Approximate Inertial Manifolds for 2D Navier–Stokes Equations

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In this paper we provide two approximate inertial manifolds for the 2D Navier–Stokes equations in the periodic case. The first one is an inertial manifold for a dissipative equation obtained by modifying the Navier–Stokes equations, namely, by enlarging the gaps of the eigenvalues of the Stokes operator. The second one, which is explicit and simple, is found by approximating the previous approximate manifold with the Euler–Galerkin scheme.

1. INTRODUCTION

In the study of the long-time behavior of the solutions of dissipative dynamical systems, absorbing sets, attractors, and inertial manifolds play an essential role. When the underlying dynamical system, usually a PDE, has an inertial manifold, its study reduces to the ODE on the inertial manifold, that is, to that of an inertial form of the PDE.

Unfortunately the existence of an inertial manifold for the Navier–Stokes equations has not been proven. Therefore one is obliged to use manifolds that are not necessarily inertial but close to the global attractor, to approximate the permanent regime of the dynamical system. These manifolds are often called approximate inertial manifolds (see [2, 4, 5, 8, 10, 13–15]).

In this paper we provide two new approximate inertial manifolds for the 2D Navier–Stokes equations in the periodic case. One is an inertial manifold for a dissipative equation obtained by modifying the Navier–Stokes equations, namely, by modifying the spectrum of the Stokes operator in a narrow band so that the new dissipative operator satisfies the standard spectrum gap condition under which the existence of an inertial manifold for the corresponding dissipative equation can be obtained. By estimating the distance between two corresponding solutions of the Navier–Stokes equations and their modified equation, we obtain that all
the solutions of the Navier-Stokes equations enter a narrow neighborhood of the inertial manifold for the modified equation and we give an estimate of the size of the neighborhood which obviously sustains the attractor of the Navier-Stokes equations.

Since the approximate inertial manifold described above is not convenient for numerical computations, we approximate it by the Euler-Galerkin scheme and obtain another approximate inertial manifold for the Navier-Stokes equations. This manifold has some advantages over the previous one in practice. It is a graph of a quadratic form which has an explicit expression and is easy for computations, while it has still the same dimension as the previous manifold and lies as close as the previous manifold to the global attractor.

This paper is organized as follows. In Section 2, we present some background material about the 2D Navier–Stokes equations in the periodic case. In Section 3, we modify the 2D Navier–Stokes equations by enlarging the gaps between its eigenvalues and prove the existence of an inertial manifold for the modified equation. In Section 4, we estimate the distance between the attractor of the Navier–Stokes equations and the approximate manifold obtained in Section 3. By approximating the approximate inertial manifold constructed in Section 3 with the Euler–Galerkin scheme, we obtain an explicit and simple approximate inertial manifold in Section 5.

2. Preliminaries

The Navier–Stokes equations of two dimensional viscous incompressible flows are written as

\[
\frac{\partial u}{\partial t} - v \Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{for} \quad (x, t) \in \Omega \times \mathbb{R}_+, \tag{2.1}
\]

\[
\nabla \cdot u = 0 \quad \text{for} \quad (x, t) \in \Omega \times \mathbb{R}_+, \tag{2.2}
\]

where \( \Omega = (0, L_1) \times (0, L_2) \), \( u = u(x, t) = \{u_1, u_2\} \) is the velocity of the particle of the fluid which is at point \( x \) at time \( t \), \( p = p(x, t) \) is the pressure of the fluid at point \( x \) at time \( t \), \( f = f(x) \) represents the density of force per unit volume at point \( x \), \( v > 0 \) is the kinematic viscosity. Equation (2.1) is the momentum conservation equation and Eq. (2.2) is the mass conservation equation (incompressibility condition). We supplement (2.1) (2.2) with periodic boundary conditions:

\[
u \text{ and } p \text{ are periodic of period } L_i \text{ in the direction } x_i, \ i = 1, 2, \tag{2.3}\]
and for simplicity we assume
\[ \int_{\Omega} u \, dx = 0. \]  

(2.4)

We denote by
\[ V = \left\{ v \in H^1_{\text{per}}(\Omega)^2 : \text{div } v = 0, \int_{\Omega} v(x) \, dx = 0 \right\} \]

and set
\[ H = \text{closure of } V \text{ in } L^2(\Omega)^2 = \left\{ u \in L^2(\Omega)^2 : \text{div } u = 0, \int_{\Omega} u \, dx = 0 \right\}. \]

If \( V \) is equipped with the norm \( \| v \|_V = \| \text{grad } v \|_{L^2(\Omega)} \) and \( H \) is equipped with the norm induced from \( L^2(\Omega)^2 \), \( V \) and \( H \) become Hilbert spaces.

Let \( P \) be the projector of \( L^2(\Omega)^2 \) onto \( H \), we define the Stokes operator \( A : D(A) = V \cap H^2(\Omega)^2 \rightarrow H \) by
\[ Au = -P A u = -A u, \quad \forall u \in D(A). \]  

(2.5)

\( A \) is a linear, self-adjoint, unbounded positive operator in \( H \) with domain \( D(A) \). Its inverse \( A^{-1} \) is compact and self-adjoint, as a result, \( H \) has an orthonormal basis \( \{ w_j \}_{j=1}^{\infty} \) consisting of eigenfunctions of the operator \( A \), with corresponding eigenvalues \( \lambda_j \), i.e.,
\[ A w_j = \lambda_j w_j, \quad j = 1, 2, \ldots, \]
\[ 0 < \lambda_1 < \lambda_2 < \cdots. \]  

(2.6)

It is well known (see [1, 11]) that the eigenfunctions \( w_j \) are related to the following sine and cosine functions:
\[ \frac{\bar{k}}{|k|} \sin \left( 2\pi \frac{k}{L} \cdot x \right), \quad \frac{\bar{k}}{|k|} \cos \left( 2\pi \frac{k}{L} \cdot x \right), \]  

(2.7)

where
\[ k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \quad \bar{k} = (k_2, -k_1), \quad \text{and} \quad \frac{k}{L} = \left( \frac{k_1}{L_1}, \frac{k_2}{L_2} \right). \]

We can define the power \( A^\alpha \) for \( \alpha \in \mathbb{R} \) with domain
\[ D(A^\alpha) = \left\{ u = \sum_{j=1}^{\infty} u_j w_j : \sum_{j=1}^{\infty} \lambda_j^{2\alpha} u_j^2 < \infty \right\}. \]

\( D(A^\alpha) \) is a Hilbert space if it is endowed with norm \( \| \cdot \|_{D(A^\alpha)} := \| A^\alpha \cdot \|_H \).
Thus $A$ becomes an isomorphism from $D(A^x) \to D(A^{x-1})$. We readily see that $D(A) = D(A^1)$, $V = D(A^{1/2})$, $H = D(A^0)$, and $V' = D(A^{1/2})$. For clarity the norm and scalar product of $H$, $L^2(\Omega)$, or $L^2(\Omega)^2$ will be denoted by $|\cdot|$ and $(,)$, respectively, and the norm $|A^{1/2} \cdot |_H$ on $V$, which is equivalent to $\| \text{grad} \cdot \|_{L^2(\Omega)^2}$, will be denoted by $\| \cdot \|$.

