On the \( n \)-fold symmetric product suspensions of a continuum

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1. Introduction

In 1979 Sam B. Nadler, Jr. introduced the hyperspace suspension of a continuum [1]. In 2004 S. Macías, defined the \( n \)-fold hyperspace suspension of a continuum [2]. We define the \( n \)-fold symmetric product suspensions of a continuum, and the purpose of this paper is to present a study of some properties of this hyperspace. The paper is divided into six sections. In Section 2, we give the basic definitions for understanding the paper. In Section 3, we give examples of geometric models for the \( n \)-fold symmetric product suspensions of given continua. In Section 4, we prove that the \( n \)-fold symmetric product suspension of a continuum is unicoherent if \( n \geq 3 \), and we present an example of a unicoherent continuum whose second symmetric product suspension is not unicoherent. In Section 5, we prove that a continuum is locally connected if and only if its \( n \)-fold symmetric product suspension is locally connected. In Section 6, we give a class of nonarcwise connected continua for which their \( n \)-fold symmetric product suspensions are arcwise connected. Also, we give a class of continua for which their \( n \)-fold symmetric product suspensions are not arcwise connected.

2. Definitions

The symbols \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) will denote the set of positive integers, real numbers and complex numbers, respectively. A \emph{continuum} is a nonempty compact, connected metric space. A \emph{subcontinuum} is a continuum contained in a space \( X \). If \( X \) is a continuum, then given \( A \subset X \) and \( \epsilon > 0 \), the open ball about \( A \) of radius \( \epsilon \) is denoted by \( V_\epsilon(A) \), and the closure of \( A \) in \( X \) by \( \text{Cl}_X(A) \). A \emph{map} means a continuous function. An onto map \( f : X \to Y \) between continua is said to be \emph{monotone} provided that for any point \( y \) of \( Y \), \( f^{-1}(y) \) is a connected subset of \( X \).
The symbol \( I \) will denote the closed interval \([0, 1]\). An arc is any space which is homeomorphic to \( I \). A simple closed curve is a space homeomorphic to \( S^1 = \{e^{it} \in \mathbb{C} : t \in \mathbb{R}\} \). A ray is a space homeomorphic to \([0, \infty)\).

A continuum \( X \) is unicoherent provided that if \( X = A \cup B \), where \( A \) and \( B \) are subcontinua of \( X \), then \( A \cap B \) is connected. For each topological space \( Y \), we define

\[
\beta(Y) = (\text{number of components of } Y) - 1
\]

if this number is finite, and \( \beta(Y) = \infty \) otherwise. The multicoherence degree, \( r(X) \), of a continuum \( X \) is defined by:

\[
r(X) = \sup \{ \beta(H \cap K) : H \text{ and } K \text{ are subcontinua of } X \text{ and } X = H \cup K \}.
\]

Given a continuum \( X \) and \( n \in \mathbb{N} \), the product of \( X \) with itself \( n \) times will be denoted by \( X^n \), the symbol \( F_n(X) \) denotes the \( n \)-fold symmetric product of \( X \); that is:

\[
F_n(X) = \{ A \subset X \mid A \text{ has at most } n \text{ points} \}
\]

topologized with the Hausdorff metric, which is defined as follows:

\[
\mathcal{H}(A, B) = \inf \{ \varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A) \}.
\]

Given a finite collection, \( U_1, \ldots, U_m \), of subsets of \( X \), \( (U_1, \ldots, U_m)_n \), denotes the follows subset of \( F_n(X) \):

\[
\left\{ A \in F_n(X) \mid A \subset \bigcup_{k=1}^m U_k \text{ and } A \cap U_k \neq \emptyset \text{ for each } k \in \{1, \ldots, m\} \right\}.
\]

It is known that the family of all subsets of \( F_n(X) \) of the form \( (U_1, \ldots, U_m)_n \), where each \( U_i \) is an open subset of \( X \), forms a basis for a topology for \( F_n(X) \) (see [3, 0.11]) called the Vietoris topology, and that the Vietoris topology and the topology induced by the Hausdorff metric coincide [3, 0.13].

Given a continuum \( X \) and \( n \in \mathbb{N} \) with \( n \geq 2 \), we define the \( n \)-fold symmetric product suspension of the continuum \( X \), denoted by \( SF_n(X) \), by the quotient space:

\[
SF_n(X) = F_n(X)/F_1(X)
\]

with the quotient topology. The fact that \( SF_n(X) \) is a continuum follows from 3.10 of [4].

**Notation 2.1.** Given a continuum \( X \), \( q^n_X : F_n(X) \to SF_n(X) \) denotes the quotient map. Also, let \( F^n_X \) denotes the point \( q^n_X(F_1(X)) \).

**Remark 2.2.** Note that \( SF_n(X) \setminus \{F^n_X\} \) is homeomorphic to \( F_n(X) \setminus F_1(X) \), using the appropriate restriction of \( q^n_X \).

### 3. Examples

In this section we present examples of geometric models of \( n \)-fold symmetric product suspensions for some given continua.

