The Symmetric Ring of Quotients of the Coproduct of Rings

WALLACE S. MARTINDALE III

Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01003

Communicated by Susan Montgomery

Received August 23, 1989

INTRODUCTION

Let $R$ be a prime ring and let $Q$ be the right quotient ring of $R$ relative to the filter $F$ of all nonzero two-sided ideals of $R$ (see [5]). $R$ is embeddable in $Q$ (via left multiplications) and given any element $q \in Q$ there exists $I \in F$ such that $qI \subseteq R$. The set $S = \{s \in Q | sI \subseteq R, Is \subseteq R$ for some $I \in F\}$ is called the symmetric ring of quotients of $R$ and was first studied by Kharchenko in [3]. The ring $S$ arises naturally in the study of $X$-inner automorphisms of $R$ ($r^\alpha = s^{-1}rs$ for some invertible $s \in Q$) and $X$-inner derivations of $R$ into itself ($r^\delta = sr - rs$ for some $s \in Q$); in each case the inducing element $s$ belongs to $S$. If $R$ is a domain it is easily seen that $S$ must also be a domain ($Q$ need not be a domain).

Throughout this paper $A$ will be a fixed division ring and $R = R_1 \bigoplus \cdots \bigoplus R_n$ will be the coproduct of arbitrary $A$-rings $R_1$ and $R_n$ with $1$ over $A$. Our goal is to show that $R = S$ unless $R_1$ and $R_n$ are of a very special nature. To describe these exceptional situations a couple of definitions are required. The first is that of a primary ring $A$: $A = A \oplus T$ where $T$ is any $A$-bimodule with $T^2 = 0$. The second, an unexpected entry into the picture, is that of what we shall call a $d$-semiprimary ring $A$: let $V$ be a $A$-bimodule, let $d: A \rightarrow V$ be a derivation, and let $A = [\begin{array}{c} \gamma \\ \delta \end{array}, \alpha]$, multiplication

$$\begin{bmatrix} \alpha & \gamma \\ \delta & \delta \end{bmatrix} \ast \begin{bmatrix} \nu \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \nu + \beta \delta - \gamma \beta & 0 \end{bmatrix}$$

(one checks that $A$, $\ast$ is indeed a $A$-ring). Setting $e_1 = [\begin{array}{c} 1 \\ 0 \end{array} 0], e_2 = [\begin{array}{c} 0 \\ 0 \end{array} 1], T = [\begin{array}{c} 0 \\ 0 \end{array} 0], e_1 = e_2 = 1, e_i e_j = \delta_{ij} e_i, e_1 T = 0 = Te_2, t = te_1 = e_2 t$ for $t \in T$, $d(\alpha) = \alpha e_1 - e_1 \alpha e_1 T$ for all $\alpha \in A$. In casc

295

0021-8693/91 $3.00
Copyright 1991 by Academic Press, Inc.
All rights of reproduction in any form reserved.
$d = 0$, i.e., $e_i$ commutes with the elements of $A$, we call $A$ semiprimary and note that it is just the matrix ring $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ under ordinary multiplication.

We are now in a position to state our main result.

**Main Theorem.** Let $R = R_1 \coprod R_2$, the coproduct of $A$-rings $R_1$ and $R_2$ with $1$ over a division ring $A$, with each of the four left and right dimensions $(R_i : A)_l$, $(R_i : A)_r$, $i = 1, 2$, greater than $2$. Then $R = S$ (i.e., $R$ is its own symmetric ring of quotients) unless one of the following occurs:

(a) Both $R_1$ and $R_2$ are primary;

(b) One $R_i$ is primary and the other is $d$-semiprimary;

(c) Each $R_i$ is $d_i$-semiprimary.

As a very special case of the theorem we have Kharchenko's result $[3, \text{Lemma}]$: if $R$ is the free noncommutative algebra over a field in two or more variables then $R = S$ (see also Passman $[10]$ and a generalization to $2$-firs by Lewin $[4]$).

The present paper is the latest in a series of articles which began with a joint venture with Montgomery $[9]$, in which we studied $X$-inner automorphisms of the coproduct of domains. Following a joint paper with Lichtman $[5]$ we proved $[6]$ the following result:

**Theorem A.** Let $R = R_1 \coprod_A R_2$, $R_1 \neq A$, $R_2 \neq A$, with

(i) each $R_i$ $1$-finite (i.e., $xy = 1$ implies $yx = 1$);

(ii) at least one of the four dimensions $(R_i : A)_l$, $(R_i : A)_r$, $i = 1, 2$, greater than $2$. Then every $X$-inner automorphism of $R$ is inner unless one of the following holds:

(I) Both $R_1$ and $R_2$ are primary.