We define a bilinear $B: V \times V \to V'$ by

$$B(u, v) = P[(u \cdot \nabla)v], \quad \forall u, v \in V.$$  \hspace{1cm} (2.8)

Under the above notations, Eqs. (2.1)-(2.3) reduce to an abstract evolution equation in $H$,

$$\frac{du}{dt} + vAu + B(u) = f,$$  \hspace{1cm} (2.9)

where $B(u) = B(u, u)$.

We recall (see \cite{9, 11}) that for $f, u_0$ given in $H$ the initial value problem (2.9), (2.10),

$$u = u_0$$  \hspace{1cm} (2.10)

possesses a unique solution $u$ defined for all $t \geq 0$ and such that

$$u \in C(\mathbb{R}_+; H) \cap L^2(0, T; V), \quad \forall T > 0.$$  

If $u_0 \in V$, then

$$u \in C(\mathbb{R}_+; V) \cap L^2(0, T; D(A)), \quad \forall T > 0.$$  

We denote by $S(t), t \geq 0$ the operators in $H: u_0 \mapsto u(t)$, where $u(\cdot)$ is the unique solution of (2.9), (2.10). The operators $S(t), t \geq 0$, form a semigroup in $H$. There exist $\rho_0, \rho_1, \rho_2$ such that the balls $B_H(0, \rho_0)$, $B_V(0, \rho_1)$, $B_{D(A)}(0, \rho_2)$ are absorbing in $H$, $V$, and $D(A)$, respectively, provided that $u_0, f \in V$. Namely, for every $R > 0$, there exists $t_*(R) > 0$ such that for $t \geq t_*(R)$, $S(t) B_H(0, R) \subset B_H(0, \rho_0)$, $S(t) B_V(0, R) \subset B_V(0, \rho_1)$, and $S(t) B_{D(A)}(0, R) \subset B_{D(A)}(0, \rho_2)$.

The following continuity properties of $B$ will be frequently used (see \cite{1, 12}):

$$|B(u, v)| \leq C_1 |u|^{1/2} \| u \|^{1/2} \| v \|^{1/2} \| Av \|^{1/2}, \quad \forall u \in V, v \in D(A),$$  \hspace{1cm} (2.11)

$$|B(u, v)| \leq C_2 |u|^{1/2} \| Av \|, \quad \forall u \in D(A), v \in V,$$  \hspace{1cm} (2.12)

$$|B(u, v, w)| \leq C_3 \left[ |u|^{1/2} \| u \|^{1/2} \| v \|^{1/2} \| Aw \|^{1/2}, \quad \forall u \in H, v \in V, w \in D(A), \right.$$

$$\left. |u|^{1/2} \| u \|^{1/2} \| v \|^{1/2} \| w \|^{1/2} \| w \|^{1/2}, \quad \forall u, v, w \in V. \right.$$  \hspace{1cm} (2.13)
We conclude this section by recalling the estimate on the growth rate of
the eigenvalues $\lambda_j$ of the Stokes operator $A$ which will be needed later (see [1]).

**Lemma 1.** The asymptotic behavior of the eigenvalues $\lambda_j$ of the operator $A$ is given by

$$\lim_{j \to \infty} j^{-1} \lambda_j = \frac{4\pi}{L_1 L_2}.$$  (2.14)

More precisely,

$$\sqrt{\frac{j + 1}{\pi L_1 L_2} - \frac{1}{2}} \ll \sqrt{\frac{\lambda_j}{2\pi}} \ll \sqrt{\frac{j + 1}{\pi L_1 L_2} + \frac{1}{2}} \frac{1}{|L|},$$  (2.15)

where $1/L = (1/L_1, 1/L_2)$.

### 3. Modified Equation

#### 3.1. The Equations

As indicated in the introduction a sufficient condition for the existence of an inertial manifold for a dissipative equation depends on the size of the gap between two sufficiently large successive eigenvalues of its dissipative operator, on which the Navier–Stokes equations in dimension two fail. We modify the abstract Navier–Stokes equation (2.9) by enlarging the gap between the eigenvalues of the Stokes operator so that the modified equation has an inertial manifold which approximates the attractor of the Navier–Stokes equations.

For any $n, m \in \mathbb{N}$, we define $\tilde{A}_{n,m} : D(\tilde{A}) (= D(A)) \to H$ by

$$\tilde{A}_{n,m}w_j = \begin{cases} 
\lambda_j w_j, & \forall 1 \leq j < n \text{ or } j > n + 2m,
\lambda_n w_j, & \forall n \leq j \leq n + m,
\lambda_{n+2m} w_j, & \forall n + m + 1 \leq j \leq n + 2m.
\end{cases}$$  (3.1)

In what follows we fix $n$ and $m$ and assume $m \leq n$. When the index $(n, m)$ is understood we write $\tilde{A} = \tilde{A}_{n,m}$. $\tilde{A}$ is a linear, self-adjoint unbounded positive operator in $H$ with domain $D(\tilde{A}) = D(A)$ and whose inverse $\tilde{A}^{-1}$ is compact, self-adjoint, and positive in $H$. Clearly $D(\tilde{A}^{\alpha}) = D(A^{\alpha})$ and $\| \cdot \|_{D(A^{\alpha})}$ is equivalent to $\| \cdot \|_{D(\tilde{A}^{\alpha})}$ for $\alpha \geq 0$ since

$$\left( \frac{\lambda_n}{\lambda_{n+m}} \right)^{\alpha} |A^{\alpha}u| \leq |\tilde{A}^{\alpha}u| \leq \left( \frac{\lambda_n + 2m}{\lambda_{n+m}} \right)^{\alpha} |A^{\alpha}u|, \quad \forall u \in D(A^{\alpha}), \alpha \geq 0.$$  (3.2)
Furthermore, for all \( \alpha, \beta \geq 0 \),
\[
\left( \frac{\lambda_n}{\lambda_{n+m}} \right)^{\alpha} |A^{(\alpha+\beta)/2}u|^2 \leq (\widetilde{A}^\alpha u, A^\beta u) \leq \left( \frac{\lambda_n+2m}{\lambda_{n+m}} \right)^{\alpha} |A^{(\alpha+\beta)/2}u|^2,
\]
\( \forall u \in D(A), \gamma = \max\{\alpha, \beta\}. \tag{3.3} \)

