines. Formally, if the given linkage contains four links B_1, B_2, B_3, B_4 such that the relative centers $C_{12}, C_{23}, C_{34}, C_{41}$ are known, then the remaining centers can be constructed by

$$C_{13} = (C_{12} \vee C_{23}) \wedge (C_{14} \vee C_{43})$$
$$C_{24} = (C_{21} \vee C_{14}) \wedge (C_{23} \vee C_{34}).$$

This is called the *rule of four*. In some references (e.g. [4,9,10]), this principle is identified with the AK-property. In Fig. 1 the rule of four is illustrated in the basic case of a cycle of 4 bars (*four-bar linkage*). This framework has 4 primary centers (joints) and 2 secondary centers.

Of course, once a center has been constructed, it becomes available as a new possible ingredient for the rule of four, and more centers may be generated. We see an example of this "domino effect" in Fig. 2.

The rule of four does not guarantee to find all the relative centers, even in the case of a (1 dof)-mechanism where their geometric positions are determined. The smallest mechanism where the rule of four fails to provide all instant centers is the *double butterfly linkage*. Some authors call such linkages *indeterminate* [4]. In Fig. 3 we show a "free framework version" of this linkage, where the originally pinned-down joints have been released and interconnected by three bars to form a rigid component.

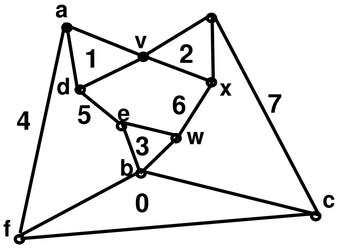

Fig. 3. If we delete the bars of triangle 0 in this framework, and after that pin down the vertices of this triangle then we obtain the famous "double butterfly linkage".

However, from [6] it follows that for 1-underbraced frameworks that can be built by a sequence of "2-valent extensions" (*Henneberg step of type* 1) from the four-bar linkage (Fig. 1), the rule of 4 always succeeds in finding all relative centers.[2]

2. *The swap principle.*

Let B_1 and B_2 be two links in a mechanism F. Let F_- be obtained by deleting a bar or link from F without affecting B_1 or B_2. Furthermore, we assume that B_1 and B_2 are not contained in a *submechanism* of F_- that has only one internal degree of freedom (a submechanism is obtained by deleting one or more links). Now the relative center C_{12} has not a fixed position; different velocities for the links of F_- will cause different locations for the relative centers. But the good news is that there exists a fixed line l_{12} such that for each infinitesimal motion of F_- it holds that $C_{12} \in l_{12}$. We call l_{12} the *relative center line* for B_1 and B_2 in F_-. Of course, the relative center C_{12} in the original mechanism F must be located on l_{12}. To our best knowledge the idea of using a relative center line in a 2-dof sublinkage in order to locate the instant centers of a given mechanism first appeared in [4], where it was an essential step in "solving" the indeterminate double butterfly linkage. In [7] the general existence of the relative center line has been proven.

If we break down the given mechanism F twice and achieve each time a 2-dof linkage, then C_{12} can be constructed as the intersection of two relative center lines, assuming that both lines are different. Alternatively, a center might be located as the intersection of an AK-line and a relative center line. We refer to [9] for a clear illustration of this last case for the *single flier eight-bar linkage*.

In [4,9] a relative center line l_{12} for a pair of links B_1 and B_2 in a 2-dof linkage has been graphically obtained by choosing two arbitrary flexings of the linkage. This has been done by selecting two arbitrary instant centers C'_{ij} and C''_{ij} for a particular pair $\{B_i, B_j\}$, implying respective instant centers C'_{12} and C''_{12} by applying the rule of four. Finally, $l_{12} = C'_{12} \vee C''_{12}$, and C_{12} is located by intersecting l_{12} by an AK-line. It should be noted that the authors explained this graphical procedure only by few examples, without claiming that such a pair of links $\{B_i, B_j\}$ always exists and without giving a general criterion how to select this pair. Furthermore, it is not clear whether two arbitrary choices C'_{ij} and C''_{ij} always lead to different C'_{12} and C''_{12}, neither how the resulting l_{12} is guaranteed not to coincide with the AK-line.