(II) One $R_i$ is primary and the other is quadratic (i.e., $2$-dimensional over $A$).

(III) Char. $A = 2$, at least one $R_i$ is not a domain, and one of the $R_i$'s is quadratic.

Our Main Theorem does not treat the case where one of the $R_i$'s is quadratic. On the other hand we have removed the restriction that the $R_i$'s are $1$-finite. In the case where at least one $R_i$ is a domain our result implies Theorem A (subject to the dimension restrictions).

We recently proved $[7]$ a result analogous to Theorem A for $X$-inner derivations, which we state as

**Theorem B.** Let $R = R_1 \coprod_A R_2$, char. $A \neq 2$, $R_1 \neq A$, $R_2 \neq A$, with at least one of the $R_i$'s neither primary nor quadratic. Then every $X$-inner derivation of $R$ is inner.
As with the case of Theorem A our result implies Theorem B if one of the \( R_i \)'s is a domain (subject to the dimension restrictions).

In Section 1 we review some needed material on coproducts in general, in Section 2 we prove the Main Theorem, and in Section 3 we construct examples illustrating the exceptional cases (a), (b), and (c) of the Main Theorem.

1. Preliminaries

In this section we recall the definition and basic properties of the notion of height as developed by Cohn in [1]. \( R \) has a filtration given by \( H^{-1} = 0, H^0 = A, H^1 = R_1 + R_2, H^n = \sum R_1 R_2 \cdots R_n, n = 2, 3, \ldots \). The height \(|r|\) of an element \( r \in R \) is defined as follows: \(|r| = n\) if \( r \neq 0, r \notin H^n, r \notin H^{n-1} \) and \(|r| = -\infty\) if \( r = 0 \). We will express the fact that an element \( r \) of \( H^n \) actually lies in \( H^{n-1} \) by saying \( r \equiv 0 \pmod{H^{n-1}} \) or simply \( r \equiv 0 \) if the context is clear. Every \( H^n \) is a \( A \)-bimodule and the bimodule \( H^n/H^{n-1} = (R_1 \otimes R_2 \otimes \cdots \otimes R_n)/(R_1 \otimes R_2 \otimes \cdots \otimes R_{n+1}) \), where \( R_i = R_i/A, i = 1, 2, R_j = R_1 \) if \( j \) is odd and \( R_j = R_2 \) if \( j \) is even. If \( n \) is even, the submodule of \( H^n \) corresponding to the first summand, namely \( R_1 R_2 R_3 \cdots R_2 \), is denoted by \( H_{n1} \) and that corresponding to the second summand, namely \( R_2 R_1 R_3 \cdots R_1 \), is denoted by \( H_{n2} \). Thus \( H^n = H_{n1} + H_{n2} \), with uniqueness of representation modulo \( H^{n-1} \). Similarly, if \( n \) is odd, we have \( H^n = H_{n11} + H_{n22} \). The elements of \( H^n \) which are of height \( n \) are called \((i, j)\)-pure. Elements of height \( \geq 1 \) which are not \((i, j)\)-pure for some \( i, j \) are called 0-pure. One always has \(|ab| \leq |a| + |b|\). Strict inequality can occur only when \( a \) is \((i, j)\)-pure and \( b \) is \((j, k)\)-pure for some \( i, j, k \). We shall frequently use the suggestive notation \( r_{ij} \) for an element of \( H^n \). If furthermore \( r_{ij} \notin H^{n-1} \) we shall indicate this by simply writing \( r_{ij} \neq 0 \). Also for a subscript \( i \) we define \( i' \) as follows: \( i' = 2 \) if \( i = 1 \) and \( i' = 1 \) if \( i = 2 \). The following lemma is an easy consequence of the preceding remarks.

**Lemma 1.** Let \( w \in H^n_{ij}, u \in H^n_{ik}, m \geq n, u \neq 0 \pmod{H^{n-1}} \). Then there exist elements \( u = u_1, u_2, \ldots, u_q \in H^n_{ik} \) right \( A \)-independent \( \pmod{H^{n-1}} \) and elements \( v_1, v_2, \ldots, v_q \in H^m_{kj} \) such that \( w \equiv \sum_{j=1}^q u_j v_i \pmod{H^{m-1}} \).

The proof uses the fact that \( u \) may be extended to a right \( A \)-basis of \( H^n_{ik} \pmod{H^{n-1}} \) and the fact that \( H^n_{ij} = H^n_{ik} H^{m-n}_{kj} \).

