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# On the time polynomial decay in elastic solids with voids <sup>☆</sup>

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## Abstract

In this paper we investigate the temporal decay behavior of the solutions of the one-dimensional problem in various theories of continua with voids. It has been proved that the coupling of the elastic structure with porous microstructure is weak in the sense that in many situations the temporal decay of solutions is slow. We have considered some theories of porous continua when the deformation-rate tensor or time-rate or porosity function or thermal effects is present. We have proved that the decay cannot be controlled by a negative exponential. The natural question now is whether there exist or not a polynomial rate of decay of the solution in some appropriate norms. In this paper we consider some cases where the decay is slow and we obtain polynomial decay estimates. In concrete we consider the case when only the viscoelastic effect is present, the case when the motion of voids is assumed to be quasi-static and the porous viscosity is present and we finish with the case of the porous-elasticity when thermal effect is coupled.

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## 1. Introduction

Elasticity problems have attracted the attention of researchers from different fields interested in the temporal decay behavior of the solutions. This interest has given many results that can be found in the literature. In the one-dimensional case, for instance, it is known that the interaction between the temperature field and elastic bodies lead to the exponential decay of the solution, see for example [8,16,17].

If elastic solids with voids are considered, as in this paper, one should look into the theory of porous elastic materials. Here we deal with the theory established by Cowin and Nunziato [3,4,13]. In their setting, the bulk density is the product of two scalar fields, the matrix material density and the volume fraction field. It is deeply discussed in the book of Ieşan [7]. It is worth recalling that porous materials have applications in many fields of engineering such as petroleum industry, material science, etc.

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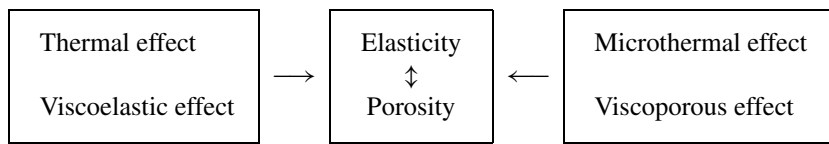
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The analysis of the temporal decay in one-dimensional porous-elastic materials was first studied by Quintanilla [15]. The author showed that the dissipation given by the porous viscosity was not powerful enough to obtain exponential stability to the solutions. For the sake of completeness we recall that the solutions generated by a semi-group  $U(t)$  are said to be *exponentially stable* if there exist two constants (independent of the initial conditions)  $C > 0$  and  $\varpi > 0$  such that  $\|U(t)\| \leq C \exp(-\varpi t)\|U(0)\|$ . We will say that the decay of the solutions is exponential if they are exponentially stable and, if they are not, we will say that the decay of the solutions is slow. Perhaps it is worth recalling the main difference between these two concepts in a thermomechanical context. If the decay is exponential, then after a short period of time, the thermomechanical deformations are very small and can be neglected. However, if the decay is slow, then the solutions weaken in a way that thermomechanical deformations could be appreciated in the system after some time. Therefore, the nature of the solutions highly determines the temporal behavior of the system and, from a thermomechanical point of view, it is relevant to be able to classify them.

In a series of recent papers [1,2,10–12] the authors have clarified the kind of decay when we combine temperature, elastic viscosity, porous viscosity and microtemperature. We recall the main conclusions with the help of a scheme:



If we take simultaneously one effect from the right square and other one from the left square, then we get exponential stability. However, if we consider two simultaneous effects from one square only, then we get slow decay.

From the above comments, it seems natural to think that the porous-elastic coupling is not very strong. In this work we want to deepen that matter and clear it up. In order to do so, we will consider some dissipation mechanisms where the temporal decay is slow and we obtain polynomial estimates for the decay.

To be concrete, we will consider the three problems: the first is the one when only the elastic viscosity of rate type is present, but the porous viscosity is absent; the second is the case when the motion of voids is assumed to be quasi-static and porous viscosity is present, but the viscoelastic effect is absent and the third is when the only dissipation effect is the thermal effect. It has been proved that the solutions decay slowly in all three cases and here we prove that this decay can be controlled by a polynomial.

Recently, Z. Liu and B. Rao [9] and A. Batkai et al. [14] found sufficient conditions to get a polynomial decay of semigroup operators. These conditions depend essentially on the regularity of the initial data and also on some estimates of the resolvent operator. One interesting point about these results is that in the two references above there exists a lack of optimality on the rate of decay. That is to say, the rate of decay is like  $1/t^{1-\epsilon}$  where such  $\epsilon$  appears for *technical* reasons.

This paper is structured as follows. In Section 2 we recall the general three-dimensional theory and we state the equations for the one-dimensional case. In Section 3 we study the theory of viscoelastic porous bodies in which the mechanical dissipation is produced by the deformation-rate tensor, and we show that the decay can be controlled by means of the inverse of a polynomial of first degree. In Section 4 we consider the case when the motion of voids is assumed to be quasi-static and the porous viscosity is present, but the viscoelastic effect is absent. A similar decay estimate is also obtained in this case. In Section 5, we prove that under suitable conditions on the constitutive material constants (see condition (5.2)) the decay is also polynomial of first degree. We use the energy method and some technical ideas to show the polynomial stability. Our decay result is optimal in the sense that no additional parameter appears in our decay estimate, that is we remove the parameter  $\epsilon$  given in [9,14]. Moreover using a result on [14] we are able to improve the polynomial rate of decay by taking more regular initial data.

