Ain Shams Engineering Journal (2015) 6, 381–389

Ain Shams University

Ain Shams Engineering Journal

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ENGINEERING PHYSICS AND MATHEMATICS

Numerical solutions of one-dimensional non-linear parabolic equations using Sinc collocation method

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Received 30 April 2014; revised 4 September 2014; accepted 8 October 2014
Available online 20 December 2014

KEYWORDS

Nonlinear parabolic equations;
Singularly perturbed equations;
Sinc collocation method;
Convergence analysis

Abstract We propose a numerical method for solving singularly perturbed one-dimensional non-linear parabolic problems. The equation converted to the nonlinear ordinary differential equation by discretization first in time then subsequently in each time level we use the Sinc collocation method on the ordinary differential equation. The convergence analysis of proposed technique is discussed, and it is shown that the approximate solution converges to the exact solution at an exponential rate as well. We know that the conventional methods for these types of problems suffer due to decreasing of perturbation parameter, but the Sinc method handles such difficulty. For efficiency and accuracy of the method, we validate the proposed method by several examples. The numerical results confirm the theoretical behavior of the rates of convergence.

1. Introduction

We consider the one dimensional non-linear parabolic partial differential equation

$$\varepsilon\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+\Psi\left(x,t,u,\frac{\partial u}{\partial x}\right), \quad (x,t)\in\Omega\equiv(0,1)\times(0,T), \tag{1}$$

subjected to the initial condition

$$u(x,0)=u_0(x), \quad 0\leqslant x\leqslant 1, \tag{2}$$

and boundary conditions

$$u(0,t)=0 \quad u(1,t)=0, \quad t>0, \tag{3}$$

where ε is a small real constant. When $0 < \varepsilon$ 1, the above equation represents a singular perturbation problem. We assume that the functions $\Psi(x,t,u,u_x)$ and $u_0(x)$ are sufficiently smooth, and suppose that $\frac{\partial\Psi}{\partial u}$ and $\frac{\partial\Psi}{\partial u_x}$ exist and $\frac{\partial\Psi}{\partial u}>0$ and also $\Psi'(z), z\in\mathbb{R}^4$ exists and bounded.

Problems of this type arise in numerous applications from science and engineering, such as fluid flow at high Reynolds number, heat transfer with small diffusion parameter, filtration of liquids, gas dynamics, heat conduction, elasticity, biological species, chemical reactions, environmental pollution, etc. [1].

In recent years, the numerical solution of (1) has been considered by many researchers. Jain et al. [2] have developed a finite difference method for solving the system of 1-D nonlinear parabolic partial differential equations using three spatial grid points subject to Dirichlet boundary conditions. Chawla

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Peer review under responsibility of Ain Shams University.

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http://dx.doi.org/10.1016/j.asej.2014.10.002
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et al. [3] and [4] have described new time-integration schemes for the linear convection-diffusion equation and for the (viscous) Burgers equation with Dirichlets and Neumann boundary conditions. Their schemes are second order in both time and space and unconditionally stable. Mohanty and Singh [5] and Mohanty [6] have developed new two-level implicit variable mesh schemes for the solution of nonlinear parabolic equations. Rashidinia et al. [7] developed a collocation method based on cubic B-splines for solving nonlinear parabolic equations, and they derived a new two-level three-point scheme of order $o(k^2 + h^2)$.

Recently, Mittal and Jiwari [8] have considered a numerical scheme based on differential quadrature method to solve nonlinear generalizations of the Fisher and Burgers equations with the zero flux on the boundary. They have mentioned that Neumann condition (zero fluxes on the boundary) is typical requirement to describe the actual processes in mathematical biology. Mohanty and Jain [9] have developed a cubic spline alternating group explicit (C-SPLAGE) method of order 2 in time and 4 in space for the solution of one-dimensional nonlinear parabolic equation on a uniform mesh. Zhou and Cheng [10] developed a linearly semi-implicit compact scheme for Burgers–Huxley equation. This scheme was fourth-order accurate in space and second-order accurate in time. They have considered Neumann boundary conditions for the solutions. Moreover, we can point to many other efficient methods for solving these problems, see [11–17].

In this paper, we discretize Eq. (1) in time direction by means of implicit Euler method and then we applied a collocation method in space direction which is based on Sinc basis functions. Sinc method has been studied extensively and found to be a very effective technique, particularly for problems with singular solutions and those on unbounded domain. In addition, Sinc function seems that be useful for problems that their solutions have oscillatory behavior in the space direction. Sinc method originally introduced by Stenger [18] which is based on the Whittaker–Shannon–Kotel'nikov sampling theorem for entire functions. The books [19] and [20] provide excellent overviews of the existing Sinc methods for solving ODEs and PDEs. This method has many advantages over classical methods that use polynomials as basis. For example, in the presence of singularities, it gives better rate of convergence and accuracy than polynomial methods.

In recent years, lot of attention has been devoted to the study of the Sinc method to investigate various scientific models. The efficiency of the Sinc method has been formally proved by many researchers Bialecki [21], Rashidinia and Zarebnia [22,23], Nurmuhammada et al. [24], El-Gamel [25], Saadatmandi and Dehghan [26] and Okayama et al. [27].

The paper is organized as follows. In Section 2, we review some basic facts about the sinc approximation. In Section 3, we discretized the temporal variable by means of implicit Euler method and then we applied the Sinc-collocation method for solving the arising second-order nonlinear boundary value problems in each time level. In Section 4, the convergence analysis of proposed method is given. Also, some numerical examples will be presented in Section 5, and at the end we conclude implementation, application and efficiency of proposed scheme.

2. Notation and background

The goal of this section is to recall notations and definitions of the Sinc function and state some known theorems that are important for this paper.