The following result is due to Cohn [1, p. 438]:

**Lemma 2.** Let \( u_1, u_2, \ldots, u_q \in H^n_{ij} \) be right \( A \)-independent \( \pmod{H^{n-1}} \) and let \( v_1, v_2, \ldots, v_q \in H^p_{jk} \). If \( \sum_{j=1}^q u_j v_i \equiv 0 \pmod{H^{n+p-1}} \), then each \( v_i \equiv 0 \pmod{H^{p-1}} \).
We now invoke Lemmas 1 and 2 in order to establish:

**Lemma 3.** Suppose \( ab \equiv cd \pmod{H^{m+n-1}} \), where \( m = |a| = |d| \), \( n = |b| = |c| \), \( n \leq m \), \( a \) \((i, j)\)-pure, \( b \) \((j', k)\)-pure, \( c \) \((i, l)\)-pure, \( d \) \((l', k)\)-pure. Then \( a \equiv ce \pmod{H^{m-1}} \), \( d \equiv eb \pmod{H^{m-1}} \), where \( e \) is \((l', j)\)-pure of height \( m - n \) (or \( e = \lambda \in A \) if \( n = m \)).

**Proof.** By Lemma 1 \( a \equiv ce + \sum_p c_p e_p \), \( c \), \( c_p \) right \( \Delta \)-independent in \( H_i \), \( e \), \( e_p \) \((i', j)\)-pure, in \( H_i \). The given congruence then becomes \( c(eb - d) + \sum_p c_p(e_p b) \equiv 0 \). By Lemma 2 \( eb \equiv d \) and \( e_p b \equiv 0 \), whence \( e_p \equiv 0 \). Thus \( a \equiv ce \) and the lemma is proved.

2. **Proof of the Main Theorem**

Throughout this section we assume that \( R = R_1 \coprod_d R_2 \), with \( (R_i : \Delta)_i > 2 \), \( (R_i : \Delta)_i > 2 \), \( i = 1, 2 \). We fix \( I \in F \) and let \( S_I = \{ s \in S | sI + Is \subseteq R \} \). Clearly \( S_I \) is an \((R, R)\)-bimodule. Next we fix \( a \in I \) such that \( a \) is \( 0 \)-pure of even height \( n = |a| > 0 \), and thus we may write \( a = a_{12} + a_{21} \). We write \( b = b_s \) and \( c = c_s \) (when the context is clear we will sometimes just write \( b = b_s \) and \( c = c_s \)). We have immediately the simple relationship

\[
sa = ab_s, \quad s \in S_I
\]  
the repeated application of which will form the basis of our proof. Since \( a \) is \( 0 \)-pure of even height it follows easily from (1) that \( |b_s| = |c_s| = m_s \), \( b_s \) is \( 0 \)-pure if and only if \( c_s \) is \( 0 \)-pure, and \( b_s \) is \((i, j)\)-pure if and only if \( c_s \) is \((i, j)\)-pure.

**Lemma 4.** If \( s \in S_I \) is such that \( m_s < n \) then \( s = 0 \).

**Proof.** If \( s \neq 0 \) choose \( r \in I \) such that \( 0 \neq rs \in R \). Therefore \( rsa \neq 0 \), whence \( b = b_s = sa \neq 0 \). We write \( b = b_{1j} + b_{2j} \), where without loss of generality we may assume \( b_{1j} \neq 0 \) (mod \( H^{m-1} \)). Now choose \( y', y'' \in R_2 \) such that \( y', y'' \) are left \( \Delta \)-independent mod \( \Delta \) (this is possible since \( (R_2 : \Delta)_i > 2 \)). Setting \( c' = ay's \in R \), we see from \((ay's)a = ay'(sa)\) that

\[
c'a = ay'b.
\]

It follows from (2) that \( |c'| = m + 1 \), and examination of the \((2, j)\)-component of (2) yields

\[
c_{2j}a_{1j} = a_{21}y'b_{1j} \pmod{H^{n+m}}.
\]

By Lemma 3 (we are assuming \( m + 1 \leq n \)) there exists \( e_{j'1} \in H_{j'1}^{n-1} \) such that \( a_{j'j} = e_{j'1}y'b_{1j} \) (mod \( H^{n-1} \)). Similarly there exists \( f_{j'1} \) such that
It follows that \( e_{j',j} = f_{j',j} y'' b_{1j} \). By Lemma 2 we see in particular that \( e_{j',j} \equiv 0 \) (since \( y', y'' \) are left \( \mathcal{A} \)-independent mod \( \mathcal{A} \)). This forces the contradiction \( a_{j,j} \equiv 0 \) and the lemma is proved.

**Lemma 5.** If \( s \in S_i \) is such that \( m_s > n \) then there exists \( g \in R \) such that \( m_{s-g} < m_s = m \).