## 2. Preliminaries

The theory of elastic solids with voids was introduced by Nunziato and Cowin [13]. Ieşan [5–7] added temperature to this theory. Let us make a short presentation of the general three-dimensional theory. The evolution equations are:

$$\begin{cases} \rho \ddot{u}_i = t_{ji,j}, \\ J \ddot{\phi} = h_{i,i} + g, \\ \rho T_0 \dot{\eta} = q_{i,i}, \end{cases}$$

where  $t_{ij}$  is the stress tensor,  $h_i$  is the equilibrated stress vector,  $g$  is the equilibrated body force,  $q_i$  is the heat flux vector,  $\eta$  is the entropy and  $T_0$  is the absolute temperature in the reference configuration. The variables  $u_i$  and  $\phi$  are, respectively, the displacement of the solid elastic material and the volume fraction. We assume that  $\rho$  and  $J$  are positive constants whose physical meaning is well known.

To state the field equations, we need first the constitutive equations. In the general case of solids with viscoelasticity, porous-viscosity, temperature we assume the following (see [6,7]):

$$\begin{cases} t_{ij} = \lambda e_{rr} \delta_{ij} + 2\mu e_{ij} + b\phi \delta_{ij} + \lambda^* \dot{e}_{rr} \delta_{ij} + 2\mu^* \dot{e}_{ij} - \beta \theta \delta_{ij}, \\ h_i = \delta \phi_{,i}, \\ g = -b e_{rr} + m\theta - \xi \phi - \tau \dot{\phi}, \\ \rho \eta = \beta e_{rr} + c\theta + m\phi, \\ q_i = k \theta_{,i}, \end{cases}$$

where  $2e_{ij} = u_{i,j} + u_{j,i}$ . Here  $\lambda, \mu, b, \lambda^*, \mu^*, \beta, \delta, d, \xi, \tau, c$  and  $k$  are the constitutive coefficients, and  $\theta$  is the temperature. From [3], the constitutive coefficients for isotropic bodies satisfy the following inequalities:

$$\mu > 0, \quad \xi > 0, \quad \delta > 0, \quad 2\mu + 3\lambda > 0, \quad (2\mu + 3\lambda)\xi > 3b^2. \tag{2.1}$$

The other coefficients satisfy the Clausius–Duhem conditions [6].

As we are considering here the one-dimensional theory, the evolution equations become easier and they are given by

$$\begin{cases} \rho \ddot{u} = t_x, \\ J \dot{\phi} = h_x + g, \\ \rho \dot{\eta} = q_x^*, \end{cases}$$

where  $q^*$  stands for  $T_0^{-1}q$ .

We use the constitutive equations:

$$\begin{cases} t = \mu u_x + b\phi + \gamma \dot{u}_x - \beta \theta, \\ h = \delta \phi_x, \\ g = -b u_x - \xi \phi + m\theta - \tau \dot{\phi}, \\ \rho \eta = \beta u_x + c\theta + m\phi, \\ q^* = k^* \theta_x, \end{cases}$$

where, abusing a little bit the notation, we write  $\mu$  instead of  $\lambda + 2\mu$  and  $\gamma$  instead of  $\lambda^* + 2\mu^*$ . We also use  $k^*$  to denote  $T_0^{-1}k$ , respectively, but in the sequel, we will omit the star.

Thus, the constitutive coefficients, in the one-dimensional case and with the new notation, satisfy the following inequalities:

$$\xi > 0, \quad \delta > 0, \quad \mu \xi > b^2. \tag{2.2}$$

It is assumed that the internal mechanical energy density is a positive definite form.

As coupling is considered,  $b$  must be different from 0, but its sign does not matter in the analysis.

When thermal effects are considered, we assume that the thermal capacity  $c$  and the thermal conductivity  $k$  are strictly positive.

Note that  $\gamma$  and  $\tau$  are nonnegative. If  $\gamma > 0$  viscoelastic dissipation is assumed in the system, and if  $\tau > 0$  porous dissipation is present.

It is known that in the one-dimensional linear theory, the equations that describe porosity and microstretch coincide. Therefore, we think it is appropriate to use the equations proposed by Ieşan [6] to describe this theory.

In this paper we analyze several problems. All of them are particular cases of the above system. However, it is worth noting that we do not consider in any place the complete system. In fact, our aim is to know if the decay can be controlled by a polynomial when only one of the damping effects is present.

To the field equations we must adjoin boundary and initial conditions. Thus, we assume that the solutions satisfy the boundary conditions

$$u(0, t) = u(\pi, t) = \phi_x(0, t) = \phi_x(\pi, t) = \theta_x(0, t) = \theta_x(\pi, t) = 0, \tag{2.3}$$

and the initial conditions

$$u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x), \quad \phi(x, 0) = \phi_0(x), \quad \dot{\phi}(x, 0) = \varphi_0(x), \quad \theta(x, 0) = \theta_0(x). \tag{2.4}$$

There are solutions (uniform in the variable  $x$ ) that do not decay. To avoid these cases, we will also assume that

$$\int_0^\pi \phi_0(x) dx = \int_0^\pi \varphi_0(x) dx = \int_0^\pi \theta_0(x) dx = 0.$$

It is worth noting that when we consider the quasi-static problem for the void motion we only need the initial condition for  $\phi$ , but not for  $\dot{\phi}$ . We finish this section, introducing a result due to Prüss [14], for which we can improve the polynomial rate of decay, by taking more regular initial data.