**Example 3.1.** K. Borsuk and S. Ulam proved that \( F_2(I) \) is homeomorphic to \( I^2 \) [5, Theorem 6]. It is known actually that there exists a homeomorphism from \( F_2(I) \) onto the triangle in the Euclidean plane which has as vertices the points \((0, 0), (0, 1)\) and \((1, 1)\) [6, Example 3.7] such that \( F_1(I) \) is homeomorphic to the segment that joins to the points \((0, 0)\) and \((1, 1)\). If we identify \( F_1(I) \) to a point, we obtain a space homeomorphic to this triangle. Hence, \( SF_2(I) \) is homeomorphic to this triangle. Thus, \( SF_2(I) \) is homeomorphic to \( F_2(I) \). Therefore, \( SF_2(I) \) is homeomorphic to \( I^2 \).

A simple triod \( T \) is a continuum which is the union of three arcs which have only one point in common. We construct a model for the hyperspace \( SF_2(T) \).

**Example 3.2.** By [6, Example 3.8], \( F_2(T) \) is homeomorphic to the Fig. 1. Which is a 2-cell \( D_0 \) with three 2-cells \( D_1, D_2 \) and \( D_3 \) glued in such a way that \( D_0 \cap D_j \) is an arc, \( j \in \{1, 2, 3\} \), and \( D_1 \cap D_2 \cap D_3 \) is a single point \( p \). Note that \( F_1(T) \) is contained in the manifold boundaries of \( D_1, D_2, D_3 \) and \( F_1(T) \cap D_0 = \{p\} \). Hence, if we identify \( F_1(T) \) to a point in \( D_1 \cup D_2 \cup D_3 \), we obtain a space homeomorphic to \( D_1 \cup D_2 \cup D_3 \). Therefore, \( SF_2(T) \) is homeomorphic to \( F_2(T) \).

In Examples 3.1 and 3.2, we give continua such that their respective 2-fold symmetric product and 2-fold symmetric product suspension are homeomorphic. In the following example we will prove that \( F_2(S^1) \) and \( SF_2(S^1) \) are topologically different. This example may be suffice to show that it is interesting to study the \( n \)-fold symmetric product suspensions of a continuum.
**Theorem 3.6.** As a space \( F_n \) then can be defined by \( H \). Hence, \( F_n \) we have that \([11, 2.23]\).

**Remark 3.4.** Observe that \( S^1 \) is an example of a nonunicoherent continuum such that \( F_2(S^1) \) is not unicoherent but \( SF_2(S^1) \) is unicoherent \([7, p. 197]\). In Example 4.4 we present a unicoherent continuum \( X \) such that \( SF_2(X) \) is not unicoherent.

A Hilbert cube, denoted by \( Q \), is a continuum which is homeomorphic to the countable cartesian product \( \prod_{i=1}^{\infty} I_i \), where each \( I_i = I \), with the product topology \([4, 1.4]\).

**Example 3.5.** If \( Q \) is the Hilbert cube, then \( SF_n(Q) \) is the Hilbert cube, for each \( n \in \mathbb{N} \) with \( n \geq 2 \). To see this let \( n \geq 2 \). Then \( F_n(Q) \) is homeomorphic to \( Q \) \([8, Theorem 2.4]\). Since \( Q \) is contractible, we have that \( F_1(Q) \) is contractible. Since \( F_1(Q) \) is nowhere dense in \( F_n(Q) \), it follows that \( F_1(Q) \) has the shape of a point \( \text{in the sense of Borsuk} \) \([9, 5.5, p. 28]\). Hence, \( F_n(Q) \setminus F_1(Q) \) is homeomorphic to \( Q \setminus \{p\} \), for some point \( p \) of \( Q \) \([10, 25.2]\). Since \( F_n(Q) \setminus F_1(Q) \) is homeomorphic to \( SF_n(Q) \setminus \{F_n^Q\} \), we obtain that \( Q \setminus \{p\} \) is homeomorphic to \( SF_n(Q) \setminus \{F_n^Q\} \). Therefore, \( SF_n(Q) \) is homeomorphic to \( Q \) \([11, 2.23]\).

A retraction is a continuous function, \( r \), from a space, \( X \), into itself such that \( r \) is the identity on its range \( \text{i.e., } r(r(x)) = r(x) \text{ for each } x \in X \). A closed subset \( A \) of \( X \) is said to be a retract of \( X \) provided that there is a retraction of \( X \) onto \( A \). A space \( X \) is called an absolute retract provided that \( X \) is a retract of every space \( Y \) containing \( X \) as a closed subset.

**Theorem 3.6.** Let \( X \) be an absolute retract and \( n \in \mathbb{N} \) with \( n \geq 2 \). Then \( SF_n(X) \) is an absolute retract.

**Proof.** Without loss of generality, we assume that \( X \) is embedded in the Hilbert cube \( Q \) \([12, 1.1.16]\). Since \( X \) is an absolute retract, there exists a retraction \( r : Q \rightarrow X \). It is easy to verify that the induced function \( SF_n(r) : SF_n(Q) \rightarrow SF_n(X) \), which can be defined by

\[
SF_n(r)(\chi) = \begin{cases} 
q_n^X(F_n(r)((q_n^Q)^{-1}(\chi)))), & \text{if } \chi \neq F_n^Q; \\
F_n^X, & \text{if } \chi = F_n^Q,
\end{cases}
\]

is a retraction. Since \( SF_n(Q) \) is homeomorphic to \( Q \) \(\text{Example 3.5}\), we have that \( SF_n(X) \) is an absolute retract \([13, Theorem 7, p. 341]\). \( \square \)

Since absolute retracts have the fixed point property \([13, Theorem 11, p. 343]\), we have:

**Corollary 3.7.** Let \( X \) be an absolute retract and \( n \in \mathbb{N} \) with \( n \geq 2 \). Then \( SF_n(X) \) has the fixed point property.