**Proof:** We let \( b = b_s \) and write \( b = b_{1j} + b_{2j} \) (possibly \( b_{1j} \equiv 0 \) or \( b_{2j} \equiv 0 \)). Likewise we have \( c = e_s = e_{1j} + e_{2j} \). Examination of the \((1, j)\)-component of \( ca = ab \) shows that

\[
e_{1j} a_{j,j} = a_{12} b_{1j}.
\]

Applying Lemma 3 to (3) we see in particular that \( b_{1j} \equiv e_{1j} a_{j,j} \pmod{H''} \) for \( e_{1j} \in H'' \). In exactly the same fashion \( b_{2j} \equiv f_{2j} a_{j,j} \) for suitable \( f_{2j} \in H'' \). Setting \( g = e_{1j} + f_{2j} \), we have

\[
(s-g)a = b - ga = b_{1j} + b_{2j} - (e_{1j} + f_{2j})(a_{j,j} + a_{j,j})
\]

\[
\equiv (b_{1j} - e_{1j} a_{j,j}) + (b_{2j} - f_{2j} a_{j,j}) \equiv 0 \pmod{H''}.
\]

In other words \( m_{s-g} < m \) and the lemma is proved.

**Lemma 6.** If \( s \in S_i \) is such that \( m_s = n \) and \( i \) is given, then there exist \( \lambda, \mu \in \mathcal{A} \) such that \( (s-\lambda)a \equiv \mu a_{ij} \).

**Proof:** We write \( b = b_s = b_{12} + b_{21} \), \( c = c_{12} + c_{21} \). We may assume that \( b_{12} \neq 0 \). From (1) we obtain \( c_{12} a_{12} \equiv a_{12} b_{12} \) and an application of Lemma 3 yields \( b_{12} \equiv a_{12} a_{12} \) for some \( a \in \mathcal{A} \). Suppose \( b_{21} \neq 0 \). Then \( s - \lambda)a = b - \lambda a = b_{12} + a_{12} a_{21} \equiv -a_{21} \). Also \( sa = b \equiv b_{12} = a_{12} \). Suppose \( b_{21} \neq 0 \), and hence \( b_{21} \equiv b_{21} \) for some \( \beta \in \mathcal{A} \). Then \( s - \lambda)a = b - \lambda a = b_{21} + a_{12} a_{21} \equiv (\beta - \lambda)a_{21} \). On the other hand \( s - \beta)a = b - \beta a = b_{12} + b_{21} - \beta a_{12} - \beta a_{21} \equiv (\mu - \beta)a_{12} \). The four cases just discussed show that the lemma has been proved.

**Lemma 7.** If \( s \in S_i \) such that \( sa = a_{ii} \), then \([s, \mathcal{A}] = 0\).

**Proof:** We may assume \( a_{ii} = a_{12} \). Now let \( \lambda \in \mathcal{A} \). From \((as)a = a_{12} \lambda a_{12} \equiv a_{12} a_{12} \equiv a_{12} \lambda (sa)\) it follows that \( a(sa - \lambda s) \equiv 0 \), whence \( s \lambda = \lambda s \) by Lemma 4.

With these lemmas to draw on we are now ready to complete the proof of the Main Theorem. Suppose for sake of argument there exists \( s \in S \) but \( s \notin R \). We have \( sI + Is \subseteq R \) for suitable \( I \in F \) and we choose \( a \in I \), a 0-pure of even height \( n \). Repeated application of Lemma 5 (if necessary) together with Lemma 4 shows that we may assume \( m_s = n \), whence by Lemma 6 we have without loss of generality \( sa \equiv \mu a_{12}, \mu \in \mathcal{A} \). In fact we may actually
assume that \( sa \equiv a_{12} \) (just replace \( s \) by \( \mu^{-1}s \)). By Lemma 7 we then know that \( s \) commutes with each element of \( A \).

We claim that \( R_1 = A \oplus T_1 \), where \( T_1 = \{ x \in R_1 \mid xs = 0 \} \) is an ideal of \( R_1 \). Indeed, for \( x \in R \) we see immediately that \( |xsa| = |xa_{12}| \leq n \).