**Theorem 2.1.** *Assume that  $-A \in \mathcal{G}(X, M, 0)$ ,  $A$  is invertible and  $\alpha$  a positive constant. Then the following statements are equivalent:*

$$\begin{aligned} \|T(t)A^{-\alpha}\| &\leq Ct^{-\beta}, \quad t > 0, \\ \|T(t)A^{-\alpha\gamma}\| &\leq C'(\gamma)t^{-\gamma\beta}, \quad t > 0, \gamma > 0. \end{aligned}$$

### 3. Viscoelasticity and porosity

In this section we study the problem determined by the system

$$\begin{cases} \rho \ddot{u} = \mu u_{xx} + b\phi_x + \gamma \dot{u}_{xx}, \\ J \ddot{\phi} = \delta \phi_{xx} - bu_x - \xi \phi, \end{cases} \tag{3.1}$$

with boundary and initial conditions given by (2.3) and (2.4), respectively. We will prove that the time decay of the solutions can be controlled by a polynomial. We note that the solutions of this problem can be generated by means of a semigroup of contractions. In fact, this semigroup is defined in

$$\mathcal{H} = \left\{ (u, v, \phi, \varphi) \in H_0^1 \times L^2 \times H^1 \times L^2, \int_0^\pi \phi(x) dx = \int_0^\pi \varphi(x) dx = 0 \right\}$$

by the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \rho^{-1}\mu D^2 & \rho^{-1}\gamma D^2 & \rho^{-1}bD & 0 \\ 0 & 0 & 0 & I \\ -J^{-1}bD & 0 & J^{-1}\delta D^2 - \xi & 0 \end{pmatrix},$$

and  $I$  is the identity operator.

Now, we recall an inner product in  $\mathcal{H}$ . If  $U^* = (u^*, v^*, \phi^*, \varphi^*)$ , then

$$\langle U, U^* \rangle = \int_0^\pi (\rho v \bar{v}^* + J \varphi \bar{\varphi}^* + \mu u_x \bar{u}_x^* + \delta \phi_x \bar{\phi}_x^* + \xi \phi \bar{\phi}^* + b(u_x \bar{\phi}^* + \bar{u}_x^* \phi)) dx.$$

Hereafter a superposed bar denotes the conjugate complex number. It is worth recalling that this product is equivalent to the usual product in the Hilbert space  $\mathcal{H}$ .

The domain of  $\mathcal{A}$  is

$$\mathcal{D}(\mathcal{A}) = \{U \in \mathcal{H}: u \in H^2, v \in H_0^1 \cap H^2, \phi \in H^2, D\phi \in H_0^1, \varphi \in H^1\}.$$

We note that  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . So, we will show that the solution of

$$U_t = \mathcal{A}U, \quad U(0) = U_0 = (u_0, v_0, \phi_0, \varphi_0) \in \mathcal{D}(\mathcal{A}) \tag{3.2}$$

decays polynomially to zero.

Our starting point is to define the first order energy as

$$E_1(t, u, \phi) = \frac{1}{2} \int_0^\pi (\rho |\dot{u}|^2 + J |\dot{\phi}|^2 + \mu u_x^2 + \delta \phi_x^2 + \xi \phi^2 + 2bu_x \phi) dx. \quad (3.3)$$

Then we introduce the second order energies,

$$E_2(t) = E_1(t, \dot{u}, \dot{\phi}), \quad (3.4)$$

and

$$E_3(t) = E_1(t, u_x, \phi_x). \quad (3.5)$$

We know that

$$\frac{dE_1}{dt} = -\gamma \int_0^\pi |\dot{u}_x|^2 dx, \quad (3.6)$$

$$\frac{dE_2}{dt} = -\gamma \int_0^\pi |\ddot{u}_x|^2 dx, \quad (3.7)$$

$$\frac{dE_3}{dt} = -\gamma \int_0^\pi |\dot{u}_{xx}|^2 dx. \quad (3.8)$$

Let us introduce the following functionals

$$\Phi(t) = \int_0^\pi \left( \rho u_x \dot{u}_x + \frac{\gamma}{2} u_{xx}^2 \right) dx,$$

$$R(t) = - \int_0^\pi \left( J \dot{\phi} u_x - \frac{\gamma \xi}{2b} u_x^2 \right) dx.$$

**Lemma 3.1.** *Let us suppose that  $(u_0, v_0, \phi_0, \varphi_0) \in D(\mathcal{A})$ , then the corresponding solution of (3.2) satisfies,*

$$\frac{d}{dt} \left( \Phi + \frac{b}{\delta} R \right) \leq -\mu \int_0^\pi u_{xx}^2 dx + \frac{b^2 - \mu \xi}{2\delta} \int_0^\pi u_x^2 dx + \epsilon \int_0^\pi |\dot{\phi}|^2 dx + c_\epsilon \int_0^\pi (|\ddot{u}|^2 + |\dot{u}_x|^2) dx.$$

**Proof.** Let us multiply Eq. (3.1)<sub>1</sub> by  $-u_{xx}$  and integrate from 0 to  $\pi$

$$\begin{aligned} \frac{d}{dt} \int_0^\pi \rho \dot{u}_x u_x dx &= \rho \int_0^\pi |\dot{u}_x|^2 dx - \rho \int_0^\pi \ddot{u} u_{xx} dx \\ &= \rho \int_0^\pi |\dot{u}_x|^2 dx - \int_0^\pi (\mu u_{xx} + b \phi_x + \gamma \dot{u}_{xx}) u_{xx} dx \\ &= \rho \int_0^\pi |\dot{u}_x|^2 dx - \mu \int_0^\pi u_{xx}^2 dx - b \int_0^\pi \phi_x u_{xx} dx - \frac{\gamma}{2} \frac{d}{dt} \int_0^\pi u_{xx}^2 dx. \end{aligned} \quad (3.9)$$

Recalling the definition of  $\Phi(t)$  we get

$$\frac{d}{dt} \Phi(t) = \rho \int_0^\pi |\dot{u}_x|^2 dx - \mu \int_0^\pi u_{xx}^2 dx - b \int_0^\pi \phi_x u_{xx} dx. \quad (3.10)$$