The Sinc function is defined on $-\infty < x < \infty$ by

$$Sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For $h > 0$ we will denote the Sinc basis functions by

$$S(j,h)(x) = sinc\left(\frac{x - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots$$

let f be a function defined on $\mathbb{R}$ then for $h > 0$ the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x),$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of Whittaker cardinal expansions have been studied and are thoroughly surveyed in Stenger [20]. These properties are derived in the infinite strip D_d of the complex plane where $d > 0$

$$D_d = \left\{ \zeta = \xi + i\eta : |\eta| < d \leqslant \frac{\pi}{2} \right\}.$$

Approximation can be constructed for infinite, semi-finite, and finite intervals. To construct approximation on the interval (0, 1), we consider the conformal map

$$\phi(z) = \ln\left(\frac{z}{z-1}\right), \tag{4}$$

which maps the eye-shaped region

$$D_E = \left\{ z = x + iy; \left| \arg\left(\frac{z}{1-z}\right) \right| < d \leqslant \frac{\pi}{2} \right\},$$

onto the infinite strip D_d.

For the Sinc method, the basis functions on the interval (0, 1) for $z \in D_E$ are derived from the composite translated Sinc function

$$S_j(z) = S(j,h) \circ \phi(z) = sinc\left(\frac{\phi(z) - jh}{h}\right). \tag{5}$$

The function

$$z = \phi^{-1}(\omega) = \frac{e^{\omega}}{1 + e^{\omega}},$$

is an inverse mapping of $\omega = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \left\{ \psi(u) = \phi^{-1}(u) \in D_E : -\infty < u < \infty \right\} = (0,1).$$

The sinc grid points $z_k \in (0,1)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image that corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{6}$$

Definition 1. Let $B(D_E)$ is the class of functions f which are analytic in D_E such that

$$\int_{\psi(u+\Sigma)} |f(z)|dz \to 0 \ as \ u \to \pm\infty \tag{7}$$

where $\Sigma = \{i\eta : |\eta| < d \leqslant \frac{\pi}{2}\}$ and satisfy

$$N(f) \equiv \int_{\partial D_E} |f(z)|dz < \infty, \tag{8}$$

where ∂D_E represents the boundary of D_E.

Definition 2. Let $L_\alpha(D_E)$ be the set of all analytic function u in D_E, for which there exists a constant C such that

$$|u(z)| \leqslant C\frac{|\rho(z)|^\alpha}{[1+|\rho(z)|]^{2\alpha}}, \quad z \in D_E, \ 0 < \alpha \leqslant 1. \tag{9}$$

where $\rho(z) = e^{\phi(z)}$.

Theorem 1 (*Stenger [20]*). *If $\phi' u \in B(D_E)$, and let*

$$\sup_{\frac{-\pi}{h}\leqslant t\leqslant\frac{\pi}{h}} \left| \left(\frac{d}{dx}\right)^l e^{it\phi(x)} \right| \leqslant C_1 h^{-l}, \quad x \in \Gamma,$$

for $l = 0, 1, \ldots, m$ with C_1 a constant depending only on m and ϕ. If $u \in L_\alpha(D_E)$ then taking $h = \sqrt{\pi d/\alpha N}$ it follows that

$$sup_{x\in\Gamma}\left| u^{(l)}(x) - \left(\frac{d}{dx}\right)^l \sum_{j=-N}^{N} u(x_j)S_j(x) \right|$$
$$\leqslant CN^{(l+1)/2}\exp(-(\pi d\alpha N)^{1/2}),$$

where C is a constant depending only on u, d, m, ϕ and α.

The Sinc-collocation method requires that the derivatives of composite Sinc function be evaluated at the nodes. We need to recall the following lemma.

Lemma 1 (*Lund and Bowers [19]*). *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (5).* Then

$$\delta_{jk}^{(0)} = [S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{10}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{(k-j)}}{k-j}, & j \neq k, \end{cases} \tag{11}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{(k-j)}}{(k-j)^2}, & j \neq k, \end{cases} \tag{12}$$

in relations (10)–(12) h is step size and x_k is sinc grid given by (6).

3. Description of method

First, we discretize equation (1) in time direction by means of the implicit Euler method with uniform step size Δt as

$$u_t(x, t_{n+1}) \approx \frac{u^{n+1} - u^n}{\Delta t}, \tag{13}$$

where $u^n = u(x, t_n)$, $u^0 = u(x, 0)$ and $t_n = n\Delta t$, $n = 1, 2, \ldots$

By substituting the approximation (13) into Eq.(1) we obtain the nonlinear ordinary differential equation as

$$\frac{u^{n+1} - u^n}{\Delta t} - \varepsilon u_{xx}^{n+1} + \Psi(x, t_{n+1}, u^{n+1}, u_x^{n+1}) = 0, \tag{14}$$

where $u^{n+1} = u(x, t_{n+1})$ is the solution of Eq. (14) at $(n + 1)$th time level.

Now we can rewrite equation (14) as

$$L\hat{u} \equiv -\varepsilon\hat{u}_{xx} + \frac{1}{\Delta t}\hat{u} + \widehat{\Psi}(x, \hat{u}, \hat{u}_x) = \hat{g}(x), \tag{15}$$

where $\hat{u} = u^{n+1}$, $\widehat{\Psi}(x, \hat{u}, \hat{u}_x) = \Psi(x, t_{n+1}, u^{n+1}, u_x^{n+1})$, $\hat{g}(x) = \frac{u^n}{\Delta t}$, and associated with boundary conditions

$$\hat{u}(a) = \hat{u}(b) = 0. \tag{16}$$

Now, we apply the Sinc-collocation method for solution of Eq. (15) with given boundary conditions (16).

