NOTE

The Existence of a Bush-type Hadamard Matrix of Order 36 and Two New Infinite Classes of Symmetric Designs

Zvonimir Janko

Mathematical Institute, University of Heidelberg, Heidelberg, Germany
Communicated by the Managing Editors
Received April 3, 2000; published online May 10, 2001

A nonsymmetric Bush-type Hadamard matrix of order 36 is constructed which leads to two new infinite classes of symmetric designs with parameters:

\[ v = 36(25^m + 25^{m-1} + \cdots + 25 + 1), \quad k = 15(25)^m, \quad \lambda = 6(25)^m, \]

and

\[ v = 36(49^m + 49^{m-1} + \cdots + 49 + 1), \quad k = 21(49)^m, \quad \lambda = 12(49)^m, \]

where \( m \) is any positive integer.

Key Words: symmetric design; Bush-type Hadamard matrix; twin design; Siamese twin design; balanced generalized weighing matrix.

1. INTRODUCTION

A symmetric \((v, k, \lambda)\) design can be described as a square \(v \times v\) \((0, 1)\)-matrix with constant row sum equal to \(k\) and constant scalar product of pairs of rows equal to \(\lambda\).

A Hadamard matrix of order \(n\) is a square \(n \times n\) matrix with entries \(\pm 1\) whose distinct rows are pairwise orthogonal. A Bush-type Hadamard matrix \([1]\) is a Hadamard matrix \(H = [H_{ij}]\) of order \(4n^2\) with blocks of size \(2n\) such that \(H_{ii} = J_{2n}\) and \(H_{ij}H_{ij} = J_{2n}\) for \(i \neq j\), \(1 \leq i, j \leq 2n\), where \(J_{2n}\) is the all-one \(2n \times 2n\) matrix.

While it is easy to construct Bush-type Hadamard of order \(16n^2\) for all values of \(n\) for which a Hadamard matrix of order \(4n\) exists (see \([6]\)), it is very difficult to decide whether such matrices exist of order \(4n^2\) if \(n\) is an odd integer, \(n > 1\).
In recent papers [3, 4] Bush-type Hadamard matrices of orders $100 = 4 \cdot 25 \ (n = 5)$ and $324 = 4 \cdot 81 \ (n = 9)$ have been constructed. They led to the discovery of four new infinite classes of symmetric designs.

It is even more difficult to decide whether such matrices exist of order $4n^2$ if $n \equiv -1 \pmod{4}$. Bussemaker et al. [2] showed that a symmetric Bush-type Hadamard matrix of order $36 \ (n = 3)$ does not exist.

In this paper, a nonsymmetric Bush-type Hadamard matrix of order $36$ is constructed. It appears that its full automorphism group has order $3$ and this group permutes the Bush 6×6 blocks without fixing any such block.

Using this Bush-type Hadamard matrix and the method of [6], two new infinite classes of symmetric designs are constructed with the following parameters:

$$v = 36(25^m + 25^{m-1} + \cdots + 25 + 1), \quad k = 15(25)^m, \quad \lambda = 6(25)^m,$$  

(1)

and

$$v = 36(49^m + 49^{m-1} + \cdots + 49 + 1), \quad k = 21(49)^m, \quad \lambda = 12(49)^m,$$  

(2)

where $m$ is any positive integer.

The design from the first class (1) are twin designs, while the Design from the second class (2) are Siamese twin designs in the sense of the following definitions. For a $(0, \pm 1)$-matrix $K$, let $K = K^+ - K^-$, where $K^+$ and $K^-$ are $(0, 1)$-matrices. A $(0, \pm 1)$-matrix $D$ is called a twin design if both $D^+$ and $D^-$ are symmetric designs with the same parameters. A $(0, \pm 1)$-matrix $S$ is called a Siamese twin design sharing the entries of matrix $I$ if $S = I + K - L$, where $I$, $K$, $L$ are nonzero $(0, 1)$-matrices and both $I + K$ and $I + L$ are symmetric designs with the same parameters.

A balanced generalized weighing matrix $\text{BGW}(v, k, \lambda)$ over a multiplicative finite group $G$ is a matrix $W = [w_{ij}]$ of order $v$ with entries from the set $G - \{0\}$ such that each row and each column of $W$ contains exactly $k$ nonzero entries and for any distinct rows $i$ and $h$, the multiset

$$\{w_{ij}^h : 1 \leq j \leq v, w_{ij} \neq 0, w_{ij}^h \neq 0\}$$

contains exactly $\lambda |G|$ copies of every element of $G$.

In this paper, we will use a balanced generalized weighing matrix $\text{BGW}(q^m + q^{m-1} + \cdots + q + 1, q^m - q^{m-1})$ over a cyclic group $G$ of order $t$, where $q$ is a prime power, $m$ is a positive integer, and $t$ is a divisor of $q - 1$. Such matrices are known to exist [5].