Multiply Eq. (3.1)<sub>2</sub> by  $u_x$

$$\begin{aligned} \frac{d}{dt} \int_0^\pi J \dot{\phi} u_x \, dx &= \int_0^\pi (\delta \phi_{xx} - bu_x - \xi \phi) u_x \, dx + J \int_0^\pi \dot{\phi} \dot{u}_x \, dx \\ &= -\delta \int_0^\pi \phi_x u_{xx} \, dx - b \int_0^\pi u_x^2 \, dx - \underbrace{\xi \int_0^\pi \phi u_x \, dx}_{:=I(t)} + J \int_0^\pi \dot{\phi} \dot{u}_x \, dx. \end{aligned} \tag{3.11}$$

But, using Eq. (3.1)<sub>1</sub> we get

$$\begin{aligned} I(t) &= \xi \int_0^\pi \phi_x u \, dx = \frac{\xi}{b} \int_0^\pi b \phi_x u \, dx = \frac{\xi}{b} \int_0^\pi (\rho \ddot{u} - \mu u_{xx} + \gamma \dot{u}_{xx}) u \, dx \\ &= \frac{\xi \rho}{b} \int_0^\pi \ddot{u} u \, dx + \frac{\mu \xi}{b} \int_0^\pi u_x^2 \, dx + \frac{\gamma \xi}{2b} \frac{d}{dt} \int_0^\pi u_x^2 \, dx. \end{aligned} \tag{3.12}$$

Recalling the definition of  $R$  and substitution of (3.12) into (3.11) we get

$$-\frac{dR}{dt} = -\delta \int_0^\pi \phi_x u_{xx} \, dx - \frac{b^2 - \mu \xi}{b} \int_0^\pi u_x^2 \, dx + \frac{\xi \rho}{b} \int_0^\pi \ddot{u} u \, dx + J \int_0^\pi \dot{\phi} \dot{u}_x \, dx. \tag{3.13}$$

From (3.10) and (3.13) we obtain that

$$\frac{d}{dt} \left( \Phi + \frac{b}{\delta} R \right) = \rho \int_0^\pi |\dot{u}_x|^2 \, dx - \mu \int_0^\pi u_{xx}^2 \, dx + \frac{b^2 - \mu \xi}{\delta} \int_0^\pi u_x^2 \, dx - \frac{\xi \rho}{\delta} \int_0^\pi \ddot{u} u \, dx - \frac{Jb}{\delta} \int_0^\pi \dot{\phi} \dot{u}_x \, dx, \tag{3.14}$$

from where our conclusion follows.  $\square$

Now let us introduce the following functionals

$$\begin{aligned} S_1(t) &= -\text{sign}(b) \int_0^\pi \rho \dot{u} \phi_x \, dx, \\ S_2(t) &= \int_0^\pi J \dot{\phi} \phi \, dx. \end{aligned}$$

Under the above conditions we have

**Lemma 3.2.** *With the same hypothesis as in Lemma 3.1 we have*

$$\frac{d}{dt} \left\{ S_1(t) - \frac{|b|}{4(\delta + \xi)} S_2(t) \right\} \leq -\frac{|b|}{4} \int_0^\pi \phi_x^2 \, dx - \frac{|bJ|}{8(\delta + \xi)} \int_0^\pi |\dot{\phi}|^2 \, dx + C \int_0^\pi (u_{xx}^2 + |\dot{u}_{xx}|^2) \, dx.$$

**Proof.** Using Eq. (3.1)<sub>1</sub> we get

$$\frac{d}{dt} \int_0^\pi \rho \dot{u} \phi_x \, dx = \int_0^\pi \rho \ddot{u} \phi_x \, dx - \int_0^\pi \dot{u}_x \dot{\phi} \, dx$$

$$\begin{aligned}
 &= \int_0^\pi (\mu u_{xx} + b\phi_x + \gamma \dot{u}_{xx})\phi_x \, dx - \rho \int_0^\pi \dot{u}_x \dot{\phi} \, dx \\
 &= b \int_0^\pi \phi_x^2 \, dx + \mu \int_0^\pi u_{xx}\phi_x \, dx + \gamma \int_0^\pi \dot{u}_{xx}\phi_x \, dx - \rho \int_0^\pi \dot{u}_x \dot{\phi} \, dx.
 \end{aligned} \tag{3.15}$$

Recalling the definition of  $S_1$  we get

$$\frac{dS_1}{dt} \leq -\frac{|b|}{2} \int_0^\pi \phi_x^2 \, dx + \text{sign}(b)\rho \int_0^\pi \dot{u}_x \dot{\phi} \, dx + C \int_0^\pi (u_{xx}^2 + |\dot{u}_{xx}|^2) \, dx, \tag{3.16}$$

where  $C$  is a positive constant.

Differentiating  $S_2$  with respect to the time and using Eq. (3.1)<sub>1</sub> we get

$$\begin{aligned}
 \frac{dS_2}{dt} &= J \int_0^\pi |\dot{\phi}|^2 \, dx + J \int_0^\pi \ddot{\phi}\phi \, dx \\
 &= J \int_0^\pi |\dot{\phi}|^2 \, dx + \int_0^\pi (\delta\phi_{xx} - bu_x - \xi\phi)\phi \, dx \\
 &\geq J \int_0^\pi |\dot{\phi}|^2 \, dx - (\delta + \xi) \int_0^\pi \phi_x^2 \, dx - b \int_0^\pi u_x \phi \, dx,
 \end{aligned} \tag{3.17}$$

where we have used the Poincaré inequality. From (3.16) and (3.17) our conclusion follows.  $\square$

Now, we are in conditions to show the main result of this section.