The approximate solution for $\hat{u}(x)$ is represented by formula

$$\hat{u}(x) \approx \hat{u}_m(x) = \sum_{j=-N}^{N} \hat{c}_j S_j(x), \quad m = 2N + 1, \tag{17}$$

where $S_j(x)$ is function $S(j, h) \circ \phi(x)$ for some fixed step size h. The unknown coefficients $\hat{c}_j$ in relation to (17) are determined by collocation method. Also the l-th derivative of the function $\hat{u}$ can be approximated as

$$\hat{u}^{(l)}(x) \approx \sum_{j=-N}^{N} \hat{c}_j \frac{d^l}{dx^l} S_j(x). \tag{18}$$

Setting

$$\frac{d^i}{d\phi^i}[S_j(x)] = S_j^{(i)}(x), \quad 0 \leqslant i \leqslant 2, \tag{19}$$

and noting that

$$\frac{d}{dx}[S_j(x)] = S_j^{(1)}(x)\phi'(x), \tag{20}$$

$$\frac{d^2}{dx^2}[S_j(x)] = S_j^{(2)}(x)[\phi'(x)]^2 + S_j^{(1)}(x)\phi''(x), \tag{21}$$

and

$$\delta_{jk}^{(l)} = h^l \frac{d^l}{d\phi^l}[S_j(x)]_{x=x_k}. \tag{22}$$

Now by substituting each term of (15) with given approximations in (17) and (18) and evaluating the result at the Sinc points x_k also using relations (19)–(22), we can obtain the discrete Sinc-collocation system of nonlinear equations to determining the unknown coefficients $\{\hat{c}_j\}_{j=-N}^{N}$ as

$$-\varepsilon\left\{\sum_{j=-N}^{N}\hat{c}_j\left(\frac{\delta_{jk}^{(2)}}{h^2}[\phi'(x_k)]^2 + \frac{\delta_{jk}^{(1)}}{h}\phi''(x_k)\right)\right\}$$
$$+\frac{1}{\Delta t}\hat{c}_k + \widehat{\Psi}\left(x_k, \hat{c}_k, \sum_{j=-N}^{N}\hat{c}_j\phi'(x_k)\frac{\delta_{jk}^{(1)}}{h}\right) = \hat{g}(x_k),$$
$$k = -N, -N+1, \ldots, N. \tag{23}$$

To obtain a matrix representation of the equations (23), let $I^{(i)}$, $i = 0, 1, 2$ be the $m \times m$ matrices whose jk-th entry is given by (10)–(12). Note that the matrices $I^{(2)}$ and $I^{(1)}$ are symmetric and skew-symmetric matrices respectively, also $I^{(0)}$ is identity matrix. We define the $m \times m$ diagonal matrix as follows:

$$D(s(x))_{ij} = \begin{cases} s(x_i), & i = j, \\ 0, & i \neq j. \end{cases}$$

Therefore, by using the above definitions and multiplying both sides of (23) in $h^2 \frac{1}{[\phi'(x_k)]^2}$, the system (23) can be represented by the following matrix form:

$$A\widehat{C} + \widehat{B} = \widehat{G}, \tag{24}$$

where $\widehat{C}$, $\widehat{B}$ and $\widehat{G}$ are m-vectors and A is $m \times m$ matrix as

$$A = -\varepsilon I^{(2)} - \varepsilon h I^{(1)} D\left(\frac{\phi''}{(\phi')^2}\right) + \frac{h^2}{\Delta t} I^{(0)} D\left(\frac{1}{(\phi')^2}\right),$$

$$\widehat{B} = h^2 D\left(\frac{1}{(\phi')^2}\right) \begin{pmatrix} \widehat{\Psi}(x_{-N}, \hat{c}_{-N}, \sum_{j=-N}^{N} \hat{c}_j \phi'(x_{-N}) \frac{\delta^{(1)}_{j-N}}{h}) \\ \widehat{\Psi}(x_{-N+1}, \hat{c}_{-N+1}, \sum_{j=-N}^{N} \hat{c}_j \phi'(x_{-N+1}) \frac{\delta^{(1)}_{j-N+1}}{h}) \\ \vdots \\ \widehat{\Psi}\left(x_N, \hat{c}_N, \sum_{j=-N}^{N} \hat{c}_j \phi'(x_N) \frac{\delta^{(1)}_{jN}}{h}\right) \end{pmatrix},$$

$$\widehat{C} = \begin{pmatrix} \hat{c}_{-N} \\ \hat{c}_{-N+1} \\ \vdots \\ \hat{c}_N \end{pmatrix}, \quad \widehat{G} = h^2 D\left(\frac{1}{(\phi')^2}\right) \begin{pmatrix} \hat{g}(x_{-N}) \\ \hat{g}(x_{-N+1}) \\ \vdots \\ \hat{g}(x_N) \end{pmatrix}.$$

The system (24) is a nonlinear system of equations which consists of m equations and m unknowns. By solving this system by means of Newton's method, we can obtain an approximate solution $\hat{u}_m(x)$ of (15) from (17).

Remark. It is worth to mention here that, if homogeneous boundary conditions are given in (3), the following nonhomogeneous boundary conditions are specified:

$$u(a, t) = p(t), \quad u(b, t) = q(t),$$

then we reformulate the problem (1)–(3) by applying the following change of variable:

$$v(x, t) = u(x, t) + \frac{x - b}{b - a} p(t) + \frac{a - x}{b - a} q(t),$$

to convert the boundary conditions to homogeneous one.