From (2.15) and the assumption \( m \leq n \) there exist constants \( \theta_1, \theta_2 > 0 \) depending only on \( \Omega \) such that \( \lambda_n/\lambda_{n+m} > \theta_1 \) and \( \lambda_{n+2m}/\lambda_{n+m} < \theta_2 \). Thus (3.2), (3.3) imply respectively that
\[
\theta_1^\alpha |A^\alpha u| \leq |\widetilde{A}^\alpha u| \leq \theta_2^\alpha |A^\alpha u|, \quad \forall u \in D(A), \alpha \geq 0, \tag{3.4}
\]
and
\[
\theta_1^\alpha |A^{(\alpha+\beta)/2}u|^2 \leq (\widetilde{A}^\alpha u, A^\beta u) \leq \theta_2^\alpha |A^{(\alpha+\beta)/2}u|^2, \quad \forall u \in D(A), \alpha, \beta \geq 0. \tag{3.5}
\]

We replace \( A \) by \( \widetilde{A} \) in (2.9) and obtain the modified Navier–Stokes equation,
\[
\frac{du}{dt} + \nu \widetilde{A}u + B(u) = f \tag{3.6}
\]
supplemented with initial condition
\[
u(0) = u_0. \tag{3.7}
\]

Following the proofs in [11] (also see [1]) for the existence and regularities of the solutions for the Navier–Stokes equations, we can prove that for any \( f, u_0 \) given in \( H \), the initial value problem (3.6), (3.7) possesses a unique solution \( u \) defined for all \( t \geq 0 \), which we denote by \( \mathcal{S}(t)u_0 \), and such that

\[
u \in C(\mathbb{R}_+; H) \cap L^2(0, T; V), \quad \forall T > 0.
\]

If \( u_0 \in V \), then
\[
u \in C(\mathbb{R}_+; V) \cap L^2(0, T; D(A)), \quad \forall T > 0.
\]

3.2. Absorbing Sets and Attractors

In this section we study the existence of the global attractor associated to the modified equation (3.6). The results obtained are similar to those for the Navier–Stokes equations.

First following the argument as in [12] for the existence of an absorbing set in \( H \) for the Navier–Stokes equation (2.9), we see that the ball
B_H (0, \tilde{\rho}_0) is absorbing in \( H \) for the modified equation (3.6) with 
\[ \tilde{\rho}_0 = \sqrt{2} \frac{|f|}{(v\lambda_1)}. \]
Specifically, for any \( R > 0 \),
\[ \mathcal{S}(t) B_H (0, R) \subset B_H (0, \tilde{\rho}_0), \quad \forall t \geq t_0 = \frac{1}{v\lambda_1} \ln \frac{2R^2 - \tilde{\rho}_0^2}{\tilde{\rho}_0^2}. \] (3.8)

We prove the existence of an absorbing set in \( V \) for the modified equation.
By taking the scalar product of (3.6) with \( Au \) in \( H \) and using the identity (see [11])
\[ (B(u), Au) = 0, \quad \forall u \in D(A), \] (3.9)
we obtain
\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 + v(\tilde{A}u, Au) = (f, Au). \] (3.10)
Using (3.5) with \( a = b = 1 \) we obtain
\[ \frac{1}{2} \frac{d}{dt} \|u\|^2 + v\theta_1 |Au|^2 \leq |f| |Au|. \]

We apply Young's inequality and have
\[ \frac{d}{dt} \|u\|^2 + v\theta_1 |Au|^2 \leq \frac{|f|^2}{v\theta_1}. \] (3.11)
Since \( |Au| \geq \sqrt{\lambda_1} \|u\| \), (3.11) yields
\[ \frac{d}{dt} \|u\|^2 + v\lambda_1 \theta_1 \|u\|^2 \leq \frac{|f|^2}{v\theta_1}. \] (3.12)

By making use of Gronwall's inequality, we obtain
\[ \|u(t)\|^2 \leq \|u(0)\|^2 \exp(-v\lambda_1 \theta_1 t) + \frac{|f|^2}{v^2 \theta_1^2 \lambda_1} (1 - \exp(-v\lambda_1 \theta_1 t)). \]

Thus for any \( R > 0 \) and \( u(0) \in B_r (0, R) \),
\[ \|u(t)\| \leq \tilde{\rho}_1, \quad \text{for} \quad t \geq t_1 = \frac{1}{v\lambda_1 \theta_1} \ln \frac{2R^2 - \tilde{\rho}_0^2}{\tilde{\rho}_0^2}, \] (3.13)
where \( \tilde{\rho}_1 = \sqrt{2} \frac{|f|}{(v\theta_1 \sqrt{\lambda_1})}. \)
Therefore \( B_r (0, \tilde{\rho}_1) \) is an absorbing set in \( V \) for equation (3.6).
Before we proceed to prove the existence of an absorbing set in $D(A)$ for the modified equation (3.6), we state the uniform Gronwall’s lemma, whose proof can be found in [3, 12], as in the following:

**Lemma 2 (The Uniform Gronwall Lemma).** Let $g, h, y$ be three positive locally integrable functions on $(t_*, + \infty)$ such that $y'$ is locally integrable on $(t_*, + \infty)$, and which satisfy

$$
\frac{dy}{dt} \leq gy + h \quad \text{for} \quad t \geq t_*,
$$

$$
\int_t^{t+r} g(s) \, ds \leq a_1, \quad \int_t^{t+r} h(s) \, ds \leq a_2, \quad \int_t^{t+r} y(s) \, ds \leq a_3, \quad \forall t \geq t_*.
$$

where $r, a_1, a_2, a_3$ are positive constants. Then

$$
y(t) \leq \left( \frac{a_3}{r} + a_2 \right) \exp a_1, \quad \forall t \geq t_* + r.
$$