In [7] it is suggested to construct a relative center line l_{12} by considering "bracings" of F_-. Such a bracing of F_- adds a new bar to F_- in order to obtain a new 1-dof mechanism F', still containing B_1 and B_2 as links. One might say that F' is obtained from the original mechanism F by "swapping a bar". The purpose is to arrive at a mechanism F' where the relative center C'_{12} can be constructed in a straightforward way (as opposed to C_{12} in F). Observe that the point C'_{12} necessarily lies on the center line l_{12} in F_-. So, by considering two appropriate bracings of F_-, F' and F'', we aim to obtain the relative center line as

$$l_{12} = C'_{12} \vee C''_{12}$$

[2] In fact, in the same article a more general result has been obtained, also allowing so-called "compound extensions".

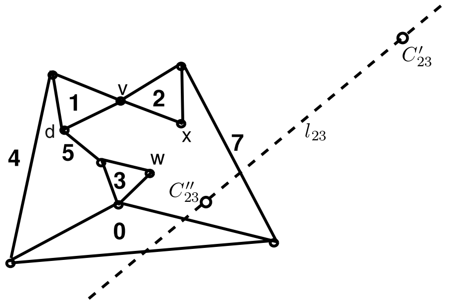

Fig. 4. The framework of Fig. 3 minus bar 6. The locus of the instant centers C_{23} for all possible infinitesimal motions is given by the relative center line l_{23}.

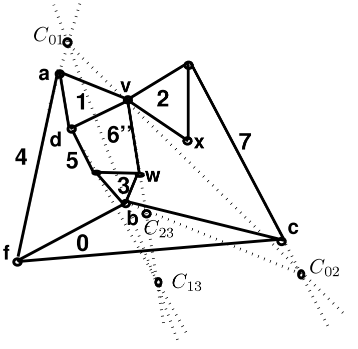

Fig. 5. If we swap bar 6 in Fig. 3 ($\rightarrow 6''$) then we obtain a 1-underbraced framework F'' that allows the construction of C''_{23} by merely using the rule of four: $C''_{13} \Rightarrow C''_{01} \Rightarrow C''_{02} \Rightarrow C''_{23}$.

which is a linear constraint for C_{12}. The procedure to construct a relative center line in this way is called the *swap principle*.

Let us illustrate this technique for the double butterfly linkage, which cannot be tackled by the rule of four. More precisely, let us locate the relative center C_{23} for the links (triangles) 2 and 3 (Fig. 3). Observe that in combination with bar 6 we have an AK-line for C_{23}, namely wx. Next, we delete bar 6 and obtain a 2-underbraced framework F_- (Fig. 4).

The center line l_{23} of the 2-underbraced framework F_- in Fig. 4 can be obtained by the swap principle. For the first bracing F' we add bar wd to F_-. This causes bar 5 and triangle 3 of F to merge in one rigid component that will be called link 3 in F'. Observe that F' has the same design as the framework of Fig. 2. So, as illustrated above, C'_{23} can be located by a sequence of applications of the rule of four.

For the second bracing we add bar vw to F_- and call it link 6 of the new mechanism F''. In Fig. 5 we see how the rule of four first locates C''_{13}, then C''_{01}, after that C''_{02}, and finally C''_{23}.

At the time of writing this article it is still not clear whether all (secondary) relative centers of a 1-underbraced framework can be found as the intersection of constructible AK-lines or relative center lines. But we will see in the next section that if we generalize the concept of line intersection by a more complicated synthetic operation, called the "Baracs construction", the previous question can be answered affirmatively.