4. Convergence analysis

In this section, we will prove that the approximate solution $\hat{u}_m(x)$ given in (17) converges exponentially to the exact solution $\hat{u}(x)$ of (15). In order to establish a bound for $|\hat{u}(x) - \hat{u}_m(x)|$, for this purpose we assume that the analytic solution of equation (15) at the Sinc points x_k denoted by $\tilde{u}(x)$ and defined by

$$\tilde{u}(x) = \sum_{j=-N}^{N} \hat{u}(x_j) S_j(x). \tag{25}$$

We first need to establish a bound for $|\hat{u}_m(x) - \tilde{u}(x)|$. So that, we need to prove the following theorem:

Theorem 2. *If $\hat{u}(x)$ be unique solution of (15), $\hat{u}(x) \in L_\alpha(D_E)$, and $\tilde{u}(x)$ defined as in (25), then there exists a constant c_1 independent of N such that*

$$\sup_{x \in \Gamma} |\hat{u}_m(x) - \tilde{u}(x)| \leq c_1 N^2 \exp(-(\pi \alpha d N)^{1/2}). \tag{26}$$

Proof. According to (25) and Cauchy–Schwarz inequality we have

$$\begin{aligned} |\hat{u}_m(x) - \tilde{u}(x)| &= \left| \sum_{j=-N}^{N} \hat{c}_j S_j(x) - \sum_{j=-N}^{N} \hat{u}(x_j) S_j(x) \right| \\ &\leq \left(\sum_{j=-N}^{N} |\hat{c}_j - \hat{u}(x_j)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=-N}^{N} |S_j(x)|^2 \right)^{\frac{1}{2}} = E, \end{aligned} \tag{27}$$

and we know that $\left(\sum_{j=-N}^{N} |S_j(x)|^2\right)^{\frac{1}{2}} \leq c_2$ where c_2 is a constant, then

$$E \leqslant c_2 \|\widehat{C} - \tilde{C}\|_2, \tag{28}$$

where $C = (\hat{u}(x_{-N}), \hat{u}(x_{-N+1}), \ldots, \hat{u}(xN))^t$.

Now we must obtain a bound for $\|\widehat{C} - \tilde{C}\|_2$, to this aim, following (24) for $\tilde{u}(x)(x)$ we have

$$A\tilde{C} + \tilde{B} = \tilde{G}, \tag{29}$$

where

$$\tilde{B} = h^2 D\left(\frac{1}{(\phi')^2}\right) \begin{pmatrix} \widehat{\Psi}\left(x_{-N}, \hat{u}(x_{-N}), \sum_{j=-N}^{N} \hat{u}(x_j) \phi'(x_{-N}) \frac{\delta^{(1)}_{j-N}}{h}\right) \\ \widehat{\Psi}\left(x_{-N+1}, \hat{u}(x_{-N+1}), \sum_{j=-N}^{N} \hat{u}(x_j) \phi'(x_{-N+1}) \frac{\delta^{(1)}_{j-N+1}}{h}\right) \\ \vdots \\ \widehat{\Psi}\left(x_N, \hat{u}(x_N), \sum_{j=-N}^{N} \hat{u}(x_j) \phi'(x_N) \frac{\delta^{(1)}_{jN}}{h}\right) \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} \hat{u}(x_{-N}) \\ \hat{u}(x_{-N+1}) \\ \vdots \\ \hat{u}(x_N) \end{pmatrix}, \quad \tilde{G} = h^2 D\left(\frac{1}{(\phi')^2}\right) \begin{pmatrix} \tilde{g}(x_{-N}) \\ \tilde{g}(x_{-N+1}) \\ \vdots \\ \tilde{g}(x_N) \end{pmatrix}$$

By subtracting (29) from (24) we have

$$A(\widehat{C} - \tilde{C}) + (\widehat{B} - \tilde{B}) = (\widehat{G} - \tilde{G}), \tag{30}$$

First, we obtain a bound for $\|\widehat{G} - \tilde{G}\|$ in the maximum norm. Applying (15) we have

$$\begin{aligned} |\hat{g}(x_k) - \tilde{g}(x_k)| &= |L\hat{u}(x_k) - L\tilde{u}(x_k)| \leq \varepsilon |\hat{u}_{xx}(x_k) - \tilde{u}_{xx}(x_k)| \\ &\quad + \frac{1}{\Delta t} |\hat{u}(x_k) - \tilde{u}(x_k)| + |\widehat{\Psi}(x_k, \hat{u}(x_k), \hat{u}_x(x_k)) \\ &\quad - \widehat{\Psi}(x_k, \tilde{u}(x_k), \tilde{u}_x(x_k))|, \end{aligned} \tag{31}$$

for $k = -N, -N+1, \ldots, N$.

From (31) and by using Theorem 1 and Theorem 9.19 [28, p. 218] we obtain

$$\left\|\widehat{G}-\tilde{G}\right\|_{\infty} \leqslant \varepsilon k_2 N^{3/2} \exp\left(-(\pi\alpha dN)^{1/2}\right) + \frac{1}{\Delta t} k_0 N^{1/2} \exp\left(-(\pi\alpha dN)^{1/2}\right) + M[|\hat{u}(x)-\tilde{u}(x)| + |\hat{u}_x(x)-\tilde{u}_x(x)|] \leqslant \exp(-(\pi\alpha dN)^{1/2}) \times \left(\varepsilon k_2 N^{3/2} + \frac{1}{\Delta t} k_0 N^{1/2} + M k_0' N^{1/2} + M k_1 N\right), \tag{32}$$

where k_0, k_0', k_1 and k_2 are constants independent on N and $\|\widehat{\Psi}'(z)\|_\infty \le M$.

Thus we have

$$\|\widehat{G}-\tilde{G}\|_\infty \leqslant K N^{3/2} \exp(-(\pi\alpha dN)^{1/2}), \tag{33}$$

where $K = \varepsilon k_2 + \frac{1}{\Delta t} k_0 + M k_0' + M k_1$.

Now to obtain a bound for $(\widehat{B}-\tilde{B})$ in (30), by using the mean value theorem we have

$$\widehat{B}-\tilde{B} = \left[h^2 \frac{\partial\widehat{\Psi}}{\partial\hat{u}}(\xi_1) D\left(\frac{1}{(\phi')^2}\right) I^{(0)} + h \frac{\partial\widehat{\Psi}}{\partial\hat{u}_x}(\xi_2) D\left(\frac{1}{\phi'}\right) I^{(1)}\right](\widehat{C}-\tilde{C}) \tag{34}$$

where $\xi_1, \xi_2 \in (0,1)$.