Now we prove the existence of an absorbing set in $D(A)$. The argument is formal and can be justified by Galerkin approximation. For simplicity we assume $f \in V$. However, the estimate can also be achieved for $f \in H$ from the analyticity of the solutions of (3.6) (see [1, 9, 11]). We take scalar product of (3.6) with $A^2u$ in $H$ and obtain an energy-type equation

$$
\frac{1}{2} \frac{d}{dt} |Au|^2 + v(\bar{A}u, A^2u) + (B(u), A^2u) = (f, A^2u).
$$

(3.17)

Using the following identities (see [11]),

$$
AB(u) = B(u, Au) - B(Au, u), \quad \forall u \in D(A^{3/2}),
$$

$$
(B(u, v), v) = 0, \quad \forall u, v \in V.
$$

(3.18)

(3.19)

we see that

$$
(B(u), A^2u) = (AB(u), Au)
$$

$$
= -(B(Au, u), Au).
$$

Since the second inequality in (2.13)

$$
|(B(u), A^2u)| \leq C_3 |Au| \|u\| |A^{3/2}u|.
$$

(3.20)

By making use of (3.5) and (3.20), (3.17) yields

$$
\frac{1}{2} \frac{d}{dt} |Au|^2 + v \theta_1 |A^{3/2}u|^2 \leq \|f\| |A^{3/2}u| + C_3 \|u\| \|Au\| |A^{3/2}u|,
$$


applying Young's inequality,

\[ \leq v\theta_1 |A^{3/2}u|^2 + \frac{1}{2v\theta_1} \|f\|^2 + \frac{C_1^2}{2v\theta_1} \|u\|^2 |Au|^2, \]

hence

\[ \frac{d}{dt} |Au|^2 \leq \frac{1}{v\theta_1} \|f\|^2 + \frac{C_1^2}{v\theta_1} \|u\|^2 |Au|^2. \] (3.21)

Assuming that \( u_0 \) belongs to a bounded set \( B_{\rho}(0, R) \) in \( V \) and that \( t \geq t_1, t_1 \) as in (3.13). We apply the uniform Gronwall lemma (Lemma 2) to (3.21) with \( g, h, y, t_* \) replaced by \( C_1^2 \|u\|^2/(v\theta_1), \|f\|^2/(v\theta_1), |Au|^2, t_1 \). Thanks to (3.13), we estimate the quantities \( a_1, a_2 \) in Lemma 2 by

\[ a_1 = \frac{C_1^2 \tilde{\rho}_1^2 r}{v\theta_1}, \quad a_2 = \frac{\|f\|^2 r}{v\theta_1}. \] (3.22)

From (3.11) and (3.13)

\[ \int_t^{t+r} |Au(s)|^2 \, ds = \frac{1}{v\theta_1} \int_t^{t+r} \left( \frac{|f|^2}{v\theta_1} - \frac{d}{ds} \|u(s)\|^2 \right) \, ds \]

\[ \leq \frac{|f|^2 r}{v^2 \theta_1} + \frac{\tilde{\rho}_1^2}{v\theta_1}. \]

Thus \( a_3 \) can be chosen as

\[ a_3 = \frac{|f|^2 r}{v^2 \theta_1} + \frac{\tilde{\rho}_1^2}{v\theta_1}. \] (3.23)

Therefore

\[ |Au(t)|^2 \leq \left( \frac{a_3}{r} + a_2 \right) \exp a_1, \quad \forall t \geq t_1 + r. \] (3.24)

Hence, by (3.4)

\[ |\tilde{A}u(t)|^2 \leq \theta_2^2 \left( \frac{a_3}{r} + a_2 \right) \exp a_1, \quad \forall t \geq t_1 + r. \] (3.25)

Let us fix \( r > 0 \), say \( r = 1 \), and denote by \( \tilde{\rho}_2 \) the right-hand side of (3.25). We then conclude that the ball \( B_{\tilde{D}(\tilde{A})}(0, \tilde{\rho}_2) \) of \( D(\tilde{A}) \) is an absorbing set in \( D(\tilde{A}) \) for the modified equation (3.6). Furthermore, for any bounded set \( B_{\tilde{D}(\tilde{A})}(0, R) \) in \( V \), \( \tilde{S}(t) B_{\tilde{D}(\tilde{A})}(0, R) \subset B_{\tilde{D}(\tilde{A})}(0, \tilde{\rho}_2) \) for \( t \geq t_2 = t_1 + 1 \).
Since the inclusion mapping from $D(\tilde{A})$ to $V$ is compact, the operators $\tilde{S}(t)$ are uniformly compact in $V$. We conclude (see [12]) that the modified equation (3.6) possesses a global attractor $\mathcal{A}$ that is compact, connected, and maximal in $V$. $\mathcal{A}$ attracts the bounded sets $V$ and $\mathcal{A}$ is also maximal among the bounded sets in $V$ which are both positively and negatively invariant. Similarly there exists an absorbing set in $D(\tilde{A})$ for the Navier–Stokes equation (2.9). From the observation of the fact that $\tilde{\rho}_1$, $\tilde{\rho}_2$ depend only on $v$, $\Omega$, and $f$, not on the choice of $n$ and $m$, we make the following remark:

**Remark.** There exist $\rho_0, \rho_1, \rho_2 > 0$, depending only on $v$, $\Omega$, and $f$ such that $B_H(0, \rho_0)$, $B_V(0, \rho_1)$, $B_{D(\tilde{A})}(0, \rho_2)$ are the absorbing sets for both the Navier–Stokes equation (2.9) and the modified equation (3.6) in $H$, $V$, $D(\tilde{A})$, respectively.

### 3.3. The Existence of an Inertial Manifold

In this section we search for an inertial manifold for the modified Navier–Stokes equation (3.6) which we expect to be an approximate inertial manifold for the Navier–Stokes equation (2.9).

From the remark in the previous section the ball $B_{D(\tilde{A})}(0, \rho_2)$ in $D(\tilde{A})$ is an absorbing set for both the Navier–Stokes equation (2.9) and its modified equation. This allows us to avoid certain technical difficulties, for large value $|\tilde{A}u|$, resulting from the nonlinear term $B(u)$ by truncating $B(u)$ and to consider instead the new equation, which is called prepared equation, with the truncated nonlinear term that provides the same symptotic behavior near the global attractor. Let $\theta: \mathbb{R}^+ \to [0, 1]$ be a fixed $C^1$ function with $\theta(s) = 1$ for $0 \leq s \leq 2$, $\theta(s) = 0$ for $s \geq 4$, and $|\theta'(s)| \leq 2$ for $s \geq 0$. We consider the prepared equation of (3.6)

$$\frac{du}{dt} + v\tilde{A}u + F(u) = 0,$$  \hspace{1cm} (3.26)

where $F(u) = \theta(|\tilde{A}u|^2/4\rho_2^2)(B(u) - f)$. The existence, uniqueness, and regularities of the solution for (3.26) together with the initial condition (3.7) are the same as for that of (3.6) and (3.7). We denote by $\tilde{S}(t)u_0$ the solution of (3.26) and (3.7). Clearly $\tilde{S}(t) = \tilde{S}(t)$ in $B_{D(\tilde{A})}(0, \rho_2)$ for all $t \geq 0$.