4. The Baracs construction

Using Henneberg sequences it is proven in [7] by induction that AK-lines and relative center lines suffice to generate all (secondary) instant centers in every 1-dof bar framework. This proof also yields a

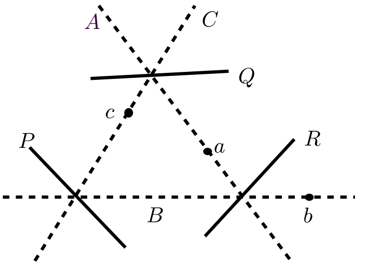

Fig. 6. Three collinear points A, B, C and three non-concurrent lines p, q, r are given. We are asked to construct lines a, b and c through A, B and C respectively, such that their pairwise intersections lie on the given lines.

general procedure for obtaining the necessary relative center lines by means of the swapping principle (Section 3). If the given mechanism is in a "sufficiently general" position then the constructions of [7] are guaranteed not to degenerate (always joining different points and intersecting different lines). However, in order to succeed for all possible mechanisms, one might need a synthetic construction that is somehow more involved than plain line intersection. In this section we recall this "classical" construction.

Let B_1, B_2, B_3, B_4 be four links in some mechanism. Suppose the positions of the relative centers C_{12}, C_{13}, C_{23} are known to us. Furthermore, for the three relative centers where B_4 is involved we are given a linear constraint: l_{14}, l_{24}, l_{34}. For finding the remaining three instant centers C_{14}, C_{24}, C_{34} we are faced with a challenging problem in descriptive geometry (recall that the AK-property states that C_{12}, C_{13}, C_{23} are collinear):

Suppose we are given a drawing of three collinear points in the projective plane: A, B and C. Furthermore, three arbitrary lines are given in that same plane: p, q and r. Now we are asked to construct three lines a, b and c with the following properties (Fig. 6):

$$A \in a, \qquad B \in b, \qquad C \in c$$
$$a \wedge b \in r, \qquad b \wedge c \in p, \qquad a \wedge c \in q.$$

The problem does not seem to allow a constructive solution at first sight. Where should we start with a geometric construction? Of course, if $\{p, q, r\}$ happen to be concurrent lines, meeting in a point S say, then we can put $a = A \vee S$, $b = B \vee S$, $c = C \vee S$, yielding an easy solution. So, let us assume that p, q and r are not concurrent.

Fortunately, this problem is a classical one, with a known solution (*cours de Topologie Structurale*, Université de Montréal, 1978, by Janos Baracs, see also [1].[3]

We start by adding an auxiliary line z through point A to the given drawing. To solve the problem we proceed as follows:

1. $v = C \vee (z \wedge q)$
2. $w = B \vee (z \wedge r)$
3. $S = v \wedge w$
4. $k = S \vee (q \wedge r)$
5. $L = k \wedge p$
6. $b = L \vee B$
7. $c = L \vee C$
8. $a = (c \wedge q) \vee A$ or $(b \wedge r) \vee A$.

Of course, as well in the given drawing, as in the construction process, we allow points or lines to lie at infinity. In the remainder of this paper we refer to this solution as the *Baracs construction*.

We note that this construction is not *intrinsic*, meaning that it does not proceed simply by the successive formation of meets of pairs of lines, and joins of pairs of points, starting from the initial

[3] Actually, the solution is based on the dual problem, which is easier to visualize as a 3D construction.

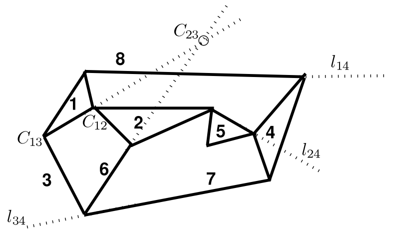

Fig. 7. A mechanism with 8 links. Using the Baracs construction the relative centers for the first 4 links can be located. The location of the remaining centers immediately follows by the rule of four.

data of point and line positions. It was essential to introduce an arbitrary line (z) passing through a given point (A). Since the solution is unique, it does not depend on the choice of z.