**Theorem 3.3.** *Assume that the hypotheses of in Lemma 3.1 hold. Then there exists a positive constant  $C$  such that*

$$E_1(t) \leq \frac{C\|(u_0, v_0, \phi_0, \varphi_0)\|_{D(\mathcal{A})}^2}{t}.$$

Moreover, if  $(u_0, v_0, \phi_0, \varphi_0) \in D(\mathcal{A}^\alpha)$ , then we have that

$$E_1(t) \leq \frac{C_\alpha\|(u_0, v_0, \phi_0, \varphi_0)\|_{D(\mathcal{A}^\alpha)}^2}{t^\alpha}.$$

**Proof.** Let us introduce the functional  $\mathcal{L}(t)$  as

$$\mathcal{L}(t) = S_1(t) - \frac{|b|}{5(\delta + \xi)} S_2(t) + N \left( \Phi(t) + \frac{b}{\delta} R(t) \right) + N_1 E_1(t) + N_2 E_2(t) + N_3 E_3(t), \tag{3.18}$$

where  $N$  and  $N_i, i = 1, 2, 3$ , are large enough such that  $\mathcal{L}(t) \geq 0$ , then from Lemmas 3.1 and 3.2 we have

$$\frac{d\mathcal{L}}{dt} \leq -\gamma_b E_1. \tag{3.19}$$

Integration over  $[0, t]$  implies

$$\mathcal{L}(t) + \gamma_b \int_0^t E_1(s) \, ds \leq \mathcal{L}(0). \tag{3.20}$$

Then

$$\frac{d}{dt}(tE_1(t)) = E_1(t) + t \frac{dE_1}{dt} \leq E_1(t). \tag{3.21}$$

A quadrature implies that

$$tE_1(t) \leq \int_0^t E_1(s) ds \leq \gamma_b^{-1} \mathcal{L}(0) \tag{3.22}$$

which implies the polynomial decay. To improve the polynomial decay we use Prüss Theorem 2.1  $\square$

#### 4. A quasi-static theory

Here we consider the case where viscoelasticity is absent, but porous dissipation is present and the void motion is quasi-static. It has been proved [12] that the decay in this situation is slow. In this section we obtain that the solutions are polynomially stable.

The system we want to study is

$$\begin{cases} \rho \ddot{u} = \mu u_{xx} + b\phi_x, \\ \tau \dot{\phi} = \delta \phi_{xx} - bu_x - \xi \phi. \end{cases} \tag{4.1}$$

From the mathematical point of view this system can be considered as the limit system (as  $J \rightarrow 0$ ) of the usual system of the porous-elasticity when only porous dissipation is present.

The boundary and initial conditions are given by (2.3) and (2.4), respectively. We suppose that the coefficients satisfies condition (2.2). Again, we note that the solutions of this problem can be generated by means of a semigroup of contractions in the Hilbert space

$$\mathcal{H} = \left\{ (u, v, \phi) \in H_0^1 \times L^2 \times H^1, \int_0^\pi \phi(x) dx = 0 \right\}.$$

Denote by  $U = (u, v, \phi)$ , and define the operator  $\mathcal{A}$  as

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ \rho^{-1} \mu D^2 & 0 & \rho^{-1} bD \\ -\tau^{-1} bD & 0 & \tau^{-1} \delta D^2 - \xi \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \{ U \in \mathcal{H}: u \in H^2, v \in H_0^1, \phi \in H^2, D\phi \in H_0^1 \}.$$

By  $I$  we denote the identity operator. Note that  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . Then the initial-boundary value problem (4.1) is equivalent to

$$U_t = \mathcal{A}U, \quad U(0) = U_0 = (u_0, v_0, \phi_0) \in \mathcal{D}(\mathcal{A}). \tag{4.2}$$

Now, we define an inner product in  $\mathcal{H}$ . If  $U^* = (u^*, v^*, \phi^*)$ , then

$$\langle U, U^* \rangle = \int_0^\pi (\rho v \bar{v}^* + \mu u_x \bar{u}_x^* + \delta \phi_x \bar{\phi}_x^* + \xi \phi \bar{\phi}^* + b(u_x \bar{\phi}^* + \bar{u}_x^* \phi)) dx.$$

To obtain the polynomial decay we define the following functions

$$E_1(t, u, \phi) = \frac{1}{2} \int_0^\pi (\rho |\dot{u}|^2 + \mu u_x^2 + \delta \phi_x^2 + \xi \phi^2 + 2bu_x \phi) dx, \tag{4.3}$$

$$E_2(t) = E_1(t, \dot{u}, \dot{\phi}), \tag{4.4}$$

and

$$E_3(t) = E_1(t, u_x, \phi_x). \tag{4.5}$$

It is not difficult to show that



$$\frac{dE_1}{dt} = -\tau \int_0^\pi |\dot{\phi}|^2 dx, \quad (4.6)$$

$$\frac{dE_2}{dt} = -\tau \int_0^\pi |\ddot{\phi}|^2 dx, \quad (4.7)$$

$$\frac{dE_3}{dt} = -\tau \int_0^\pi |\dot{\phi}_x|^2 dx. \quad (4.8)$$

Let us introduce the functional

$$S(t) = \int_0^\pi \frac{\tau}{2} \phi_x^2 dx + \frac{b\rho}{\mu} \int_0^\pi \dot{u} \phi_x dx. \quad (4.9)$$

**Lemma 4.1.** *If the initial data  $U_0 = (u_0, v_0, \phi_0) \in D(\mathcal{A})$  then the functional  $S$  satisfies*

$$\frac{dS}{dt} = -\delta \int_0^\pi \phi_{xx}^2 dx - \left( \xi - \frac{b^2}{\mu} \right) \int_0^\pi \phi_x^2 dx + \frac{b\rho}{\mu} \int_0^\pi \dot{u} \phi_x dx. \quad (4.10)$$