4. Unicoherence

**Theorem 4.1.** Let \( X \) be a continuum and let \( n \geq 3 \) be a positive integer. Then \( SF_n(X) \) is a unicoherent continuum.

**Proof.** By \([14, Theorem 8]\), \( F_n(X) \) is unicoherent. We note that \( q_n^X \) is a monotone map. Then, by \([4, Corollary 13.34]\), we have that \( SF_n(X) \) is a unicoherent continuum. \( \square \)

For the case \( n = 2 \), it is known that \( r(F_2(X)) \leq 1 \) \(\text{see [14, p. 181]}\). Since \( q_n^X \) is a monotone map, by \([4, Theorem 13.33]\), we have that \( r(SF_2(X)) \leq 1 \). Hence, we have the following:

**Theorem 4.2.** Let \( X \) be a continuum. Then \( r(SF_2(X)) \leq 1 \).
In Example 4.4 we will present a continuum $X$ such that $r(SF_2(X)) = 1$. In order to give this example, first we will prove the following lemma:

**Lemma 4.3.** Let $X$ be a continuum and let $A, B, M$ be subcontinua of $X$ such that $X = A \cup B$ and $A \cap B$ is not connected. If there is a component $K$ of $A \cap B$ such that $K \cap M = \emptyset$, then the quotient space $X/M$ is not unicoherent.

**Proof.** Let $K$ be a component of $A \cap B$ such that $K \cap M = \emptyset$. We will consider two cases:

1. Suppose that no component of $A \cap B$ intersects $M$. Then $M \subset X \setminus (A \cap B)$. Considering the quotient map, $q_M : X \to X/M$, we have that $X/M = q_M(A) \cup q_M(B)$, $q_M(A \cap B) = q_M(A) \cap q_M(B)$ and $q_M(A \cap B)$ is homeomorphic to $A \cap B$. Since $A \cap B$ is not connected, $q_M(A) \cap q_M(B)$ is not connected. Therefore, $X/M$ is not unicoherent.

2. Suppose that there exists a component $L$ of $A \cap B$ such that $L \cap M \neq \emptyset$. It is clear that $q_M(A)$ and $q_M(B)$ are subcontinua of $X/M$ such that $X/M = q_M(A) \cup q_M(B)$. We will prove that $q_M(A) \cap q_M(B)$ is not connected. Let $N = \bigcup \{C : C$ is a component of $A \cap B \text{ and } C \cap M \neq \emptyset\}$. We have that $L \subset N$, thus $N \neq \emptyset$.

We note that $N$ is a closed subset of $X$ and no connected subset of $A \cap B$ intersects both $N$ and $K$. Then, by [4, Theorem 5.2], there exist two closed subsets $P$ and $Q$ of $A \cap B$ such that $A \cap B = P \cup Q$, $K \subset P$, $N \subset Q$ and $P \cap Q = \emptyset$. We have that $q_M(A) \cap q_M(B) = q_M(P) \cup q_M(Q)$. Since $P \cap M = \emptyset$, it follows that $q_M(P) \cap q_M(Q) = \emptyset$. Hence, $q_M(A) \cap q_M(B)$ is not connected. Therefore, $X/M$ is not unicoherent. □

E. Castañeda [15, Example 2.1] gives a unicoherent continuum $X$ whose $F_2(X)$ is not unicoherent. With very similar techniques, we give another proof of that $F_2(X)$ is not unicoherent. We use our proof to verify that $SF_2(X)$ is not unicoherent.

**Example 4.4.** Let $S_2 = \{3e^{it} \in \mathbb{C} : t \in \mathbb{R}\}$, $Y = \{(1 + \frac{1}{t+2})e^{it} \in \mathbb{C} : t \in \mathbb{R}\}$ and $X = S^1 \cup S_2 \cup Y$. Then $X$ is a unicoherent continuum (see Fig. 2) such that $SF_2(X)$ is not unicoherent.

We define

$$A = \left\{(z, w) \in F_2(X) : \text{Re} \left( \frac{z}{w} \right) \geq 0 \right\},$$

$$B = \left\{(z, w) \in F_2(X) : \text{Re} \left( \frac{z}{w} \right) \leq 0 \right\},$$

where $\text{Re}(\frac{z}{w})$ is the real part of the complex number $\frac{z}{w}$. Since $\text{Re}(\frac{z}{w}) \geq 0$ if and only if $\text{Re}(\frac{z}{w}) \geq 0$, we have that $A$ and $B$ are well defined.