In Fig. 7 we show a 1-underbraced framework where the relative centers can be located by the aid of the Baracs construction. We restrict our analysis to links 1, 2, 3 and 4. The positions of the relative centers C_{12}, C_{13}, C_{23} are immediately found as primary centers and by the rule of four. Also, for link 4 we find three AK-lines: l_{14}, l_{24}, l_{34}. This exactly matches the conditions of the Baracs construction, yielding the location of C_{14}, C_{24}, C_{34}.

5. Implicit bars

Let F be an arbitrary planar bar framework with e bars. Let $\gamma = (C_1, \dots, C_e)$ be a given infinitesimal motion of F, that is, C_i is the center of motion for the ith bar such that the joint conditions are satisfied (Section 2). If the ith bar of F connects joints V and W, then $C = C_i$ satisfies simultaneously the joint conditions in V and in W. So, for each bar P incident to V and for each bar Q incident to W:

$$(C - C_P) \times (V, 1) = (C - C_Q) \times (W, 1) = 0. \tag{1}$$

Next, suppose that joints V and W are not linked by a bar in F. If for each motion γ of F we can still find an additional center C such that Eq. (1) holds for each bar P incident to V and each bar Q incident to W, then we call VW an *implicit bar*. In this context, ordinary bars are referred to as *explicit*. We refer to [5] for details on "bases", "dependencies" and "closure" in the more general setting of *matroids*.

Theorem 1. *Let F be a planar framework with non-isolated joints V_1 and V_2 ($V_1 \neq V_2$). Then the following three statements are equivalent:*

1. *V_1V_2 is an implicit or explicit bar of F.*
2. *For each motion $\gamma = (C_1, \dots, C_e)$ of F, there exists a bar P_1 incident to V_1, and a bar P_2 incident to V_2, such that*
$$[(C_{P_1} - C_{P_2}) \times (V_1, 1)] \cdot (V_2, 1) = 0.$$
3. *For each motion $\gamma = (C_1, \dots, C_e)$ of F, for each bar P_1 incident to V_1, and for each bar P_2 incident to V_2, it holds that*
$$[(C_{P_1} - C_{P_2}) \times (V_1, 1)] \cdot (V_2, 1) = 0.$$

Proof. $1 \Rightarrow 3$: Let $\gamma = (C_P)_P$ be a motion of the framework F, and let C be the center of motion for the implicit or explicit bar V_1V_2. Further, let P_i be a bar incident to joint V_i ($i = 1, 2$). Then, by the joint conditions,

$$\begin{aligned}[C_{P_1} \times (V_1, 1)] \cdot (V_2, 1) &= [C \times (V_1, 1)] \cdot (V_2, 1)\\ &= -[(V_1, 1) \times C] \cdot (V_2, 1)\\ &= -[(V_1, 1) \times C_{P_2}] \cdot (V_2, 1)\\ &= [C_{P_2} \times (V_1, 1)] \cdot (V_2, 1).\end{aligned}$$

$3 \Rightarrow 2$: Obvious, as we assumed that V_1 and V_2 are non-isolated.

$2 \Rightarrow 1$: Because

$$[(C_{P_1} - C_{P_2}) \times (V_1, 1)] \cdot (V_2, 1) = 0$$

and because $V_1 \neq V_2$, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$C_{P_1} - C_{P_2} = \lambda_2(V_2, 1) - \lambda_1(V_1, 1).$$

(If $C_{P_1} = C_{P_2}$ then we choose $\lambda_1 = \lambda_2 = 0$.) Now we can define the following center of motion for V_1V_2:

$$C = C_{P_1} + \lambda_1(V_1, 1) = C_{P_2} + \lambda_2(V_2, 1).$$

Next, let Q_i be an arbitrary bar of F that is incident to joint V_i ($i = 1, 2$). It follows that

$$\begin{aligned}(C - C_{Q_i}) \times (V_i, 1) &= (C_{P_i} + \lambda_i(V_i, 1) - C_{Q_i}) \times (V_i, 1)\\ &= (C_{P_i} - C_{Q_i}) \times (V_i, 1) = 0\end{aligned}$$

for $i = 1, 2$, where we applied the joint conditions in V_i. □

In the next section we will use the idea of implicit bars to describe so-called "dead-point positions" in linkages.