**Proof.** Let us multiply the second equation of (4.1) by  $-\phi_{xx}$ , to get

$$\frac{\tau}{2} \frac{d}{dt} \int_0^\pi \phi_x^2 dx = -\delta \int_0^\pi \phi_{xx}^2 dx + b \int_0^\pi u_x \phi_{xx} dx - \xi \int_0^\pi \phi_x^2 dx. \quad (4.11)$$

Using the first equation of (4.1) we have

$$\begin{aligned} b \int_0^\pi u_x \phi_{xx} dx &= -\frac{b}{\mu} \int_0^\pi \phi_x (\rho \ddot{u} - b \phi_x) dx \\ &= -\frac{b\rho}{\mu} \frac{d}{dt} \int_0^\pi \dot{u} \phi_x dx + \frac{b\rho}{\mu} \int_0^\pi \dot{u} \dot{\phi}_x dx + \frac{b^2}{\mu} \int_0^\pi \phi_x^2 dx. \end{aligned} \quad (4.12)$$

Finally, by substituting (4.12) into (4.11) our conclusion follows.  $\square$

Let us introduce the functional

$$R(t) = \text{sign}(b)\tau \int_0^\pi \phi u_x dx - \frac{|b|}{6\mu} \int_0^\pi \rho \dot{u} u dx. \quad (4.13)$$

**Lemma 4.2.** *With the same hypotheses as in Lemma 4.1, we have*

$$\frac{dR}{dt} \leq -\frac{|b|}{4} \int_0^\pi u_x^2 dx - \frac{\rho|b|}{12\mu} \int_0^\pi |\dot{u}|^2 dx + C_0^* \int_0^\pi (\phi_x^2 + \phi_{xx}^2) dx.$$

**Proof.** Let us multiply the second equation of (4.1) by  $u_x$ . We see that

$$\text{sign}(b)\tau \frac{d}{dt} \int_0^\pi \phi u_x dx = \text{sign}(b) \left( \tau \int_0^\pi \dot{\phi} u_x dx + \tau \int_0^\pi \phi \dot{u}_x dx \right)$$

$$\begin{aligned}
 &= \text{sign}(b) \left( \int_0^\pi (\delta\phi_{xx} - bu_x - \xi\phi)u_x dx - \tau \int_0^\pi \phi_x \dot{u} dx \right) \\
 &= \text{sign}(b) \left( \delta \int_0^\pi \phi_{xx}u_x dx - b \int_0^\pi u_x^2 dx - \xi \int_0^\pi u_x\phi dx - \tau \int_0^\pi \phi_x \dot{u} dx \right) \\
 &\leq -\frac{|b|}{2} \int_0^\pi u_x^2 dx - \tau \int_0^\pi \phi_x \dot{u} dx + \frac{\delta^2 + \xi^2}{|b|} \int_0^\pi (\phi_x^2 + \phi_{xx}^2) dx.
 \end{aligned} \tag{4.14}$$

Multiplying the first equation of (4.1) by  $u$  we get

$$\begin{aligned}
 \frac{d}{dt} \int_0^\pi \rho \dot{u}u dx &= \int_0^\pi \rho \ddot{u}u dx + \rho \int_0^\pi |\dot{u}|^2 dx = \int_0^\pi (\mu u_{xx} + b\phi_x)u dx + \rho \int_0^\pi |\dot{u}|^2 dx \\
 &= -\mu \int_0^\pi u_x^2 dx + b \int_0^\pi \phi_x u dx + \rho \int_0^\pi |\dot{u}|^2 dx.
 \end{aligned} \tag{4.15}$$

So we have

$$-\frac{d}{dt} \int_0^\pi \rho \dot{u}u dx \leq -\rho \int_0^\pi |\dot{u}|^2 dx + \frac{3\mu}{2} \int_0^\pi u_x^2 dx + C_1 \int_0^\pi \phi_x^2 dx, \tag{4.16}$$

where  $C_1$  is a positive constant that can be calculated. Finally, multiplying relation (4.16) by  $|b|/6\mu$  and summing up with relation (4.14) our conclusion follows.  $\square$

Now, we are in conditions to show the main result of this section.

**Theorem 4.3.** *Assume that the hypotheses of in Lemma 4.1 hold. Then there exists a positive constant  $C$  such that*

$$E_1(t) \leq \frac{C \|(u_0, v_0, \phi_0)\|_{D(\mathcal{A})}^2}{t}.$$

Moreover, if  $(u_0, v_0, \phi_0) \in D(\mathcal{A}^\alpha)$ , for  $\alpha > 0$ , then we have that

$$E_1(t) \leq \frac{C_\alpha \|(u_0, v_0, \phi_0)\|_{D(\mathcal{A}^\alpha)}^2}{t^\alpha}.$$

**Proof.** Let us introduce the functional  $\mathcal{L}(t)$  as

$$\mathcal{L}(t) = R(t) + NS(t) + N_1E_1(t) + N_2E_2(t) + N_3E_3(t), \tag{4.17}$$

where  $N$  is sufficiently great and  $N_i, i = 1, 2, 3$ , are also sufficiently greater to guarantee that  $\mathcal{L}(t)$  is positive. Using Lemmas 4.1 and 4.2 we conclude that

$$\frac{d\mathcal{L}}{dt} \leq -\gamma_b E_1. \tag{4.18}$$

We can finish the proof of the polynomial decay in a similar way as in Section 3.  $\square$