As in [7] we define an inertial manifold $\mathcal{M}$ for the prepared equation (3.26) in $H$ to be a subset of $H$ satisfying the following properties:

1. $\mathcal{M}$ is a finite-dimensional Lipschitz manifold;
2. $\mathcal{M}$ is positively invariant, i.e., $\tilde{S}(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$;
3. for all $R > 0$, $\tilde{S}(t)u_0$ uniformly converges to $\mathcal{M}$ at an exponential rate for every $u_0 \in B_H(0, R)$. 

Clearly the global attractor is contained in $\mathcal{M}$. We will apply the results in [8] to study the existence of an inertial manifold for the prepared equation (3.26).

We denote by $R(u) = B(u) - f$, $R(\cdot)$ is a nonlinear differential map from $D(A)$ to $D(A^{1/2})$ and

$$R'(u)v = B(v, u) + B(u, v), \quad \forall u, v \in D(A). \quad (3.27)$$

The following proposition gives two continuity properties of $R'$.

**Proposition 3.**

$$|R'(u)v| \leq C_1 |Au| |v|, \quad \forall u, v \in D(A), \quad (3.28)$$

$$\|R'(u)v\| \leq C_2 |Au| |Av|, \quad \forall u, v \in D(A), \quad (3.29)$$

where $C_1 = (C_1 + C_2)\lambda_1^{-1/2}$ and $C_2 = (C_2 + C_3)\lambda_1^{1/2}$.

**Proof.** Using (2.11) and (2.12), we find that

$$|R'(u)v| \leq |B(v, u)| + |B(u, v)|$$

$$\leq C_1 |v|^{1/2} |v|^{1/2} |u|^{1/2} |Au|^{1/2} + C_2 |u|^{1/2} |Au|^{1/2} |v|$$

$$\leq (C_1 + C_2) \lambda_1^{-1/2} |Au| |v|.$$

So (3.28) is obtained.

To derive (3.29), we take any $w \in V$ and write

$$(A^{1/2}R'(u)v, A^{1/2}w) = (AR'(u), w),$$

from (3.27)

$$= (A(B(u, v) + B(v, u)), w),$$

by bilinearity of $B$

$$= (A(B(u + v) - B(v) - B(u)), w),$$

using identity (3.18) and bilinearity of $B$

$$= (B(u, Av), w) + (B(v, Au), w) - (B(Au, v), w) - (B(Av, u), w). \quad (3.30)$$

From the identity

$$(B(u, v), w) = -(B(u, w), v), \quad \forall u, v, w \in V, \quad (3.31)$$
(3.30) becomes

\[(A^{1/2}R'(u)v, A^{1/2}w) = - \left[ (B(u, w), Av) + (B(v, w), Au) + (B(Au, w), v) + (B(Av, w), u) \right].\]

Apply (2.11), (2.12), (2.13); we have

\[
\left| (A^{1/2}R'(u)v, A^{1/2}w) \right| \\
\lesssim |B(u, w)| |Av| + |B(v, w)| |Au| + |B(Au, w)| |v| + |B(Av, w)| |u| \\
\lesssim C \{ |u|^{1/2} |Av|^{1/2} |w|, |Av| + |v|^{1/2} |Av|^{1/2} |w| |Au| \\
+ C_3 |Av| |w| |v|^{1/2} |Av|^{1/2} + C_3 |Av| |w| |u|^{1/2} |Av|^{1/2} \}
\lesssim 2(C_2 + C_3) \lambda^{-1/2} |Av| |Av| |w|^{1/2}.
\]

Thus

\[|R'(u)v| \lesssim 2(C_2 + C_3) \lambda^{-1/2} |Av| |Av| |w|^{1/2}.
\]

From (3.4) we easily see that (3.28) and (3.29) imply respectively

\[
|R'(u)v| \lesssim M_0 |\tilde{A}u| |\tilde{A}v|, \quad \forall u, v \in D(\tilde{A}),
\]

\[
|\tilde{A}^{1/2}R'(u)v| \lesssim M_1 |\tilde{A}u| |\tilde{A}v|, \quad \forall u, v \in D(\tilde{A}),
\]

where \(M_0 = C_1 \theta^{1/2}, M_1 = C_2 \theta_2 \theta^{-1} \).

We denote by \(P_{n+m} \) the orthogonal projection from \(H \) onto \(H_{n+m} := \text{span}\{w_1, \ldots, w_{n+m}\} \) and by \(Q_{n+m} = I - P_{n+m} \). We define \(\mathcal{F}_{b,l} \) to be the set of all functions \(\Phi: P_{n+m} D(\tilde{A}) \rightarrow Q_{n+m} D(\tilde{A}) \) that satisfies

\[
\text{Support } \Phi \subset \{ p \in H_{n+m}: |\tilde{A}p| \leq \rho_2 \},
\]

\[
|\tilde{A}\Phi(p_1) - \tilde{A}\Phi(p_2)| \leq l|\tilde{A}p_1 - \tilde{A}p_2|, \quad \forall p_1, p_2 \in P_{n+m} D(\tilde{A}),
\]

\[
|\tilde{A}\Phi(p)| \leq b, \quad \forall p \in P_{n+m} D(\tilde{A}).
\]

We apply the results in [8] for the existence of an inertial manifold of the preparated equation (3.26) and summarize in the following.