In case \( |xsa| < n \) then by Lemma 4 \( xs = 0 \) and we are finished. If \( |xsa| = n \) then by Lemma 6 there exist \( \alpha, \beta \in A \) such that \( (xs - x)a \equiv \beta a_{12} \equiv \beta sa \), i.e., \( (xs - x - \beta)s = 0 \) (mod \( H^{n-1} \)). By Lemma 4 \( xs - x - \beta s = 0 \); i.e., \( (x - \beta)s = \alpha \). If \( x \neq 0 \) we set \( x_0 = x^{-1}(x - \beta) \) and note that \( x_0s = l \). From this we obtain the contradiction \( a = x_0sa = x_0a_{12} + \cdots \) (just compare the \((2,1)\)-component of both sides). Therefore we are left with \( \alpha = 0 \) and accordingly \( (x - \beta)s = 0 \). This places \( x - \beta \in T_1 \) and so we have shown \( R_1 = A + T_1 \). Now let \( x \in T_1 \) and \( r \in R_1 \). By what we have just shown we can write \( r = y + r, y \in A, r \in T_1 \). Using Lemma 7 we have \( xrs = xys + xs = xys = 0 \), whence \( T_1 \) is an ideal of \( R_1 \) and our claim has been established.

We next claim that \( R_1 = A \oplus T_2 \), where \( T_2 = \{ x \in R_1 \mid sx = x \} \) is an ideal of \( R_1 \). Indeed, since \( a(1 - s) = a - us = a_{12} + a_{21} - u_{12} = u_2s_1 \) the obvious analogue of the preceding claim may be invoked, with \( 1 - s \) playing the role of \( s \).

For \( t_1 \in T_1, t_2 \in T_2 \) the observation \( 0 = t_1st_2 = t_1t_2 \) shows that \( T_1T_2 = 0 \). Since \( T_1 \) and \( T_2 \) (regarded as either left or right \( A \)-spaces) each have codimension 1 it follows that \( T = T_1 \cap T_2 \) has codimension either 1 or 2. Furthermore we know that \( T^2 = 0 \).

If \( T \) has codimension 1 then \( T = T_1 = T_2 \) and we see that \( R_1 = A \oplus T \) is a primary ring.

We may therefore assume that \( T \) has codimension 2. The ring isomorphism \( T_1/T \cong R_1/T_2 \cong A \) shows that \( T_1/T \) has an identity element, and since \( T^2 = 0 \) it is well known that \( T_1 \) has an idempotent \( e_1 \neq 0 \). Likewise \( T_2 \) has an idempotent \( e_2 \neq 0 \). Using \( e_1e_2 = 0 \) we may replace \( e_2 \) by the idempotent \( e_2 = e_2 - e_2e_1 \), noting now that \( e_1e_2 = e_2e_1 = 0 \). If \( e_1 + e_2 \in T_1 \), say \( T_1 \), then \( e_2 = t_1 - e_1 \in T_1 \cap T_2 = T \), which is impossible. Therefore, since \( R/T_1 \cong A \), we have \( e_1 + e_2 \equiv 1 \) (mod \( T \)) which forces \( e_1 + e_2 = 1 \). For \( t \in T \) we note next that \( t e_1 = t(1 - e_2) = t = (1 - e_1)t = e_2t \). Finally, for \( \lambda \in A \), we observe from \( [\lambda, e_1 + e_2] = 0 \) that \( d(\lambda) = [\lambda, e_1] = [e_2, \lambda] \in T_1 \cap T_2 = T \). Clearly \( R = e_1A \oplus e_2A \oplus T \) and in view of the various observations we have made \( R \) is indeed a \( d \)-semiprimary ring.

Since \( sa = a_{12} = as \) the same arguments used to obtain the structure of \( R_1 \) can equally well be used to show that \( R_2 \) is either primary or \( d \)-semiprimary. The proof of the Main Theorem is now complete.
3. Examples

Our purpose in this section is to show that the possibilities (a), (b), and (c) in the Main Theorem actually occur. It will be useful to have an internal characterization of the elements of $S$ and so we first begin with a result about prime rings in general.

Lemma 8. Let $R$ be prime, $a \neq 0 \in R$, $g$ and $h$ set-theoretic maps of $R$ into itself such that

$$axh(y)=g(x)ya$$

(*)

for all $x, y \in R$. Then there exists $s \in S$ such that $sy = h(y)$ and $axs = g(x)$ (it follows that $g$ and $h$ must be additive). Conversely, every $s \in S$ satisfies (*), where $h(y) = sy$ and $g(x) = axs$.

Proof. We set $I = RaR$ and first show that the map $f: I \to R$ given by $\sum y_iaw_i \mapsto \sum h(y_i)v_i$, $y_i, v_i \in R$, is well defined. Indeed, if $\sum y_iaw_i = 0$ then, making use of (*), we see that $ax \sum h(y_i)v_i = g(x) \sum y_iaw_i = 0$ for all $x \in R$. Since $R$ is prime we conclude that $h(y_i)v_i = 0$. As $f: I \to R$ is clearly a right $R$-module map we have produced an element $s = [f, I] \in R$. Considering $R$ as a subring of $S$ we see that $sy = h(y)$, $y \in R$ and so $sI \subseteq R$. Now let $x, y \in R$ and note that $axsya = ash(y) = g(x)ya$. Thus $(axs - g(x))I = 0$, whence $axs = g(x)$, which says that $Is \subseteq R$ and accordingly $s \in S$. The converse is clear: given $s \in S$ choose $0 \neq I$ such that $Is + sI \subseteq R$, fix $0 \neq a \in I$, and simply note that $(axs)ya = ax(sy)$.