First, we prove that $A$ is compact. We define the following subset of $X^2$:

$$A_1 = \{(z, w) \in X^2 : \text{Re} \left( \frac{z}{w} \right) \geq 0 \}.$$
We note that $A_1$ is a closed subset of $X^2$. We consider the map $f_2 : X^2 \to F_2(X)$ given by $f_2(z, w) = \{z, w\}$, for each $(z, w) \in X^2$ (see [16, Lemma 1]). It follows that $f_2(\mathcal{A}_1) = \mathcal{A}$. This implies that $\mathcal{A}$ is a closed subset of $F_2(X)$; and hence, compact. In a similar way, we have that $\mathcal{B}$ is compact.

To prove that $\mathcal{A}$ is connected, we define $\mathcal{D} = \{(z, w) \in \mathcal{A} : z, w \in S^1\}$, $\mathcal{E} = \{(z, w) \in \mathcal{A} : z, w \in Y\}$ and $\mathcal{F} = \mathcal{D} \cup \mathcal{E}$.

We see that $\mathcal{D}$ is connected. We take the point $[1] \in \mathcal{D}$, and let $P = [z, w] \in \mathcal{D}$. Then there exist $r, s \in \mathbb{R}$ such that $z = e^{ir}$ and $w = e^{is}$. Without loss of generality, we assume that $r \leq s$. Since $P \in \mathcal{A}$, it follows that $\text{Re}(\frac{z}{w}) = \text{Re}(e^{i(r-s)}) \geq 0$. Thus, $\text{cos}(s-r) \geq 0$. Then we have two possibilities: there exists $m \in \mathbb{N} \cup \{0\}$ such that $-(\frac{4m+5}{2}) \leq r-s \leq -\pi (\frac{2m+3}{2})$ or $\frac{\pi}{2} \leq r-s \leq 0$. We suppose that $-(\frac{4m+5}{2}) \leq r-s \leq -\pi (\frac{2m+3}{2})$ (similarly, if $-\frac{\pi}{2} \leq r-s \leq 0$). Since $-(\frac{4m+5}{2}) \leq -2\pi (m+1) \leq -\pi (\frac{4m+5}{2})$, we can assume that $-(\frac{4m+5}{2}) \leq r-s \leq -2\pi (m+1)$ (similarly, if $-2\pi (m+1) \leq r-s \leq -\pi (\frac{4m+5}{2})$). Then $-\frac{\pi}{4} \leq \frac{r-s}{2} \leq 0$. We define

$$\mathcal{M} = \left\{ \left\{ e^{i(r-t)}, e^{i(s+t)} \right\} \in F_2(X) : t \in \left[ -\frac{r-s+2\pi (m+1)}{2}, 0 \right] \right\}.$$

Note that $\mathcal{M}$ is connected. To prove that $\mathcal{M}$ is a subset of $\mathcal{A}$, let $t \in \left[ -\frac{r-s+2\pi (m+1)}{2}, 0 \right]$. Then $-\frac{\pi}{4} \leq t \leq 0$, and hence, $0 \leq -2t \leq \frac{\pi}{4}$. Thus, $-(\frac{4m+5}{2}) \leq -2t+r-s \leq -\pi (\frac{2m+3}{2})$. This implies that $\text{cos}(-2t+r-s) \geq 0$. It follows that $\text{Re}(\frac{z}{w}) \geq 0$. Consequently $\mathcal{M} \subset \mathcal{A}$. Hence, $\mathcal{M} \subset \mathcal{D}$. Let $l = \frac{r-s+2\pi (m+1)}{2}$ and $Q = \{e^{il}\}$. Thus, for $t = \frac{r-s+2\pi (m+1)}{2}$, $Q \in \mathcal{M}$. Also, $P \in \mathcal{M}$. Let

$$\mathcal{N} = \left\{ \left\{ e^{i(l+t)} \right\} \in F_2(X) : t \in \mathbb{R} \right\}.$$

Then $\mathcal{N}$ is a connected subset of $\mathcal{D}$ such that $Q \cdot \{1\} \subset \mathcal{N}$. This implies that $\mathcal{M} \cup \mathcal{N}$ is a connected subset of $\mathcal{D}$ containing the points $P$ and $1$. Therefore, $\mathcal{D}$ is connected.

On the other hand, let $P = [z, w] \in \mathcal{E}$. To prove that $\mathcal{F}$ is connected it is sufficient to verify that there exists a connected subset $\mathcal{P}$ of $\mathcal{F}$ containing the point $P$ and $\mathcal{P} \cap \mathcal{D} \neq \emptyset$. Let $r, s \in \mathbb{R}$ such that $z = (\frac{r}{1+|r|} + 2)e^{ir}$ and $w = (\frac{1}{1+|r|} + 2)e^{is}$. We assume that $r \leq s$. Since $P \in \mathcal{A}$, it follows that $\text{cos}(s-r) \geq 0$. Hence, either there exists $m \in \mathbb{N} \cup \{0\}$ such that $-(\frac{4m+5}{2}) \leq r-s \leq -\pi (\frac{2m+3}{2})$ or $\frac{\pi}{2} \leq r-s \leq 0$. We suppose that $-(\frac{4m+5}{2}) \leq r-s \leq -\pi (\frac{2m+3}{2})$ (similarly, if $\frac{\pi}{2} \leq r-s \leq 0$). We assume that $-(\frac{4m+5}{2}) \leq r-s \leq -2\pi (m+1)$ (a similar argument, if $-2\pi (m+1) \leq r-s \leq -\pi (\frac{4m+5}{2})$). Let

$$\mathcal{M} = \left\{ \left\{ e^{i(r-t)} \left( \frac{1}{\left| r-t \right| + 2 \right)} e^{i(l+t)} \left( \frac{s+t}{1+|s+t|} + 2 \right) e^{i(s+t)} \right\} \in F_2(X) : t \in \left[ -\frac{r-s+2\pi (m+1)}{2}, 0 \right] \right\}.$$