From (30) and (34) we have

$$R\left(\widehat{C}-\tilde{C}\right) = \widehat{G}-\tilde{G}, \tag{35}$$

where

$$R = A + h^2 \frac{\partial\widehat{\Psi}}{\partial\hat{u}}(\xi_1) D\left(\frac{1}{(\phi')^2}\right) I^{(0)} + h \frac{\partial\widehat{\Psi}}{\partial\hat{u}_x}(\xi_2) D\left(\frac{1}{\phi'}\right) I^{(1)}, \tag{36}$$

therefore, by taking L^2-norm from (35) and using (33) we have

$$\|\widehat{C}-\tilde{C}\|_2 \leqslant \|R^{-1}\|_2 \|\widehat{G}-\tilde{G}\|_2 \leqslant \|R^{-1}\|_2 \sqrt{3N} \|\widehat{G}-\tilde{G}\|_\infty \leqslant \sqrt{3} K N^2 \|R^{-1}\|_2 \exp(-(\pi\alpha dN)^{1/2}). \tag{37}$$

Finally, by applying (27), (28) and (37) we can obtain

$$\sup_{x\in\Gamma} |\hat{u}_m(x)-\tilde{u}(x)| \le c_1 N^2 \exp(-(\pi\alpha dN)^{1/2}). \tag{38}$$

where $c_1 = \sqrt{3} c_2 K \|R^{-1}\|_2$.

Now we need to derive an upper bound for $\|R^{-1}\|_2$, where the matrix R is defined by (36). $\square$

Lemma 3. *Let the matrix R be defined by (36), and for $x \in \phi^{-1}((-\infty,\infty))$, we let*

$$\frac{R+R^*}{2} = -\varepsilon I^{(2)} + \frac{1}{2} H,$$

where $(\cdot)^*$ states the conjugate transpose of a matrix and also

$$H = -\varepsilon h\left[I^{(1)} D\left(\frac{\phi''}{(\phi')^2}\right) - D\left(\overline{\frac{\phi''}{(\phi')^2}}\right)\right] + \frac{h^2}{\Delta t} D\left(2Re\frac{1}{(\phi')^2}\right) + h^2 \frac{\partial\widehat{\Psi}}{\partial\hat{u}}(\xi_1) D\left(2Re\frac{1}{(\phi')^2}\right) + h \frac{\partial\widehat{\Psi}}{\partial\hat{u}_x}(\xi_2)\left[D\left(\frac{1}{\phi'}\right) I^{(1)} - I^{(1)} D\left(\overline{\frac{1}{\phi'}}\right)\right], \tag{39}$$

If eigenvalues of matrix H are nonnegative, then there exists a constant c_3, independent of N, such that

$$\|R^{-1}\|_2 \leqslant \frac{4N^2}{\varepsilon\pi^2}\left(1+\frac{c_3}{N}\right), \tag{40}$$

holds for a sufficiently large N.

Proof. Let δ_i $(i = 1, 2, \ldots, 2N+1)$ be the singular values of the matrix R satisfying $\delta_i \leqslant \delta_{i+1}$ and $\lambda_i(\cdot)$ be the eigenvalues of a matrix as ordered $\lambda_i(\cdot) \leqslant \lambda_{i+1}(\cdot)$. It is well-known that the eigenvalues of the matrix $I^{(2)}$ are bounded from below by $4\sin^2\left(\frac{\pi}{4(N+1)}\right)$

Theorem 4.18 [19], from [29] we have

$$\delta_1 \geqslant \min_{1\leqslant i\leqslant 2N+1}\left|\lambda_i\left(\frac{R+R^*}{2}\right)\right| = \min_{1\leqslant i\leqslant 2N+1}\left|\lambda_i(-\varepsilon I^{(2)} + \frac{1}{2}H)\right| \geqslant \varepsilon \min_{1\leqslant i\leqslant 2N+1} \lambda_i(I^{(2)}) \geqslant 4\varepsilon\sin^2\left(\frac{\pi}{4(N+1)}\right),$$

therefore, above relations lead to the estimate in (40). $\square$

Theorem 4. *Let $\hat{u}(x)$ be the exact solution of (15) and $\hat{u}_m(x)$ be its Sinc approximation defined by (17). Then, under the assumptions of* Theorems 1 and 2 *and* Lemma 3, *there exists a constant c, independent of N, such that*

$$\sup_{x\in\Gamma} |\hat{u}_m(x)-\hat{u}(x)| \leqslant c N^4 \exp(-(\pi\alpha dN)^{1/2}). \tag{41}$$

Proof. By making use of the triangular inequality we have

$$|\hat{u}_m(x)-\hat{u}(x)| \leqslant |\hat{u}_m(x)-\tilde{u}(x)| + |\tilde{u}(x)-\hat{u}(x)|, \tag{42}$$

we note that $\tilde{u}(x)$ be the solution of equation (15) at the Sinc points x_k and it has been defined in (25).