**Theorem 4.** Let \(b = \rho_2/4, 0 \leq l \leq 1/32, \) and \(\gamma > 1 \). We assume that

\[
\sqrt{\lambda_{n+2m}} + \sqrt{\lambda_n} \geq \max \{4K_2(1+l)/l, 3K_3(1+\gamma^{-1} + \gamma), K_1/b \},
\]

where \(K_1 = \theta_2^{1/2} \|f\| + 8\rho_2^2 M_1, \ K_2 = 4\rho_2 M_1 + 2K_1 \rho_2^{-1}, \) and \(K_3 = 2\rho_2 M_0.\)
Then there exists $\Phi \in \mathcal{F}_{b,t}$ whose graph $\mathcal{M}$ is an inertial manifold for the prepared equation (3.26). Namely, for any solution $u(t)$ of (3.26) and (3.7), or (3.6) and (3.7) with $|Au_0| \leq \rho_2$, 
\[
\text{dist}_H(u(t), \mathcal{M}) \leq K_4 \exp(-\mu t),
\]
where $K_4 = 2(1 + \gamma^{-1})\lambda_1^{-1/4} \rho_1$ and $\mu = \lambda_{n+2m} - K_3 \lambda_{n+2m}^{1/2} - K_3 \gamma^{-1/4} \lambda_{n+2m}^{1/4} \geq \lambda_{n+2m}/2$.

Let us denote by $K$ the right-hand side of (3.35). The following theorem provides a condition on the choice of $n$ and $m$, under which (3.35) holds.

**Theorem 5.** Let $C_K = (\pi L_1 L_2/2)((K/2\pi) + |1/L|)^2$. If $n$ and $m$ satisfy
\[
m \geq C_K + \sqrt{2C_K(n+1)},
\]
then the modified equation (3.6) and its prepared equation (3.26) have an inertial manifold $\mathcal{M}$.

**Proof.** We write
\[
\lambda'_{n+2m} = \frac{L_1 L_2}{4\pi} \lambda_{n+2m}, \quad \lambda'_n = \frac{L_1 L_2}{4\pi} \lambda_n,
\]
\[
K' = \sqrt{\frac{L_1 L_2}{4\pi}} K, \quad E = \frac{\sqrt{\pi L_1 L_2}}{2} \left| \frac{1}{L} \right|.
\]
Under the above notations, (3.35) is equivalent to
\[
\sqrt{\lambda'_{n+2m}} - \sqrt{\lambda'_n} \geq K'.
\]
From (2.15) in Lemma 1,
\[
\sqrt{\lambda'_{n+2m}} \geq \sqrt{n+2m+1} - E \quad \text{and} \quad \sqrt{\lambda'_n} \leq \sqrt{n+1} + E.
\]
So (3.38) holds if
\[
\sqrt{n+2m+1} - \sqrt{n+1} - 2E \geq K',
\]
that is
\[
\sqrt{n+2m+1} - \sqrt{n+1} \geq \sqrt{2C_K}.
\]
By rationalizing (3.39), we obtain
\[
m^2 - 2C_K m + C_K^2 - 2(n+1) C_K \geq 0.
\]
Easy to see that (3.40) holds when

\[ m \geq C_k + \sqrt{2(n + 1)C_k}. \]

Therefore, Theorem 4 applies.

In the numerical simulation of the inertial manifold, the size of the computations needed depends on the dimension of the manifold. In order for the dimension of \( \mathcal{M} \) to be as low as possible, in addition to the assumption \( m \leq n \) we choose \( m = -\lfloor -C_k - \sqrt{2(n + 1)C_k} \rfloor \), the smallest integer \( m \) satisfying (3.37). Therefore in what follows we fix \( n \) and \( m \) such that

\[ n \geq m = -\lfloor -C_k - \sqrt{2(n + 1)C_k} \rfloor. \tag{3.41} \]

4. THE DISTANCE OF THE NAVIER–STOKES ORBITS TO \( \mathcal{M} \)

Our aim is to use the inertial manifold \( \mathcal{M} \) of the modified equation obtained in the previous section to approximate the attractor for the Navier–Stokes equations. Because \( \mathcal{M} \) attracts exponentially all the orbits of the modified equation, we need only to estimate the distance of each solution \( u \) of the Navier–Stokes equation (2.9) to the solution \( v \) of the modified equation (3.6) with same initial value \( v(0) = u(0) \). Since both the orbit of the Navier–Stokes equation (2.9) and that of its modified equation (3.6) enter the absorbing set \( B_{D(\mathcal{A})}(0, \rho_2) \) after finite time as shown in Section 3.2, we may assume \( u(0) \) and \( v(0) \) are in \( B_{D(\mathcal{A})}(0, \rho_2) \).

We write \( w = u - v \), then \( w \) satisfies

\[ \frac{dw}{dt} + v Aw = v(A - A)v - B(u, w) - B(w, v). \tag{4.1} \]

By taking the scalar product of (4.1) with \( w \) in \( H \) and using identity (3.19), we obtain

\[ \frac{1}{2} \frac{d|w|^2}{dt} + v \|w\|^2 = v((A - A)v, w) - (B(w, v), w) \]

\[ = v((A - A)A^{-3/2}Av, A^{1/2}w) - (B(w, v), w), \]

from the second inequality of (2.12)

\[ \leq v \| (A - A)A^{-3/2}Av \|_{L(H)} \| Av \| \| w \| + C_3 \| v \| \| w \| \| w \| \]

by Young's inequality

\[ \leq v \| w \|^2 + \frac{\varepsilon^2}{2} + \frac{\alpha}{2} |w|^2, \tag{4.2} \]
where \( \varepsilon = \sqrt{\theta_1^{-1} \rho_2 \| (\tilde{A} - A) A^{-3/2} \|_{\mathcal{L}(H)} } \) and \( \alpha = C_3 \rho_1^2 / \nu \). Hence

\[
\frac{d |w|^2}{dt} - \alpha |w|^2 \leq \varepsilon^2,
\]

by Gronwall's lemma

\[
|w|^2 \leq \frac{\varepsilon^2}{\alpha} (\exp(\alpha t) - 1). \quad (4.3)
\]

Equations (4.3) and (3.36) imply

\[
\text{dist}_H(u(t), \mathcal{M})^2 \leq \frac{2\varepsilon^2}{\alpha} (\exp(\alpha t) - 1) + 2K_4^2 \exp(-2\alpha t), \quad \forall t \geq 0. \quad (4.4)
\]

We denote the right-hand side of (4.4) by \( E(t) \). \( E(t) \) assumes its minimum at \( T = (1/(\alpha + 2\mu)) \ln(2\mu K_4^2 / \varepsilon^2) \) and

\[
E(T) = 2 \frac{\varepsilon^2}{\alpha} \left( \left( \frac{2\mu K_4^2}{\varepsilon^2} \right)^{\alpha/(\alpha + 2\mu)} - 1 \right) + 2K_4^2 \left( \frac{2\mu K_4^2}{\varepsilon^2} \right)^{-2\alpha/(\alpha + 2\mu)}. \quad (4.5)
\]