6. Dead-center positions

The issue of a *dead-center* position (or a *dead point*) is of great importance for linkages, both during the performance as in the stage of designing. In such a position the linkage loses its mobility. A dead point might be part of the functionality of the mechanism, but mostly it is an unwanted obstacle. We refer to [13] for more references to the literature. In this section we will describe dead-center positions in terms of implicit bars, and hence, due to Theorem 1, in terms of instant centers.

In this section, we restrict to the case of mechanisms that are "pinned-down to the ground" (read the introduction). The joints that belong to the ground link are called *base joints*. W.l.o.g. we assume that each link is a bar or a collection of bars. Furthermore, by introducing bars that connect the base joints, the given linkage is formally transformed to a bar framework F.

As in Section 5 we represent a motion of a framework F (with e bars) by an ordered tuple $(C_1, \ldots, C_e)$ of motion centers for each bar (subject to the joint conditions). Automatically, identical centers are assigned to bars that belong to the same link. Furthermore, we demand zero centers for the bars that constitute the ground link.

In practice, the motion of a linkage is controlled by driving a specific link that is attached (joint J_0) to the ground link. This link is called the *driving link* or *input link*.

Now we can formally define a *dead-center position* of F as a position where every possible motion assigns a zero center to the (bars of the) driving link. Notice that this position is defined w.r.t. the driving link of the mechanism. This definition of a dead point of a (planar) mechanism is the most common one in recent publications [13,8]. However, the reader should be aware of the existence of other definitions (e.g., [17]).

So, in a dead-point position there might still exist mobility, but this cannot be exploited since we can only realize motions that are caused by the driving link. We could also say that in a dead-point position the driving link and the ground link merge into one link. Let $J_1 \neq J_0$ be another base joint, and let $P \neq J_0$ be another joint that belongs to the driving link, then PJ_1 is not a bar in F, because otherwise the driving link and the ground link would be identical in each position. However, in a dead-point position of F, it follows from the definition that PJ_1 must be an implicit bar. Conversely, if PJ_1 is implicit, and if moreover P, J_0, J_1 are not collinear then F is in a dead-point position. Indeed, a (non-degenerated) triangle is infinitesimally rigid, causing the driving link to be locked by the ground link. We conclude in

Theorem 2. *Let F be a pinned-down mechanism, J_0 a base joint that is incident to the driving link of F and and let P be any joint of the driving link different from J_0. Then*

1. *If F is in a dead-center position then PJ_1 is an implicit bar for each base joint $J_1 \neq J_0$.*
2. *If there exists a base joint J_1, not collinear with P and J_0, such that PJ_1 is an implicit bar then F is in a dead-center position.*

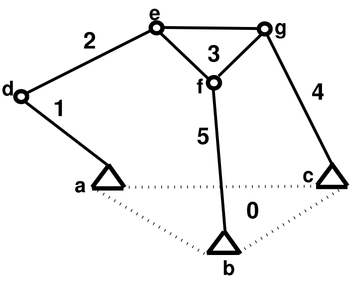

Fig. 8. A dead-center position for the Stephenson III linkage occurs if *BD* or *CD* becomes an implicit bar.

Theorem 2 can always be translated into a geometric condition on instant centers. Indeed, let B be a link attached to joint P but different from the driving link and let I be the relative center of B w.r.t. the ground link (= center of zero velocity). According to Theorem 1 the bar PJ_1 becomes implicit if I is collinear with P and J_1.

In [17] we found a different approach for finding dead-center positions, also making use of relative centers. However, care should be taken if one compares both results, because the authors use another definition for a dead-center position, depending on the choice for the *output link* as well. In addition to our convention, in [17] the output link is not allowed to have zero velocity in a dead point. In this reference, expressed in the previous notation and taking B as output link, the geometric condition for a dead-center position is the coincidence of I and P. This is a special case of our collinearity condition, as should be, because the set of dead-center positions in the definition of [17] is a subset of our dead-center positions.