By applying Theorem 1 for second term of right hand side of (42) there exists a constant c_4 independent of N such that

$$\sup_{x\in\Gamma} |\tilde{u}(x)-\hat{u}(x)| \le c_4 N^{\frac{1}{2}} \exp(-(\pi\alpha dN)^{1/2}). \tag{43}$$

Also, by using Theorem 2 and Lemma 3 for first term in the right hand side of (42) we can obtain

$$\sup_{x\in\Gamma} |\hat{u}_m(x)-\tilde{u}(x)| \le c_5 N^4 \exp(-(\pi\alpha dN)^{1/2}), \tag{44}$$

where c_5 is a constant independent of N. $\square$

Finally, by applying relations (42)–(44) we have

$$\sup_{x\in\Gamma} |\hat{u}_m(x)-\hat{u}(x)| \le c N^4 \exp(-(\pi\alpha dN)^{1/2}), \tag{45}$$

where $c = \max\{c_4, c_5\}$. $\square$

Now, if we suppose that $u(x, t)$ be the exact solution and $U(x, t)$ be the numerical approximation (1) by our numerical process, then we have

$$\|u(x,t) - U(x,t)\|_\infty \leqslant \rho(\exp(-(\pi d\alpha N)^{1/2}) + \Delta t),$$

where ρ is a constant.

Table 1 RMS errors and rate of convergence for Example 1 with $p = 2N$ and $\Delta t = \frac{2.56}{p^2}$ at $t = 1$.

$\varepsilon\rightarrow$ $N\downarrow$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
8	2.23×10^{-5}	1.79×10^{-6}	1.79×10^{-7}	1.81×10^{-8}	1.79×10^{-9}
	4.15	4.24	4.89	4.90	4.89
16	1.06×10^{-6}	6.50×10^{-8}	5.10×10^{-9}	5.07×10^{-10}	5.07×10^{-11}
	5.82	4.71	5.33	7.67	7.67
32	2.71×10^{-8}	1.33×10^{-9}	4.64×10^{-11}	2.51×10^{-12}	2.51×10^{-13}
	7.34	7.17	6.36	6.86	11.38
64	1.68×10^{-10}	8.73×10^{-12}	3.80×10^{-13}	4.71×10^{-15}	9.80×10^{-17}

Table 2 RMS errors and rate of convergence for Example 1 with $p = 2N$ and $\Delta t = \frac{2.56}{p^2}$ at $t = 3$.

$\varepsilon\rightarrow$ $N\downarrow$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
8	5.65×10^{-5}	3.15×10^{-6}	2.08×10^{-7}	2.72×10^{-8}	2.72×10^{-9}
	3.83	3.91	4.15	4.42	4.61
16	2.68×10^{-6}	8.12×10^{-8}	9.36×10^{-9}	9.16×10^{-10}	9.15×10^{-11}
	5.38	5.67	5.82	7.21	7.32
32	7.67×10^{-8}	6.90×10^{-9}	8.35×10^{-11}	4.70×10^{-12}	4.47×10^{-13}
	7.21	6.89	6.28	6.35	9.72
64	4.64×10^{-10}	4.41×10^{-11}	3.60×10^{-12}	1.85×10^{-14}	1.67×10^{-16}

5. Numerical results

In this section, the application of the Sinc method for Eq. (1) is tested on several problems to verify the theoretical results discussed in the previous sections. In all of the examples that considered in this paper, we choose $\alpha = 1$ and $d = \frac{\pi}{2}$ which yield $h = \frac{\pi}{\sqrt{2N}}$, also the errors are reported on uniform grids

$$U = \{z_0, z_1, \dots, z_p\}, \quad z_r = \frac{r}{p}, \quad r = 0, 1, \dots, p. \tag{46}$$

The pointwise error can be calculated as

$$e_\varepsilon^{m,\Delta t}(z_r, t_n) = |u(z_r, t_n) - u_m(z_r, t_n)| \tag{47}$$

And the maximum pointwise error is given by

$$E_\varepsilon^{m,\Delta t}(z_r, t_n) = \max_{r,n} e_\varepsilon^{m,\Delta t}. \tag{48}$$

Also, the numerical rate of convergence is given as

$$P_{\varepsilon,m} = \frac{\ln E^{m,\Delta t} - lnE^{2m,\Delta t/2}}{\ln 2} \tag{49}$$

Example 1. We consider the following equation

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 - u^2 = f(x,t), \quad 0 < x < 1, t > 0, \tag{50}$$

subjected to initial and boundary conditions

$$u(x,0) = \varepsilon\sin(\pi x)\cos(\pi x), \quad u(0,t) = u(1,t) = 0,$$

where $f(x, t)$ is taken such that the exact solution is

$$u(x,t) = \varepsilon e^{-\varepsilon^2 t}\sin(\pi x)\cos(\pi x).$$

The root mean square (RMS) errors and rate of convergence for this example at $t = 1$ and $t = 3$ with $\Delta t = \frac{2.56}{p^2}$ are given in Tables 1 and 2 for various values of ε and N. The results show that the method has enough accuracy and efficiency even for small values of perturbation parameter. Moreover, we observe that the method does not lose its accuracy with increasing time. Convergence curves for this example are plotted for various values of ε in Fig. 1. This figure indicates that the maximum errors decrease at an exponential rate with respect to N especially for small values of ε, and this graph confirms the theoretical results.

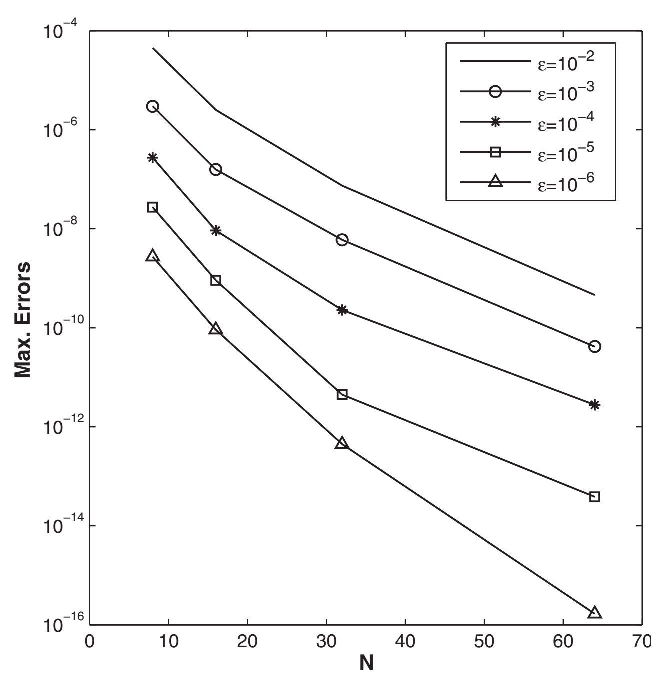

Figure 1 Convergence curves of the method for Example 1 for various values of ε with $\Delta t = \frac{2.56}{p^2}$ at $t = 1$.