The Kronecker product of two matrices $A = [a_{ij}]$ and $B$, denoted $A \otimes B$, is defined by $A \otimes B = [a_{ij}B]$. For a matrix $A = [a_{ij}]$, denote by $|A|$ the matrix $[|a_{ij}|]$. Throughout the paper $-I$ represents $-1$. $I_6$ is the unit matrix of order 6 and $J_6$ is all-one matrix of order 6.
2. A BUSH-TYPE HADAMARD MATRIX OF ORDER 36

**Theorem 1.** There exists a nonsymmetric Bush-type Hadamard matrix $H = [H_{ij}]$ of order 36, where $1 \leq i, j \leq 6$ and $H_{ij}$ are Bush blocks of order 6. The full automorphism group $\langle s \rangle$ of $H$ has order 3, where $s$ permutes the Bush blocks and does not fix any such blocks.

**Proof.** Using the automorphism $s$ of order 3 we obtain the following Bush-type Hadamard matrix $H = [H_{ij}]$ of order 36 ($\ast$ means $-1$):

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 
\end{bmatrix}
\]
The full automorphism group of \(H\) has order 3 and is generated by \(s\) acting fixed-point-free on the 36 Bush blocks \(H_{ij}\) as follows:

\[
\begin{align*}
\sigma &= (H_{11}, H_{22}, H_{33})(H_{12}, H_{23}, H_{31})(H_{13}, H_{21}, H_{32}) \\
& (H_{14}, H_{25}, H_{36})(H_{15}, H_{26}, H_{34})(H_{16}, H_{24}, H_{35}) \\
& (H_{41}, H_{52}, H_{63})(H_{42}, H_{53}, H_{61})(H_{43}, H_{51}, H_{62}) \\
& (H_{44}, H_{55}, H_{66})(H_{45}, H_{56}, H_{64})(H_{46}, H_{54}, H_{65}).
\end{align*}
\]

The matrix \(H\) is nonsymmetric. Our theorem is proved.

Remark. It is interesting to note that the \((0, \pm 1)\)-matrix \(M = H - I_6 \otimes J_6\) is a twin design. Both \(M^+\) and \(M^-\) are symmetric designs with parameters \((36, 15, 6)\) but they are nonisomorphic.

### 3. Two Classes of Symmetric Designs

Let \(SP_6\) be the set of all signed permutation matrices of order 6. Let \(U = \text{circ}(0, 1, 0, 0, 0, 0)\) be the circulant matrix of order 6 with the first row \((0, 1, 0, 0, 0, 0)\) and \(N = \text{diag}(-1, 1, 1, 1, 1, 1)\) be the diagonal matrix of order 6 with \(-1\) at the \((1, 1)\)-position and 1 elsewhere on the diagonal. Let \(E = UN\); then \(E\) is in \(SP_6\). Let \(G_{12} = \{E^i \otimes I_6 : i = 1, 2, \ldots, 12\}\). Then \(G\) is a cyclic group of order 12. Note that 12 divides 25\(^{-1}\) and also 12 divides 49\(^{-1}\).

**Theorem 2.** For every positive integer \(m\), there exists a twin design with parameters

\[
v = 36(25^m + 25^{m-1} + \cdots + 25 + 1), \quad k = 15(25)^m, \quad \lambda = 6(25^m),
\]

and a Siamese twin design with parameters

\[
v = 36(49^m + 49^{m-1} + \cdots + 49 + 1), \quad k = 21(49)^m, \quad \lambda = 12(49)^m.
\]

**Proof.** Let \(m\) be a positive integer and \(H\) the Bush-type Hadamard matrix of order 36 from the previous section. Let \(W = [W_{ij}]\) be the balanced generalized weighing matrix \(\text{BGW}(v, k, \lambda)\) over the group \(G_{12}\), where \(v = 25^m + 25^{m-1} + \cdots + 25 + 1\), \(k = 25^m\), \(\lambda = 25^m - 25^{m-1}\). Let \(D = [MW_{ij}]\), where \(M = H - I_6 \otimes J_6\). This is a twin design with parameters (3). To see that, we compute that

\[
D^+ = \frac{1}{2} \left[ P \langle w_{ij} \rangle + MW_{ij} \right]
\]
and

\[ D^+ = \frac{1}{2} [ P \ | w_{ij} \ | - M w_{ij} ], \]

where \( P = J_{36} - I_6 \otimes J_6 \) are both symmetric designs with parameters (3).

Now, let \( S = [ s_{ij} ] \) be the balanced generalized weighing matrix \( \text{BGW}(v, k, \lambda) \) over the group \( G_{12} \), where \( v = 49^m + 49^{m-1} + \ldots + 49 + 1 \), \( k = 49^m \), \( \lambda = 49^m - 49^{m-1} \). Let \( M = H - I_6 \otimes J_6 \). Then

\[ \frac{1}{2} [ M s_{ij} + ((J_6 + I_6) \otimes J_6) \ | s_{ij} \ ] \]

and

\[ \frac{1}{2} [ -M s_{ij} + ((J_6 + I_6) \otimes J_6) \ | s_{ij} \ ] \]

are both symmetric designs with parameters (4), sharing the entries of \( \{ (I_6 \otimes J_6) \ | s_{ij} \} \). The theorem is proved.

**REFERENCES**

3. Z. Janko, H. Kharaghani, and V. D. Tonchev, Bush-type Hadamard matrix of order 100 and two infinite classes of symmetric designs, submitted for publication.