Let $l = \frac{r-s+2\pi (m+1)}{2}$ and $Q = \{(\frac{1}{1+|r|} + 2)e^{il}, (\frac{1+2\pi (m+1)}{1+|r|+2\pi (m+1)} + 2)e^{i(2\pi (m+1)+l)} \}$. Hence, for $t = \frac{r-s+2\pi (m+1)}{2}$, $Q \in \mathcal{M}$. Then $\mathcal{M}$ is a connected subset of $\mathcal{F}$ such that $P, Q \in \mathcal{M}$. Let

$$\mathcal{N} = \left\{ \left\{ e^{i(l+t)} \left( \frac{1}{\left| l+t \right| + 2 \right)} e^{i(l+t)} \left( \frac{l+2\pi (m+1)+t}{1+|l+2\pi (m+1)+t|} + 2 \right) e^{i(l+2\pi (m+1)+t)} \right\} : t \in \mathbb{R} \right\}.$$

We note that $\mathcal{N}$ is a connected subset of $\mathcal{E}$ such that $Q \in \mathcal{N}$.

For each $j \in \mathbb{N}$, let $t_j = -2j\pi$ and

$$B_j = \left\{ \left\{ e^{i(l+t_j)} \left( \frac{1}{\left| l+t_j \right| + 2 \right)} e^{i(l+t)} \left( \frac{l+2\pi (m+1)+t_j}{1+|l+2\pi (m+1)+t_j|} + 2 \right) e^{i(l+2\pi (m+1)+t_j)} \right\} : t \in \mathbb{R} \right\}.$$

Hence, $\{B_j\}_j$ is a sequence contained in $\mathcal{N}$ converging to $\{e^{il}\}$. Then $\{e^{il}\} \in \text{Cl}_{F_2(X)}(\mathcal{N})$. It follows that $\mathcal{N} \cup \{e^{il}\}$ is a connected subset of $\mathcal{F}$. Thus, $\mathcal{P} = \mathcal{M} \cup \mathcal{N} \cup \{e^{il}\}$ is a connected subset of $\mathcal{F}$, $P \in \mathcal{P}$ and $\mathcal{P} \cap \mathcal{D} \neq \emptyset$. Therefore, $\mathcal{F}$ is connected.

Note that $\mathcal{A} = \text{Cl}_{F_2(X)}(\mathcal{F})$. Since $\mathcal{F}$ is connected, we have that $\mathcal{A}$ is connected. Hence, $\mathcal{A}$ is a subcontinuum of $F_2(X)$.

With similar arguments to the ones given to prove that $\mathcal{A}$ is connected, we can verify that $\mathcal{B}$ is connected. Therefore, $\mathcal{B}$ is a subcontinuum of $F_2(X)$.

To see that $\mathcal{A} \cap \mathcal{B}$ is not connected, we define

$$\mathcal{H} = \left\{ \{z, w\} \in F_2(X) : \frac{z}{w} = it, t \geq 0 \right\},$$

$$\mathcal{K} = \left\{ \{z, w\} \in F_2(X) : \frac{z}{w} = it, t \leq 0 \right\}.$$
5. Local connectedness

In this section we present results about of the $n$-fold symmetric product suspensions of locally connected continua.

Lemma 5.1. Let $X$ be a continuum and let $n \geq 2$ be a positive integer. If $F_n(X) \setminus F_1(X)$ is locally connected, then $X$ is locally connected.

Proof. Let $x$ be a point in $X$ and let $U$ be an open subset of $X$ such that $x \in U$. Let $\delta > 0$ be such that the open ball about $x$ of radius $\delta$, $V_\delta(x)$, is contained in $U$. By [4, Corollary 5.5], there exists a subcontinuum $K$ of $X$ such that $\{x\} \subseteq K \subseteq V_\delta(x)$. Let $y \in K$ be such that $x \neq y$. Let $\alpha > 0$ be such that $V_\alpha(x) \cap V_\alpha(y) = \emptyset$. We denote $A = \{x, y\}$ and let $\epsilon > 0$ be such that $\epsilon < \min\{\delta, \alpha\}$. Let $U_1 = V_\epsilon(x)$ and $U_2 = V_\epsilon(y)$. Hence, $(U_1, U_2)_\emptyset$ is an open subset of $F_n(X)$. Then $(U_1, U_2)_\emptyset \cap (F_n(X) \setminus F_1(X))$ is an open subset of $F_n(X) \setminus F_1(X)$ such that $A \in (U_1, U_2)_\emptyset \cap (F_n(X) \setminus F_1(X))$. Since $F_n(X) \setminus F_1(X)$ is locally connected, there exists a connected open subset $C$ of $F_n(X) \setminus F_1(X)$ (and hence of $F_n(X)$) such that $A \in C \subseteq (U_1, U_2)_\emptyset \cap (F_n(X) \setminus F_1(X))$. Then, by [17, Lemma 6.1], $\bigcup C$ is an open subset of $X$ and $(\bigcup C) \cap U_1$ is a connected subset of $X$. This implies that $(\bigcup C) \cap U_1$ is a connected open subset of $X$ such that $x \in (\bigcup C) \cap U_1 \subset U$. It follows that $X$ is locally connected at the point $x$. Therefore, $X$ is locally connected.