We write \( \delta = \sqrt{E(T)} \) and have from (4.4),

\[
\text{dist}_H(u(T), \mathcal{M}) \leq \delta. \quad (4.6)
\]

Now for any \( t \geq T \), by replacing \( u(0) \) and \( v(0) \) by \( u(t - T) \) and then repeating the above argument we obtain

\[
\text{dist}_H(u(t), \mathcal{M}) \leq \delta. \quad (4.7)
\]

It is easy to see that

\[
\| (\tilde{A} - A) A^{-3/2} \|_{\mathcal{L}(H)} = \max \left\{ \frac{\lambda_{n+m} - \lambda_n}{\lambda_n^{3/2}}, \frac{\lambda_{n+2m} - \lambda_{n+m+1}}{\lambda_{n+m+1}^{3/2}} \right\}.
\]

From Lemma 1 and (3.41), there exists \( K_5 \) (and the following \( K_j, j = 6, 7, 8 \)) depending only on \( \Omega, \nu, \) and \( f \) such that

\[
\varepsilon \leq K_5 n^{-1}. \quad (4.8)
\]

Since \( \lambda_{n+m}/2 \leq \mu \leq \lambda_{n+m} \), together with (4.8), we find that

\[
T \leq K_6 n^{-1} \ln n \quad (4.9)
\]

and

\[
E(T) \leq K_7 n^{-2}(n^{1/n} - 1) + n^{-3} = O(n^{-3} \ln n). \quad (4.10)
\]
Therefore

\[ \delta \leq K_n n^{-3/2} (\ln n)^{1/2}. \]  

(4.11)

We have proven the following

**THEOREM 6.** After finite time \( T \) in the order of \( n^{-1} \ln n \) each solution of the Navier–Stokes equations (2.9), (2.10) with \( |\tilde{A}u_0| \leq \rho_2 \) enters the \( \delta \)-neighborhood of the Lipschitz manifold \( \mathcal{M} \) of dimension \( n + m \) with \( \delta \) in the order of \( n^{-3/2} (\ln n)^{1/2} \).

5. **APPROXIMATE INERTIAL MANIFOLD \( \Sigma \)**

The approximate inertial manifold \( \mathcal{M} \) constructed in Section 3 is not practical for the sake of numerical computations. Following [8], we will apply the Euler–Galerkin method, which is very simple and easy for numerical computations, to construct an explicit approximate manifold \( \Sigma \) with the same dimension as \( \mathcal{M} \) and lying as close as \( \mathcal{M} \) to the global attractor in \( H \) of the Navier–Stokes equation.

In order to follow [8, Sects. 6 and 7] we need to verify the assumption (7.1) therein. We assume \( f \in D(A) \) in this section. Using (3.18), (2.11), (2.12), we have

\[
|AR(u)| = |A(B(u) - f)| \\
\leq |B(Au, u)| + |B(u, Au)| + |Af| \\
\leq C_1 |u|^{1/2} |Au| |A^{3/2}u|^{1/2} + C_2 |u|^{1/2} |Au|^{1/2} |A^{3/2}u| + |Af| \\
\leq C'_1 |Au| |A^{3/2}u| + |Af|,
\]

where \( C'_1 \) is as in Proposition 3. As a result,

\[
|AF(u)| \leq 4C'_1 \rho_2 |A^{3/2}u| + |Af|.
\]

Thus, by (3.4)

\[
|\tilde{A}F(u)| \leq M_2 |\tilde{A}^{3/2}u| + M_3,
\]

(5.1)

where \( M_2 = 4C'_1 \rho_2 \theta_2 \theta^{-3/2} \) and \( M_3 = \theta_2 |Af| \).

Let \( d \) be a fixed positive integer which will be given later. Let \( K_2 \) and \( l \) be as in Theorem 4. We set

\[
\tau = (2\lambda_{n+2m} - 2K_2 (1 + l) \lambda_{n+2m}^{1/2})^{-1} \ln \frac{\lambda_{n+m+d+1}}{\lambda_1}
\]

(5.2)
and in addition to (3.41), we assume
\[ \tau \leq \min \left\{ \frac{1}{8K_2^2}, \frac{1}{3M_2^2} \right\}. \quad (5.3) \]

We consider the Galerkin approximation to (3.26),
\[ \frac{du}{dt} + \bar{A}u + F_{n+m+d}(y) = 0 \quad \text{in } H_{n+m+d}, \quad (5.4) \]
where \( F_{n+m+d}(u) := P_{n+m+d}F(u) \). For any given initial value \( u_0 \in H_{n+m+d} \)
the ordinary differential system (5.4) possesses a unique solution \( \bar{S}_{n+m+d}(t)u_0 \) for all \( t \in \mathbb{R} \). Furthermore, \( |\bar{A}\bar{S}_{n+m+d}(t)u_0| \leq \rho_2 \) for all \( t \geq 0 \) whenever \( |\bar{A}u_0| \leq \rho_2 \).

From [8, Theorems 6.1 and 6.3, Corollary 6.11] there exists a function \( \Psi_\tau \) in
\[ \mathcal{F}_{b,l,n+m+d} = \{ \psi \in \mathcal{F}_{b,l} : H_{n+m} \rightarrow Q_{n+m}H_{n+m+d} = \text{span}\{w_{n+m+1}, \ldots, w_{n+m+d}\} \} \]
such that \( \text{graph}(\Psi_\tau) = \bar{S}_{n+m+d}(\tau)H_{n+m} \) and
\[ \| \Psi_\tau - \Phi \| \leq \left( \frac{1}{4K_1} + \frac{\sqrt{\lambda_1 \rho_2}}{4} \right) \lambda^{-1/2} \quad (5.5) \]
Consequently, for any \( u_0 \in B_{D(\lambda_j)}(0, \rho_2) \), when \( t \geq T(=O(n^{-1} \ln n)) \),
\[ \text{dist}_H(S(t)u_0, \Psi_\tau) < K_9 n^{-3/2}(\ln n)^{1/2}, \quad (5.6) \]
where \( K_9 \) (and \( K_j, j \geq 10 \) in the sequel) depends only on \( \Omega, \nu, f, \) and \( M_j, j = 0, 1, 2, 3 \).