We now proceed to construct examples illustrating (a), (b), and (c) of the Main Theorem. These examples will be built up from primary and semiprimary rings and the reader may refer to the beginning of this paper for the definition of and notational devices used for these rings. For simplicity we will take $A$ to be a field.

(a) An example of an $X$-inner automorphism which is not inner has previously been given in [6], but for completeness we present an example here. Let $R_1 = A + T$, $R_2 = A + U$ be primary rings. Then $R = R_1 \coprod R_2$ may be written as a vector space direct sum in two ways: $R = A + RU + RT = A + UR + TR$. We fix $t_0 \in T$, $u_0 \in U$, and set $a = t_0u_0 + u_0t_0$. Maps $g, h: R \to R$ are then defined as

$$g(1) = t_0u_0 \quad h(1) = t_0u_0$$
$$g(ru) =aru \quad h(ur) = 0 \quad u \in U, \quad t \in T, \quad r \in T$$
$$g(rt) = 0 \quad h(tr) = tra$$
and then extended by linearity. We remark that \( g(R) \subseteq RU \), \( L(R) \subseteq TR \), and proceed to verify

\[
axh(y) = g(x)h(y) = g(x)ya \quad x, y \in R.
\]

We note that since \( g \) and \( h \) are linear it is enough to verify (*) on a basis (brief hints will sometimes be given in parentheses on the right):

\[
\begin{align*}
(\ast) & \quad (Y) = (\delta \delta o + \delta \delta o^2) (Y) = \delta \delta o(Y) (4 \ast) \in TR) \\
& \quad \text{for} \quad (Y) = g(r u)h(y) \\
& \quad \text{for} \quad (Y) = g(r t)h(y) \\
& \quad \text{for} \quad (Y) = g(x)(u r) = 0 = g(x)h(u r) \\
& \quad \text{for} \quad (Y) = g(x)(t r) = g(x)h(t r).
\end{align*}
\]

By Lemma 8 there is an element \( s \in S \) such that \( sa = h(1) = t_0 u_0 \). If \( s \in R \), since \( a \) is \( 0 \)-pure, we have \( |s| = 0 \); i.e., \( s \in A \), which is impossible. Therefore \( s \notin R \).

(b) Let \( R_1 = \Delta e_1 + \Delta e_2 + T \) be a semiprimary in which, setting \( T_1 = \Delta e_2 + T \), we have \( T_1 T_2 = 0 \), \( t_1 e_1 = t_1 \), \( e_2 t_2 = t_2 \). Let \( R_2 = A + U \) be a primary ring. We fix \( u_0 \in U \) and set \( a = e_2 u_0 + u_0 e_1 \). We decompose \( R \) in two ways:

\[
\begin{align*}
R &= \Delta 1 + \Delta e_1 + RU e_2 + RU \oplus RT \\
&= \Delta 1 + \Delta e_1 + e_1 UR \oplus UR \oplus TR
\end{align*}
\]

and define \( g, h : R \to R \) as

\[
\begin{align*}
g(1) &= e_2 u_0 \\
h(1) &= e_2 u_0 \\
g(e_2) &= e_2 u_0 \\
h(e_1) &= (e_1 - 1)u_0 e_1 \\
g(r u e_2) &= a r u \\
h(e_1 u r) &= (e_1 - 1)u r a \\
g(r u) &= a r u \\
h(u r) &= 0 \\
g(r t) &= 0 \\
h(t r) &= t r a,
\end{align*}
\]

where \( t \in T \), \( u \in U \), \( r \in R \). We remark that \( g(R) \subseteq RU \), \( h(R) \subseteq T_2 R \), and proceed to a case by case verification of