### 5. Porous-elasticity and heat conduction

In this section, we consider the problem determined by the system of the porous-elasticity when the only dissipation mechanism is the heat conduction. The system of equations is

$$\begin{cases} \rho \ddot{u} = \mu u_{xx} + b\phi_x - \beta\theta_x, \\ J \ddot{\phi} = \delta\phi_{xx} - bu_x - \xi\phi + m\theta, \\ c\dot{\theta} = k\theta_{xx} - \beta\dot{u}_x - m\dot{\phi}. \end{cases} \tag{5.1}$$

The solutions of this system with the initial and boundary conditions determined by (2.3), (2.4) are generated by a semigroup of contractions which is defined in the Hilbert space

$$\mathcal{H} = \left\{ (u, v, \phi, \varphi, \theta) \in H_0^1 \times L^2 \times H^1 \times L^2 \times L^2, \int_0^\pi \phi \, dx = \int_0^\pi \varphi \, dx = \int_0^\pi \theta \, dx = 0 \right\},$$

by the operator

$$A = \begin{pmatrix} 0 & Id & 0 & 0 & 0 \\ \rho^{-1}\mu D^2 & 0 & \rho^{-1}bD & 0 & -\rho^{-1}\beta D \\ 0 & 0 & 0 & Id & 0 \\ -J^{-1}bD & 0 & J^{-1}(\delta D^2 - \xi) & 0 & J^{-1}m \\ 0 & -c^{-1}\beta D & 0 & -c^{-1}m & c^{-1}kD^2 \end{pmatrix},$$

with domain

$$D(A) = \{U \in \mathcal{H}; u \in H^2(0, \pi), v \in H_0^1(0, \pi), D\phi, D\varphi, D\theta \in H_0^1(0, \pi)\}.$$

In fact, we consider the scalar product

$$\begin{aligned} & ((u, v, \phi, \varphi, \theta), (u^*, v^*, \phi^*, \varphi^*, \theta^*))_{\mathcal{H}} \\ &= \int_0^\pi (\rho v \bar{v}^* + J \varphi \bar{\varphi}^* + c \theta \bar{\theta}^* + \mu u_x \bar{u}_x^* + \alpha \phi_x \bar{\phi}_x^* + \xi \phi \bar{\phi}^* + b(u_x \bar{\phi}^* + \bar{u}_x^* \phi)) \, dx. \end{aligned}$$

Therefore, system (5.1) is equivalent to solving the Cauchy problem

$$U_t = AU, \quad U(0) = U_0 = (u_0, v_0, \phi_0, \varphi_0, \theta_0) \in D(A). \tag{5.2}$$

In this section we prove the polynomial decay of solutions whenever

$$m(\beta b - m\mu) > 0. \tag{5.3}$$

For this system, we can define the following energies

$$E_1(t, u, \phi, \theta) = \frac{1}{2} \int_0^\pi (\rho |\dot{u}|^2 + \mu u_x^2 + J |\dot{\phi}|^2 + \delta \phi_x^2 + 2b\phi u_x + c\theta^2) \, dx, \tag{5.4}$$

$$E_2(t, u, \phi, \theta) = E_1(t, \dot{u}, \dot{\phi}, \dot{\theta}), \tag{5.5}$$

$$E_3(t, u, \phi, \theta) = E_1(t, u_x, \phi_x, \theta_x). \tag{5.6}$$

After several integrations by parts, we can see that

$$\frac{dE_1}{dt} = -k \int_0^\pi \theta_x^2 \, dx, \tag{5.7}$$

$$\frac{dE_2}{dt} = -k \int_0^\pi |\dot{\theta}_x|^2 \, dx \tag{5.8}$$

and

$$\frac{dE_3}{dt} = -k \int_0^\pi \theta_{xx}^2 \, dx. \tag{5.9}$$

Let us introduce the functional

$$R(t) = \int_0^\pi (c\theta + m\phi)\dot{u}_x \, dx + c \int_0^\pi \theta_x \dot{u} \, dx + \frac{m\mu}{b} \int_0^\pi u_x \dot{u}_x \, dx. \tag{5.10}$$

**Lemma 5.1.** *Let us suppose that the initial data  $(u_0, v_0, \phi_0, \varphi_0, \theta_0) \in D(\mathcal{A})$ . Then there exists positive constants  $\gamma_0$  and  $C_0$  such that the solution of problem (5.2) satisfies*

$$\frac{dR}{dt} \leq -\gamma_0 \int_0^\pi (|\dot{u}_x|^2 + |\ddot{u}|^2) dx + C_0 \int_0^\pi (\theta_x^2 + \theta_{xx}^2) dx.$$

**Proof.** We can rewrite the third equation in (5.1) as

$$c\dot{\theta} + m\dot{\phi} = k\theta_{xx} - \beta\dot{u}_x.$$

Multiplying by  $\dot{u}_x$  and integration from 0 to  $\pi$  we find that

$$\begin{aligned} \frac{d}{dt} \int_0^\pi (c\theta + m\phi)\dot{u}_x dx &= \int_0^\pi (c\dot{\theta} + m\dot{\phi})\dot{u}_x dx + \int_0^\pi (c\theta + m\phi)\ddot{u}_x dx \\ &= \int_0^\pi (k\theta_{xx} - \beta\dot{u}_x)\dot{u}_x dx + c \int_0^\pi \theta\ddot{u}_x dx + m \int_0^\pi \phi\ddot{u}_x dx \\ &= k \int_0^\pi \theta_{xx}\dot{u}_x dx - \beta \int_0^\pi |\dot{u}_x|^2 dx - c \frac{d}{dt} \int_0^\pi \theta_x \dot{u} dx + c \int_0^\pi \dot{\theta}_x \dot{u} dx - m \int_0^\pi \phi_x \ddot{u} dx. \end{aligned} \tag{5.11}$$