Theorem 5.2. A continuum $X$ is locally connected if and only if $SF_n(X)$ is locally connected for each positive integer $n \geq 2$.

Proof. We suppose that $X$ is locally connected. By [16, Lemma 2], we have that $F_n(X)$ is locally connected. Since $q_\alpha^n(F_n(X)) = SF_n(X)$, by [4, Proposition 8.16], it follows that $SF_n(X)$ is locally connected.

If $SF_n(X)$ is locally connected, then, by [13, Theorem 3, p. 230], $SF_n(X) \setminus \{F_1^n\}$ is locally connected. Since $SF_n(X) \setminus \{F_1^n\}$ is homeomorphic to $F_n(X) \setminus F_1(X)$, we have that $F_n(X) \setminus F_1(X)$ is locally connected. Therefore, by Lemma 5.1, $X$ is locally connected.

6. Arcwise connectedness

Given an arc $A$, let $\alpha : [0, 1] \to A$ be a homeomorphism and let $\alpha(0) = p$ and $\alpha(1) = q$. In this case, we say that $A$ is an arc from $p$ to $q$ (the end points of $A$), or we say that $A$ is an arc joining the points $p$ and $q$. A continuum $X$ is said to be arcwise connected provided that any two points can be joined by an arc contained in $X$.

Let $X$ be a continuum. It is known that if $X$ is an arcwise connected continuum, then $F_n(X)$ is an arcwise connected continuum [18, Proposition 2.7]. Hence, we have the following:

Theorem 6.1. Let $X$ be an arcwise connected continuum and let $n \geq 2$ be a positive integer. Then $SF_n(X)$ is an arcwise connected continuum.

Theorem 6.2. Let $X$ be a continuum and let $n \geq 2$ be a positive integer. Then the following are equivalent:

1. $X$ contains an arc.
2. $F_n(X)$ contains an arc.
3. $SF_n(X)$ contains an arc.

Proof. If $X$ contains an arc, then $X$ contains two disjoint arcs, from which we see that $F_n(X)$ contains an arc missing $F_1(X)$; clearly, then $SF_n(X)$ contains an arc. Hence, (1) implies (3).

(3) implies that there is an arc $\Gamma$ in $SF_n(X) \setminus \{F_1^n\}$; hence, $(q_\alpha^n)^{-1}(\Gamma)$ is an arc in $F_n(X)$.

Now we assume that $A$ is an arc in $F_n(X)$. Then, by [19, Lemma 2.2], $\bigcup A$ is a nondegenerate locally connected compact subset of $X$. Thus, $\bigcup A$ contains an arc [4, Theorem 8.23]. This proves the theorem.

As an easy consequence of Theorem 6.2, we have the following:

Remark 6.3. Let $X$ be a continuum and let $n \geq 2$. If either $X$ is the pseudo-arc [4, 1.23], or $X$ is hereditarily indecomposable [4, 1.23], or $X$ is constructed as in [4, 2.27] which is a hereditarily decomposable continuum that contains no arc, then $SF_n(X)$ is not arcwise connected.

The following theorem proves that the converse of Theorem 6.1 is not true.

Theorem 6.4. Let $X$ be a compactification of $[0, \infty)$ with a nondegenerate arcwise connected continuum $L$ as remainder, and let $n \geq 2$ be a positive integer. If there exists a retraction $r : X \to L$, then $SF_n(X)$ is an arcwise connected continuum.
Proof. Let $A \in SF_n(X) \setminus \{F^n_w\}$. To prove that $SF_n(X)$ is arcwise connected, we will check that there exists an arcwise connected subset of $SF_n(X)$ containing the points $A$ and $F^n_X$.

Let $A \in F_n(X) \setminus F_1(X)$ such that $q^n_A(A) = A$. Using the arc components of $X$, namely $L$ and $X \setminus L$, we can find a point $B \in F_2(X) \setminus F_1(X)$ and an arcwise connected subset $\Gamma_1$ of $F_n(X)$ such that $A, B \in \Gamma_1$. Hence, $q^n_A(\Gamma_1)$ is an arcwise connected subset of $SF_n(X)$ joining to $A$ and $q^n_B(B)$. Since $X$ have two arc components, we have the following cases:

**Case (1).** $B$ is in a single arc component of $X$. Let $F_2$ be an arcwise connected subset of $F_n(X)$ joining to $B$ and $F_1(X)$. Hence, $q^n_A(\Gamma_1) \cup q^n_F(\Gamma_2)$ is an arcwise connected subset of $SF_n(X)$ joining to $A$ and $F^n_X$.