\[
\begin{align*}
axh(y) &= g(x)h(y) = g(x)ya, \\
(\ast) & \quad x, y \in R \\
a \cdot 1 \cdot h(y) &= (e_2 u_0 + u_0 e_1)h(y) \\
&= e_2 u_0 h(y) = g(1)h(y) \\
&= (h(y) \in T_2 R; T_1 T_2 = 0)
\end{align*}
\]
\[ ae_2 h(y) = ah(y) = g(1) h(y) = g(e_2) h(y) \quad (e_2 t_2 = t_2) \]
\[ a(rue_1) h(y) = aruh(y) = g(rue_2) h(y) \quad (e_2 t_2 = t_2) \]
\[ a(ru) h(y) = g(ru) h(y) \]
\[ a(rt) h(y) - 0 = g(rt) h(y) \quad (T_1, T_2 = 0) \]
\[ g(x) \cdot 1 \cdot a = g(x) e_2 u_0 = g(x) h(1) \quad (g(x) \in RU, U^2 = 0) \]
\[ g(x) e_1 a = g(x) e_1 u_0 e_1 = g(x)(e_1 - 1) u_0 e_1 \]
\[ = g(x) h(e_1) \quad (g(x) u_0 = 0) \]
\[ g(x)(e_1 ur)a = g(x)(e_1 - 1) ura \]
\[ = g(x) h(e_1 ur) \quad (g(x) u = 0) \]
\[ g(x)(ur)a = 0 = g(x) h(ur) \quad (U^2 = 0) \]
\[ g(x)(tr)a = g(x) h(tr). \quad (*) \]

By Lemma 8, (*) determines an element \( s \in S \) such that \( sa = e_2 u_0 \). If \( s \in R \) then \( |s| = 0 \), i.e., \( s \in \Delta \), which is impossible. Therefore \( s \notin S \).

(c) Let \( R_1 = \Delta e_1 \oplus \Delta e_2 \oplus T \) be a semiprimary ring where, setting \( T_i = \Delta e_i + T \), we have \( T_1, T_2 = 0 \), \( t_1 e_1 = t_1 \), \( e_2 t_2 = t_2 \). Let \( R_2 = \Delta f_1 \oplus \Delta f_2 \oplus U \) also be a semiprimary ring where, setting \( U_i = \Delta f_i + U \) and reversing the subscripts 1 and 2, we have \( U_2 U_1 = 0 \), \( f_1 u_1 = u_1 \), \( u_2 f_2 = u_2 \). We set \( R = R_1 \bigcup_1 R_2 \). We will make use of the following notational device: if \( r = r_1 r_2 \cdots r_n \) then \( r^* = (r_1 - 1)(r_2 - 1) \cdots (r_n - 1) \).

Let \( P \) be the set of all alternating monomials in \( f_1 \) and \( e_2 \), e.g., \( e_2 f_1 e_2 f_1 \). For \( p \in P \) we define
\[
p' = \begin{cases} 
1 & \text{if } p = e_2 \\
p^* & \text{if } p \text{ ends in } f_1 \\
p_0^* & \text{if } p = p_0 e_2 \text{ ends in } e_2.
\end{cases}
\]

We write \( R \) as a direct sum
\[ R = \Delta 1 \oplus \Delta P \oplus R(T + U) P \oplus RU \oplus RT \]
and define \( g: R \to R \) as
\[
g(1) = e_2 f_2 \\
g(p) = \begin{cases} 
e_2 f_2 p' & \text{if } p = e_2 p_0 \\
f_1 e_1 p' & \text{if } p = f_1 p_0
\end{cases}
\]
\[
g(rzp) = arzp' \\
g(ru) = aru \\
g(rt) = 0.
\]
where \( u \in U, \ t \in T, \ z = u \) if \( p = e_2 p_0, \ z = t \) if \( p = f_1 p_0 \). We observe that \( g(R) \subseteq RU_2 \).

Let \( Q \) be the set of all alternating monomials in \( e_1 \) and \( f_2 \). For \( q \in Q \) we define

\[
q' = \begin{cases} 
1 & \text{if } q = f_2 \\
q^* & \text{if } q \text{ begins with } e_1 \\
q_0^* & \text{if } q = f_2 q_0 \text{ begins with } f_2.
\end{cases}
\]

We write \( R \) as a direct sum

\[
R = U_1 \oplus U Q \oplus Q(T + U) R \oplus U R \oplus T R
\]

and define \( h: R \to R \) as

\[
h(1) = e_2 f_2
\]

\[
h(q) = \begin{cases} 
q' e_2 f_2 & \text{if } q = q_0 f_2 \\
q f_1 e_1 & \text{if } q = q_0 e_1
\end{cases}
\]

\[
h(qzr) = q' zra
\]

\[
h(tr) = tra
\]

\[
h(ur) = 0,
\]

where \( u \in U, \ t \in T, \ z = u \) if \( q = q_0 e_1, \ z = t \) if \( q = q_0 f_2 \). We note that \( h(R) \subseteq T_2 R \).