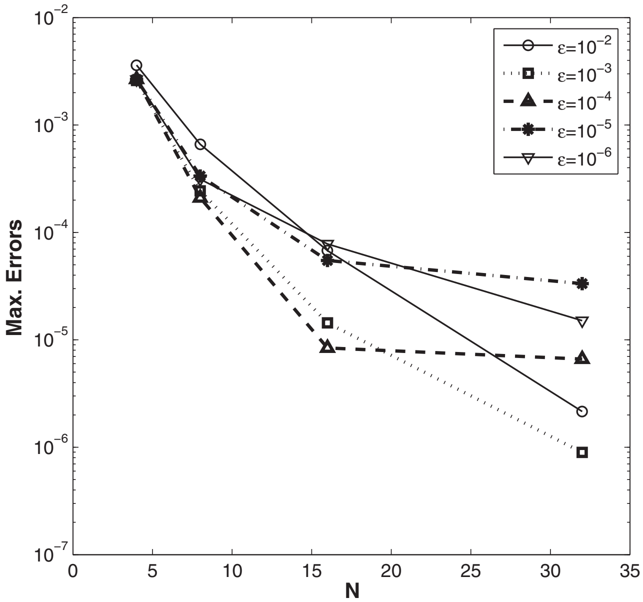

Figure 2 Convergence curves of the method for Example 2 for various values of ε with $\Delta t = \frac{0.64}{p^2}$ at $t = 1$.

Example 2. We consider the following problem:

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}u^{1/2} = 0, \quad x \in (0,1), t > 0, \tag{51}$$

with initial and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = \sin(\pi x).$$

This problem cannot be solved analytically, therefore for each ε the pointwise errors are estimated as

$$\bar{e}_{\varepsilon}^{m,\Delta t}(z_r, t_n) = |u_m(z_r, t_n) - u_{2m}(z_r, t_n)|. \tag{52}$$

The maximum pointwise error is given by

$$\bar{E}_{\varepsilon}^{m,\Delta t}(z_r, t_n) = \max_{r,n} \bar{e}_{\varepsilon}^{m,\Delta t}. \tag{53}$$

The maximum pointwise errors and rate of convergence for this example are reported in Table 3 for various values ε and N at $t = 1$ with $\Delta t = \frac{0.64}{p^2}$. The results show that the errors decrease with increasing N. Moreover, data in this table show that with increasing N the maximum error does not greatly reduce especially when value of N increases from 16 to 32 for $\varepsilon = 10^{-4}$, 10^{-5} and 10^{-6}. The reason is that when ε is small the arising coefficient matrix becomes singular with increasing N for this example. Also, the convergence curves are plotted for various values of ε in Fig. 2. This figure shows that the treatment of maximum errors is exponential with increasing N especially for $\varepsilon = 10^{-2}$ and 10^{-3}.

Table 4 RMS errors and rate of convergence for Example 3 with $p = 2N$ and $\Delta t = \frac{1.6}{p^2}$ at $t = 1$.

$N\downarrow$	$\gamma = 10$		$\gamma = 20$	
	[31]	Our scheme	[31]	Our scheme
4	1.21×10^{-4}	4.18×10^{-3}	4.07×10^{-5}	8.61×10^{-4}
		2.67		1.69
8	7.50×10^{-6}	9.73×10^{-4}	2.53×10^{-6}	1.51×10^{-4}
		3.52		2.39
16	4.63×10^{-7}	7.98×10^{-5}	1.57×10^{-7}	7.19×10^{-5}
		3.97		3.91
32	2.87×10^{-9}	2.46×10^{-6}	9.76×10^{-9}	1.18×10^{-6}

Example 3. We consider test problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \gamma u\frac{\partial u}{\partial x} + f(x,t), \quad (x,t) \in (0,1) \times (0,1) \tag{54}$$

with boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = \sin(\pi x),$$

where $f(x, t)$ is taken so that the exact solution is $u(x, t) = e^{-t}\sin(\pi x)$.

This example has been considered by many authors with different methods. Mohanty and Dahiya in [31] applied a new two-level implicit cubic spline method of $O(k^2 + kh^2 + h^4)$ for the solution of this example. Mohanty in [6] used an implicit high accuracy variable mesh scheme for this problem and also Arora et al. in [30] chose this example for test problem of their method that was based on a stable variable mesh method.

The RMS errors and rate of convergence are listed in Table 4 for various value of N and γ at $t = 1$ with $\Delta t = \frac{1.6}{p^2}$. This table shows that by increasing N the errors in our scheme decrease slowly as compared with the method in [31]. Also, for this problem the RMS errors for various values of γ are plotted in Fig. 3. This figure represents convergence of method for values of γ and confirms the results obtained in section 4.

Example 4. We consider a nonlinear reaction-diffusion equation with polynomial reaction term as

Table 3 The maximum pointwise errors and rate of convergence for Example 2 with $p = 2N$ and $\Delta t = \frac{0.64}{p^2}$ at $t = 1$.