**Case (2).** $B \cap L \neq \emptyset$ and $B \cap (X \setminus L) \neq \emptyset$. We assume that $B \cap (X \setminus L) = \{p\}$ and $B \cap L = \{q\}$. Let $\gamma : [0, 1) \to (X \setminus L)$ be a homeomorphism. Let $I_2$ be an arcwise connected subset of $F_n(X)$ joining to $\{p, r(p)\}$ and $B = \{p, q\}$. It follows that $q^n_A(I_2)$ is an arcwise connected subset of $SF_n(X)$ joining to $q^n_A(p, r(p))$ and $q^n_A(B)$. We consider the map $\delta : [0, 1) \to F_n(X)$ be given by $\delta(t) = (\gamma(t), r(\gamma(t)))$. Let $\mathcal{M} = \delta([0, 1))$. We note that $\delta$ is a homeomorphism onto $\mathcal{M}$ such that $\{p, r(p)\} \in \mathcal{M}$. Let $I_3 = Cl_{F_n(X)}(\mathcal{M})$. Then $I_3$ is a compactification of $[0, 1)$. Let $R = I_3 \setminus \mathcal{M}$ be the remainder. Since $R \subset F_1(X)$, we have that $q^n_R(R) = \{F^n_X\}$. Also, since $\mathcal{M} \cap F_1(X) = \emptyset$, $q^n_R(\mathcal{M})$ is a ray. Then $q^n_A(I_3) = q^n_R(\mathcal{M}) \cup \{F^n_X\}$ is an arc joining the points $q^n_A(\{p, r(p)\})$ and $F^n_X$. Thus, $q^n_A(I_1) \cup q^n_A(I_2) \cup q^n_A(I_3)$ is an arcwise connected subset of $SF_n(X)$ containing to $A$ and $F^n_X$. Therefore, $SF_n(X)$ is an arcwise connected continuum. □

The following corollary is an easy consequence of Theorem 6.4 and [20, p. 30]:

**Corollary 6.5.** Let $X$ be a compactification of $[0, \infty)$ with a nondegenerate locally connected continuum as remainder, and let $n \geq 2$ be a positive integer. Then $SF_n(X)$ is an arcwise connected continuum.

We define $S^+ = \{(1 + e^{-\theta}, \theta) : \theta \geq 0\}$, $S^- = \{(1 + e^{\theta}, -\theta) : \theta \geq 0\}$, and let

$$W = S^1 \cup S^+ \cup S^-.$$  

The Waraszkiewicz spirals are subcontinua of $W$ [12, 2.6.35]. However, $W$ is not a Waraszkiewicz spiral.

**Remark 6.6.** Note that Theorem 6.4 implies that the $n$-fold symmetric product suspensions of the Elsa continua [21, p. 329] and the Waraszkiewicz spirals [22] are arcwise connected continua.

Next we will prove that $SF_n(W)$ is an arcwise connected continuum.

**Example 6.7.** Let $n \in \mathbb{N}$ be such that $n \geq 2$. Then $SF_n(W)$ is an arcwise connected continuum.

To see this let $A \in SF_n(W) \setminus \{F^n_W\}$. Then there exists $A \in F_n(W) \setminus F_1(W)$ such that $q^n_A(A) = A$. Let $\gamma : [0, 1) \to S^1$ be a homeomorphism, $p = \gamma(0)$ and $q = \phi(p)$, where $\phi : W \to S^1$ is the projection map given by $\phi(r, \theta) = (1, \theta)$. We have the following cases:

**Case (1).** $A$ is in a single arc component of $X$. Then there exists an arcwise connected subset of $SF_n(X)$ joining to $A$ and $F^n_X$.

**Case (2).** $A \cap S^1 \neq \emptyset$ and $A \cap (S^+ \cup S^-) \neq \emptyset$. Since $S^1$ and $S^+ \cup S^-$ are arcwise connected subsets of $W$, we have an arcwise connected subset $\Gamma_1$ of $F_n(W)$ such that $A, \{p, q\} \in \Gamma_1$. It follows that $q^n_W(\Gamma_1)$ is an arcwise connected subset of $SF_n(W)$ containing to $A$ and $q^n_W(\{p, q\})$.

Let $\delta : [0, 1) \to F_n(W)$ be the map given by $\delta(t) = \{\gamma(\gamma(t)), r(t)\}$. Let $\mathcal{M} = \delta([0, 1))$. Then $\delta$ is a homeomorphism onto $\mathcal{M}$ such that $\{p, q\} \in \mathcal{M}$. Let $I_2 = Cl_{F_n(W)}(\mathcal{M})$. Thus, $I_2$ is a compactification of $[0, 1)$. Let $R = F_2 \setminus \mathcal{M}$ be the remainder. Note that $R \subset F_1(W)$ and $\mathcal{M} \cap F_1(W) = \emptyset$. It follows that $q^n_W(I_2)$ is an arc joining the points $q^n_W(\{p, q\})$ and $F^n_W$. Hence, $q^n_W(I_1) \cup q^n_W(I_2)$ is an arcwise connected subset of $SF_n(W)$ containing to $A$ and $F^n_W$. Therefore, $SF_n(W)$ is an arcwise connected continuum.

With the next theorem we give a class of continua, for which their $n$-fold symmetric product suspensions are not arcwise connected.

**Theorem 6.8.** Let $X$ be a continuum, and let $n \in \mathbb{N}$ be such that $n \geq 2$. If there exist two different closed arc components of $X$, then $SF_n(X)$ is not arcwise connected.