The following observation is crucial to our verification of (*):

**Lemma 9.** If \( p \in P, \ q \in Q \) then

(a) \( t_1 p t_2 = t_1 p' t_2, \ t_i \in T_i, \ p = f_1 p_0 \)

(b) \( u_2 q u_1 = u_2 q' u_1, \ u_i \in U_i, \ q = q_0 e_1 \)

Proof. We prove (a) by induction on the "length" of \( p \) (leaving the analogous proof of (b) to the reader). If \( p = f_1 \) then \( t_1 p t_2 = t_1 f_1 t_2 - t_1 (f_1 - 1) t_2 = t_1 p' t_2 \) (using \( T_1 T_2 = 0 \)). If \( p = e_2 \) then \( u_2 p t_2 = u_2 e_2 t_2 = u_2 t_2 = u_2 p' t_2 \) (using \( e_2 t_2 = t_2 \) and \( p' = 1 \)). This completes the initial step and we move on to the inductive step. If \( p = f_1 p_0 \) then \( t_1 f_1 p_0 t_2 = t_1 (f_1 - 1) p_0 t_2 = t_1 (f_1 - 1) p_0 t_2 = t_1 p' t_2 \) (using \( t_1 p_0 = 0 \)). If \( p = e_2 p_0 \) then \( u_2 e_2 p_0 t_2 = u_2 (e_2 - 1) p_0 t_2 = u_2 (e_2 - 1) p_0 t_2 = u_2 p' t_2 \) (using \( p_0 \in U_1 R \) and \( U_2 U_1 = 0 \)).
We now give a case verification of \( axh(y) = g(x)h(y) \), \( x, y \in R \). Indeed,

\[
axh(y) = (e_2 f_2 + f_1 e_1)h(y) = e_2 f_2 h(y) - g_1(x)h(y)
\]

\[
(h(y) \in T_2 R, e_1 T_2 = 0)
\]

\[
a(e_2 p_0)h(y) = e_2 f_2(e_2 p_0)h(y) = e_2 f_2(e_2 p_0)'h(y)
\]

\[
= g(e_2 p_0)h(y) \quad \text{(Lemma 9(a))}
\]

\[
a(f_1 p_0)h(y) = f_1 e_1(f_1 p_0)h(y) = f_1 e_1(f_1 p_0)'h(y)
\]

\[
= g(f_1 p_0)h(y) \quad \text{(Lemma 9(a))}
\]

\[
a(rue_2 p_0)h(y) = ar[ue_2 p_0 h(y)] = ar(e_2 p_0)'h(y)
\]

\[
= g(rue_2 p_0)h(y) \quad \text{(Lemma 9(a))}
\]

\[
a(rtf_1 p_0)h(y) = ar[tf_1 p_0 h(y)] = art(f_1 p_0)'h(y)
\]

\[
= g(rtf_1 p_0)h(y) \quad \text{(Lemma 9(a))}
\]

\[
aru h(y) = g(rh)h(y)
\]

\[
ar th(y) = 0 = g(rt)h(y).
\]

We next verify \( g(x)y a = g(x)h(y) \), \( x, y \in R \).

\[
g(x) \cdot 1 \cdot a - g(x)(e_2 f_2 + f_1 e_1) - g(x)e_2 f_2 - g(x)h(1)
\]

\[
(g(x) \in RU_2, U_2 U_1 = 0)
\]

\[
g(x)(q_0 f_2)a = g(x)(q_0 f_2)e_2 f_2 - g(x)(q_0 f_2)'e_2 f_2
\]

\[
= g(x)h(q_0 f_2) \quad \text{(Lemma 9(b))}
\]

\[
g(x)(q_0 e_1)a = g(x)(q_0 e_1)f_1 e_1 = g(x)(q_0 e_1)'f_1 e_1
\]

\[
= g(x)h(q_0 e_1) \quad \text{(Lemma 9(b))}
\]

\[
g(x)(q_0 e_1 ur)a = [g(x)q_0 e_1 u]ra = g(x)(q_0 e_1)'ura
\]

\[
= g(x)h(q_0 e_1 ur) \quad \text{(Lemma 9(b))}
\]

\[
g(x)(q_0 f_2 tr)a = [g(x)q_0 f_2 t]ra = g(x)(q_0 f_2)'tra
\]

\[
= g(x)h(q_0 f_2 tr) \quad \text{(Lemma 9(b))}
\]

\[
g(x)(tr)a = g(x)h(tr)
\]

\[
g(x)(ur)a = 0 = g(x)h(ur) \quad (U_2 U = 0).
\]

This completes the verification of (*) and so by Lemma 8 we have produced an element \( s \in S \) such that \( sa = e_2 f_2 \). If \( s \in R \) we again are forced into the contradiction that \( s \in A \), and therefore we conclude that \( s \notin R \).
REFERENCES