$\varepsilon\rightarrow$ $N\downarrow$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
4	3.61×10^{-3}	2.60×10^{-3}	2.67×10^{-3}	2.66×10^{-3}	2.65×10^{-3}
	2.44	3.41	3.66	2.98	3.8
8	6.62×10^{-4}	2.44×10^{-4}	2.10×10^{-4}	3.35×10^{-4}	3.13×10^{-4}
	3.28	4.08	4.64	4.61	2.25
16	6.77×10^{-5}	1.44×10^{-5}	8.41×10^{-6}	5.48×10^{-5}	7.81×10^{-5}
	4.97	4.00	0.34	0.71	2.37
32	3.15×10^{-6}	8.96×10^{-7}	6.62×10^{-6}	3.34×10^{-5}	1.51×10^{-5}

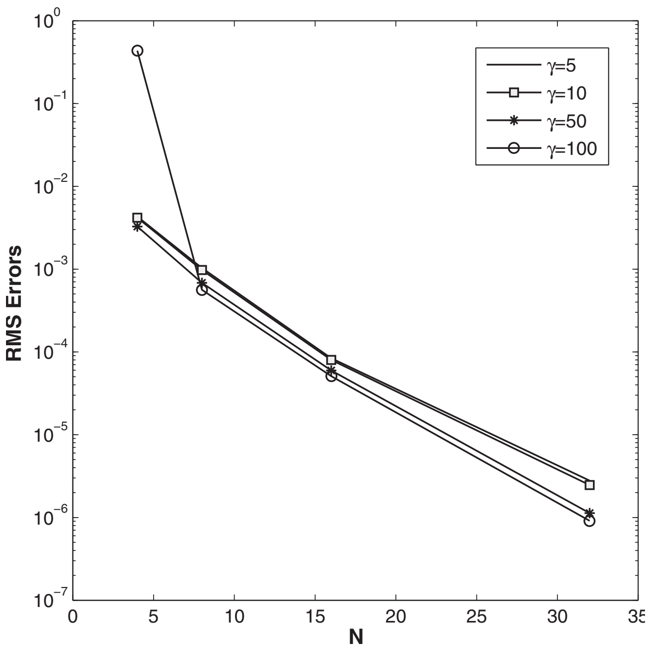

Figure 3 Convergence curves of the method for Example 3 for various values of γ with $\Delta t = \frac{1.6}{p^2}$ at $t = 1$.

Table 5 RMS errors and rate of convergence for Example 4 with $p = 2N$ and $\Delta t = \frac{0.64}{p^2}$ at $t = 1$.

$\varepsilon\rightarrow$ $N\downarrow$	10^0	10^{-1}	10^{-2}
4	6.02×10^{-7}	3.18×10^{-3}	2.70×10^{-2}
	1.33	3.48	1.64
8	2.39×10^{-7}	2.85×10^{-4}	8.66×10^{-3}
	3.28	3.96	2.78
16	2.46×10^{-8}	1.83×10^{-5}	1.18×10^{-3}
	5.06	5.03	4.12
32	7.37×10^{-10}	5.66×10^{-7}	6.76×10^{-5}

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} - u^3 = 0, \quad x \in (0,1), t > 0, \tag{55}$$

with boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = \sin\left(\frac{7\pi x + \pi}{4}\right).$$

It is impossible to find analytic solution for equation (55). Therefore, the pointwise errors and maximum pointwise errors are computed using (52) and (53). The RMS errors and rate of convergence for this example are tabulated in Table 5 for various values of ε and N at $t = 1$ with $\Delta t = \frac{0.64}{p^2}$. The obtained results show the efficiency and accuracy of the method, also the magnitude of RMS errors increases by decreasing perturbation parameter.

Example 5. We consider a problem with singular coefficient as

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}u\frac{\partial u}{\partial x} + u^2 = f(x,t), \quad (x,t) \in (0,1)\times(0,1) \tag{56}$$

with boundary and initial conditions

$u(0,t) = u(1,t) = 0, \quad u(x,0) = 0,$

where $f(x, t)$ is taken so that the exact solution is

$u(x,t) = tx(1-x).$

The numerical results for this problem are shown in Table 6 for various values of N and ε at $t = 1$ with $\Delta t = \frac{0.64}{p^2}$. These results demonstrate the accuracy of method and the errors reduce with increasing N. Also, with decreasing perturbation parameter the magnitude of errors increases slightly, and this is one of the difficulties of singularly perturbed problems.

6. Conclusions

In this article, a numerical method was employed successfully for solving singularly perturbed one-dimensional non-linear parabolic problems. This approach is based on the implicit Euler method for temporal discretization and the Sinc collocation method in the spatial direction. The convergence analysis of the proposed method is presented and an exponential convergence is achieved as well. Results from numerical experiments indicate the efficiency and accuracy of proposed method. Also, from the Figs. 1–3 of convergence curves, we get some useful information about the convergence such that an exponential order is observed.

Acknowledgment

The authors are very grateful to the anonymous referees for their careful reading of this paper, and for providing helpful suggestions to improve this work.

Table 6 The maximum pointwise errors and the rate of convergence for Example 5 with $p = 2N$ and $\Delta t = \frac{0.64}{p^2}$ at $t = 1$.

$\varepsilon\rightarrow$ $N\downarrow$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
4	6.53×10^{-4}	1.79×10^{-4}	2.03×10^{-4}	9.12×10^{-4}	1.65×10^{-3}
	2.07	3.70	2.15	4.18	2.39
8	1.51×10^{-4}	1.37×10^{-5}	6.13×10^{-5}	3.11×10^{-5}	3.13×10^{-4}
	3.41	1.99	2.66	1.87	2.00
16	1.42×10^{-5}	3.44×10^{-6}	9.67×10^{-6}	8.48×10^{-6}	7.81×10^{-5}
	4.93	2.52	3.66	2.66	3.19
32	4.65×10^{-7}	5.96×10^{-7}	7.62×10^{-7}	1.34×10^{-6}	8.51×10^{-6}

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