We illustrate Theorem 2 by verifying the dead-center positions of the Stephenson III linkage [13]. We refer to Fig. 8.

The base joints are denoted by A, B and C. Together they form link 0 (ground link). Link 1 is the driving link. So, in the formulation of Theorem 2 we put $P = D, J_0 = A$ and for J_1 we can choose between B and C. This linkage is in dead-center position if BD is implicit with A, B, D not collinear, or if CD is implicit with A, C, D not collinear. By Theorem 1, this is implied if the relative center C_{02} sits at the line BD or CD. Note that AD is an AK-line for the pair $\{0, 2\}$, such that C_{02} must sit at joint D in a dead-center position.

Applying the rule of four twice:

$$C_{03} = BF \wedge CG$$
$$C_{02} = AD \wedge (E \vee C_{03}).$$

Consequently, C_{02} sits at joint D
$\Leftrightarrow$ C_{03} sits at line DE
$\Leftrightarrow$ ED, BF, CG are concurrent.

This conclusion matches the result of [13] (the other dead-center positions listed by the authors are special cases of the concurrency condition).

In principle, for applying Theorem 2 it is irrelevant whether the involved instant center has been located by AK-lines only or by more advanced procedures. But of course, if the instant center has been obtained by a complex geometric construction, it will be more difficult to have a geometric understanding of the dead-center position. Take for example the double butterfly linkage (Fig. 3). With link 0 as ground link and link 4 as input link we can choose $J_0 = f, P = a$ and $J_1 = b$ in the terminology of Theorem 2. A necessary and sufficient condition for ab to become an implicit bar is for example the collinearity of the instant center $I_{13} = \pi(C_{13})$ with a and b. As described in Section 3, I_{13} can be constructed as the intersection of de (link 5) with the relative center line l_{23} (Fig. 4). Although this yields a geometric characterization for all dead-center positions, it is not a simple one, due to the complex construction for I_{13}. However, simple special cases can be deduced from it easily. For example, if the mechanism is in a position where a, b, d, e lie on one line then it reaches a dead end, because $I_{13} \in de$.

7. Conclusions and further research

The first order motions of a planar mechanism are determined by the location of the relative motion centers for each pair of links. Furthermore, a synthetic construction for these centers, rather than numerical computations, provides qualitative insight for the kinematic behaviour of the design of the linkage (and not just of a particular realization). During the past decades several graphical procedures for obtaining the relative centers have been presented, spread over the literature and over several disciplines. Mathematicians and engineers used each their own terminology, methods and even environments (projective versus Euclidean plane). This paper presents several techniques from several disciplines in one uniform framework. Roughly speaking, most of the constructions operate by means of linear constraints for the relative centers, where we distinguish two types: AK-lines and relative center lines. It is still an open problem whether there exists an algorithm that for every mechanism constructs the necessary AK-lines and relative center lines in order to locate the relative centers by line intersection. This would avoid the Baracs construction, yielding much simpler synthetic formulas for these centers. An alternative direction of research toward a simple general graphical procedure for locating relative centers could be to generalize the joint-joining principle of Dijksman for general mechanisms.

Because the AK-property is equally valid for screw motion centers in 3D, it would be interesting to investigate how the previous constructions can be lifted to a higher dimension.

A second contribution of this paper is to characterize implicit bars in special positions of a given mechanism in terms of relative motion centers. This turns out to be a "collinearity condition", formulated in Theorem 1, which has never been proved in previous publications. The graphical procedures for relative centers in the first part of the paper now imply a geometric condition (synthetic formula) for the existence of implicit bars.

A third contribution of this paper is to provide a new characterization of the dead-center positions of mechanisms with a given topology (design) in terms of implicit bars (Theorem 2). Combining the graphical procedures for relative centers with Theorem 1 now yields a geometric condition on dead points. Consequently, each future simplification in the graphical procedures for relative centers will facilitate the geometric understanding of these dead-center positions.

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