**Proof.** Let $L_1$ and $L_2$ be two different closed arc components of $X$. Let $a_1 \in L_1$, $a_2 \in L_2$, and let $A = \{a_1, a_2\}$. Let $A$ be an arc in $F_n(X)$ such that $A \in A$. First we will prove that $A \subset (L_1 \cup L_2)_n$. By [19, Lemma 2.2], we have that $\bigcup A$ is a compact locally connected subset of $X$. If $\bigcup A$ is a subcontinuum of $X$, then by [4, Theorem 8.23], $\bigcup A$ is an arcwise connected subcontinuum of $X$. This implies that $L_1 \cap L_2 = \emptyset$, this is a contradiction. Hence, $\bigcup A$ is not a subcontinuum of $X$. On the
other hand, by [17, Lemma 2.2], \( \bigcup A \) has at most two components. Let \( \bigcup A = C_1 \cup C_2 \), where \( C_1 \) and \( C_2 \) are the components of \( \bigcup A \). We note that \( C_1 \) and \( C_2 \) are arcwise connected subcontinua of \( X \). Thus, neither \( A \subset C_1 \) nor \( A \subset C_2 \). Hence, we can assume that \( a_1 \in C_1 \) and \( a_2 \in C_2 \). This implies that \( C_1 \subset L_1 \) and \( C_2 \subset L_2 \). It follows that \( \bigcup A \subset L_1 \cup L_2 \).

Now let \( b \in A \). Then \( B \subset L_1 \cup L_2 \). If \( B \cap L_1 = \emptyset \), then \( B \subset L_2 \). Let \( b \in B \). Since \( F_n(X) \) is an arcwise connected subset of \( F_n(X) \) [18, Proposition 2.7], there exists an arc \( B \) contained in \( F_n(X) \) joining to \( B \) and \( \{b\} \). It follows that \( A \cup B \) is a locally connected subcontinuum of \( F_n(X) \) [13, Theorem 1, p. 230] of \( F_n(X) \). By [19, Lemma 2.2] and [17, Lemma 2.2], we have that \( \bigcup (A \cup B) \) is a locally connected subcontinuum of \( X \). Hence, by [4, Theorem 8.23], \( \bigcup (A \cup B) \) is an arcwise connected subcontinuum of \( X \) such that \( a_1, a_2 \in \bigcup (A \cup B) \). Therefore, we have a contradiction. Hence \( B \cap L_1 \neq \emptyset \). Similarly \( B \cap L_2 \neq \emptyset \). This implies that \( B \in (L_1, L_2)_n \). Thus, \( A \subset (L_1, L_2)_n \). Therefore, if \( A \) is an arc in \( F_n(X) \) such that \( A \in A \), then \( \bigcup (A \subset (L_1, L_2)_n \).

Since \((L_1, L_2)_n \) is closed subset of \( F_n(X) \) and \((L_1, L_2)_n \cap F_1(X) = \emptyset \), let \( U \) be an open subset of \( F_n(X) \) such that \( F_1(X) \subset U \) and \( U \cap (L_1, L_2)_n = \emptyset \). It follows that \( F^n_\alpha(U) \) is an open subset of \( F^n_\alpha(X) \) such that \( F^n_\alpha(U) \) is arcwise connected. Since \( A \subset F_n(X) \setminus F_1(X), q^n(A) \subset F_n(X) \setminus \{F^n(\alpha)\} \). Let \( \Gamma \) be an arc in \( F_n(X) \) joining to \( q^n(A) \) and \( F^n \), and let \( \alpha : [0, 1] \to \Gamma \) be a homeomorphism such that \( \alpha(0) = q^n(A) \) and \( \alpha(1) = F^n \). We take a point \( r \in [0, 1] \) such that \( \alpha(r) \in q^n(U) \subset \{F^n(\alpha)\} \). Let \( D \subset F_n(X) \) such that \( q^n(D) = \alpha(r) \). We consider the subarc \( \alpha([0, r]) \) of \( \Gamma \). Then \( (q^n)^{-1}(\alpha([0, r])) \) is an arc contained in \( F_n(X) \) such that \( A_\alpha \subset (q^n)^{-1}(\alpha([0, r])) \). Then \( (q^n)^{-1}(\alpha([0, r])) \subset (L_1, L_2)_n \). On the other hand, \( D \subset U \equiv (q^n)^{-1}(q^n(U)) \). Thus \( U \cap (L_1, L_2)_n = \emptyset \). Therefore, we have a contradiction. This implies that \( F_n(X) \) is not arcwise connected.

\[ \square \]

**Remark 6.9.** Let \( X \) be the continuum of Example 4.4 and let \( n \geq 2 \) be a positive integer. As a consequence of Theorem 6.8, we have that \( F_n(X) \) is not arcwise connected. On the other hand, related to Theorem 6.4, we note that there exist compactifications of \([0, \infty)\) whose \( n \)-fold symmetric product suspensions are not arcwise connected, namely: let \( Z \) be a compactification of \([0, \infty)\) with the continuum \( X \) of Example 4.4 as remainder. By Theorem 6.8, it follows that \( F_n(Z) \) is not arcwise connected.

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**References**