Therefore we may replace \( M \) by \( \bar{S}_{n+m+d}(\tau)H_{n+m} \) as an approximate inertial manifold for the Navier–Stokes equations. This enables us to approximate \( M \) by approximating the solution of (5.4) with the initial condition
\[ u(0) = \bar{p} \in H_{n+m}. \quad (5.7) \]
Equations (5.4), (5.7) are equivalent to (5.8), (5.9),
\[ \frac{d\omega}{dt} + \bar{A}\omega + F_{n+m+d}(e^{-t\lambda} \bar{p} + \omega) = 0 \quad \text{in } H_{n+m+d}, \quad (5.8) \]
\[ \omega(0) = 0, \quad (5.9) \]
since \( u(t) = e^{-\tau^3} \rho + \omega(t) \) is a solution of (5.4) and (5.7) if and only if \( \omega(t) \) is a solution of (5.8) and (5.9).

We approximate \( \omega(t) \) and \( u(t) \) by \( w(t) \) and \( v(t) \), respectively, applying the one-step Euler's implicit scheme which is called the Euler-Galerkin scheme,

\[
w(t) + t\mathcal{A}w(t) + tF_{n+m+d}(v) = 0, \quad \text{in } H_{n+m+d},
\]

\[
v(t) = e^{-\tau^3} \rho + w(t).
\]

We write

\[
\Sigma = \{ v(\tau) \in H_{n+m+d} : v(\tau) = e^{-\tau^3} \rho + w(\tau; \rho) \text{ for some } \rho \in H_{n+m} \}
\]

where \( w(\cdot; \rho) \) is the solution of (5.10), (5.11).

Following the discussion as in [8], we know that there exists a map

\[
\chi : H_{n+m} \to Q_{n+m} H_{n+m+d}
\]

such that

\[
\text{graph } \chi = \Sigma,
\]

\[
|\mathcal{A}(\chi_\tau(p_1) - \chi_\tau(p_2))| \leq K_2 \sqrt{\tau} |\mathcal{A}(p_1 - p_2)|, \quad \forall p_1, p_2 \in H_{n+m},
\]

\[
|\mathcal{A}(\chi_\tau(p) - \Psi_\tau(p))| \leq K_{10} \lambda_{n+m} \tau^{3/2}, \quad \forall p \in H_{n+m}.
\]

The manifold \( \chi_\tau(p) \) is given by the implicit relation,

\[
\chi_\tau(p) = -\tau(1 + \tau\mathcal{A}P_{n+m+d})^{-1}Q_{n+m}F_{n+m+d}(p + \chi_\tau).
\]

We set

\[
\chi(p) = -\tau(1 + \tau\mathcal{A}P_{n+m+d})^{-1}Q_{n+m}F_{n+m+d}(p)
\]

\[
= -\theta |\mathcal{A}p|^2 / 4\rho_2^3 \tau \sum_{j=n+m+1}^{n+m+d} (1 + \tau \lambda_j)^{-1} (B(p, p), w_j) w_j,
\]

where \( \lambda_j = \lambda_{n+2m} \) for \( n+m+1 \leq j \leq n+2m \) and \( \lambda_j = \lambda_j \) for \( n+2m+1 \leq j \leq n+m+d \); then from [8, Theorem 7.4],

\[
\sup_{p \in H_{n+m}} |\mathcal{A}(\chi_\tau(p) - \chi(p))| \leq K_1 K_2^2 \tau.
\]

Combining (5.2), (5.6), (5.15), and (5.18) we have, for any \( u_0 \in B_{D(\lambda)}(0, \rho_2) \) the solution of (2.9), (2.10), \( S(t)u_0 \) satisfies

\[
dist_H(S(t)u_0, \text{graph(\chi)}) \leq K_{11} n^{-3/2} [\ln(n+m+d)]^{3/2}.
\]
It is easy to check that for all \( p \in H_{n+m} \)

\[
(B(p, p), w_{n+m+j}) = 0,
\]

if \( j \) satisfies

\[
\sqrt{\lambda_{n+m+j}} > 2 \sqrt{\lambda_{n+m}}. \tag{5.20}
\]

That is, the nonlinear term does not produce any \( \lambda_{n+m+j} \) mode for \( j \) satisfying the condition (5.20). On the other hand, the nonlinear term does produce \( \lambda_{n+m+j} \) modes from \( H_{n+m} \) for some \( j \)'s not satisfying (5.20).

Therefore, hoping to capture this feature of the nonlinear term, we choose \( d \) to be the smallest integer \( j \) satisfying (5.20). Even though such choice of \( d \) does not improve the rate of the closeness of \( \text{graph}(\chi) \) to the attractor, we believe it is a better choice in practice.

Using (2.15) we can easily find that (5.20) holds if

\[
j \geq 3(n+m+1) + 6 \sqrt{\pi L_1 L_2 (n+m+1) \left| \frac{1}{L} + \frac{9}{4} \pi L_1 L_2 \right|^2}.
\]

Thus we obtain the following estimate of the upper bound of \( d \):

\[
d \leq 3(n+m+1) + 6 \sqrt{\pi L_1 L_2 (n+m+1) \left| \frac{1}{L} + \frac{9}{4} \pi L_1 L_2 \right|^2}. \tag{5.21}
\]

We sum up this section in the following.

**Theorem 7.** Suppose \( n \) and \( m \) satisfy (3.41) and (5.3), and \( d \) is the largest integer satisfying (5.21). Let \( \Sigma \) denote the graph of \( \chi \) defined by (5.17). Then the solution \( S(t)u_0 \) of the Navier–Stokes equation (2.9) with \( u_0 \in B_{\rho_1(\partial\Omega)}(0, \rho_2) \) satisfies

\[
\text{dist}_H(S(t)u_0, \Sigma) \leq K_{12} n^{-3/2}(\ln n)^{3/2}
\]

when \( t \geq T (= 0(n^{-1} \ln n)) \).

Since we are only interested in the case \( |\bar{\alpha}u| \leq \rho_2, \theta(|\bar{\alpha}p|^2/4\rho_2^2) \) can be taken equal to one in (5.17), and

\[
\chi(p) = - \sum_{j=n+m+1}^{n+m+d} \tau(1 + \tau\bar{\lambda}_j)^{-1}(B(p, p), w_j)w_j, \quad \forall p \in H_{n+m},
\]

which is a quadratic form from \( H_{n+m} \) to \( Q_{n+m}H_{n+m+d} \).
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REFERENCES

8. C. Foias, G. R. Sell, and E. S. Titi, Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations, J. Dynamics Differential Equations 1, No. 2 (1989), 199–244.