The last term on the right-hand side can be estimated in the following way

$$\begin{aligned} -m \int_0^\pi \phi_x \ddot{u} dx &= -\frac{m}{b} \int_0^\pi (\rho\ddot{u} - \mu u_{xx} + \beta\theta_x)\ddot{u} dx \\ &= -\frac{m\rho}{b} \int_0^\pi |\ddot{u}|^2 dx - \frac{m\mu}{b} \int_0^\pi u_x \ddot{u}_x dx - \frac{m\beta}{b} \int_0^\pi \theta_x \ddot{u} dx \\ &= -\frac{m\rho}{b} \int_0^\pi |\ddot{u}|^2 dx - \frac{m\mu}{b} \frac{d}{dt} \int_0^\pi u_x \dot{u}_x dx + \frac{m\mu}{b} \int_0^L |\dot{u}_x|^2 dx - \frac{m\beta}{b} \int_0^\pi \theta_x \ddot{u} dx. \end{aligned} \tag{5.12}$$

Recalling the definition of  $R$  we get

$$\frac{dR}{dt} = k \int_0^\pi \theta_{xx}\dot{u}_x dx - \underbrace{\left(\beta - \frac{m\mu}{b}\right)}_{:=I_1} \int_0^\pi |\dot{u}_x|^2 dx + c \int_0^\pi \dot{\theta}_x \dot{u} dx - \underbrace{\frac{m\rho}{b}}_{:=I_2} \int_0^\pi |\ddot{u}|^2 dx - \frac{m\beta}{b} \int_0^\pi \theta_x \ddot{u} dx. \tag{5.13}$$

By hypotheses we have that  $I_1$  and  $I_2$  have the same sign. Therefore we can assume that  $\beta b > m\mu$ . Otherwise, we take  $-R$  instead of  $R$ . From where our conclusion follows.  $\square$

Finally we are able to show the polynomial decay.

**Theorem 5.2.** *Assume that the hypotheses of in Lemma 5.1 hold. Then there exists a positive constant  $C$  such that*

$$E_1(t) \leq \frac{C \|(u_0, v_0, \phi_0, \varphi_0, \theta_0)\|_{D(\mathcal{A})}^2}{t}.$$

Moreover, if  $(u_0, v_0, \phi_0, \varphi_0, \theta_0) \in D(\mathcal{A}^\alpha)$ , then we have that

$$E_1(t) \leq \frac{C_\alpha \|(u_0, v_0, \phi_0, \varphi_0, \theta_0)\|_{D(\mathcal{A}^\alpha)}^2}{t^\alpha}.$$

**Proof.** Let us multiply the first equation of (5.1) by  $u$ , then we have

$$\begin{aligned} \frac{d}{dt} \int_0^\pi \rho u \dot{u} \, dx &= \rho \int_0^\pi |\dot{u}|^2 \, dx + \int_0^\pi \rho \ddot{u} u \, dx \\ &= \rho \int_0^\pi |\dot{u}|^2 \, dx - \int_0^\pi \mu u_x^2 \, dx + b \int_0^\pi \phi_x u \, dx - \beta \int_0^\pi u \theta_x \, dx, \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \frac{d}{dt} \int_0^\pi c \theta \dot{\phi} \, dx &= \int_0^\pi c \dot{\theta} \dot{\phi} \, dx + c \int_0^\pi \theta \ddot{\phi} \, dx \\ &= k \int_0^\pi \theta_{xx} \dot{\phi} \, dx - \beta \int_0^\pi \dot{u}_x \dot{\phi} \, dx - m \int_0^\pi |\dot{\phi}|^2 \, dx + \frac{c}{J} \int_0^\pi \theta (\delta \phi_{xx} - b u_x - \xi \phi + m \theta) \, dx. \end{aligned} \tag{5.15}$$

Denoting by

$$Q(t) = \text{sign}(m) \int_0^\pi c \theta \dot{\phi} \, dx, \tag{5.16}$$

we find that

$$\begin{aligned} \frac{dQ}{dt} &\leq -\frac{|m|}{2} \int_0^\pi |\dot{\phi}|^2 \, dx + C_1 \int_0^\pi (|\dot{u}_x|^2 + |\theta_{xx}|^2 + |\theta_x|^2) \, dx \\ &\quad - \frac{\text{sign}(m)c}{J} \left( \int_0^\pi \delta \theta_x \phi_x \, dx + \int_0^\pi b \theta u_x \, dx + \int_0^\pi \xi \phi \theta \, dx \right), \end{aligned} \tag{5.17}$$

where  $C_1$  is a positive constant that can be estimated.

Now, we can see that

$$\begin{aligned} \frac{d}{dt} \int_0^\pi J \phi \dot{\phi} \, dx &= J \int_0^\pi \phi \ddot{\phi} \, dx + J \int_0^\pi |\dot{\phi}|^2 \, dx \\ &= -\delta \int_0^\pi \phi_x^2 \, dx - b \int_0^\pi \phi u_x \, dx - \xi \int_0^\pi \phi^2 \, dx + m \int_0^\pi \theta \phi \, dx + J \int_0^\pi |\dot{\phi}|^2 \, dx. \end{aligned} \tag{5.18}$$

Let us introduce the functional

$$S(t) = \int_0^\pi (\rho u \dot{u} + J \phi \dot{\phi}) \, dx. \tag{5.19}$$

Using (5.14) and (5.18) we arrive to

$$\begin{aligned} \frac{dS}{dt} &= \rho \int_0^\pi |\dot{u}|^2 \, dx + J \int_0^\pi |\dot{\phi}|^2 \, dx - \mu \int_0^\pi u_x^2 \, dx - \delta \int_0^\pi \phi_x^2 \, dx - \xi \int_0^\pi \phi^2 \, dx - 2b \int_0^\pi \phi u_x \, dx \\ &\quad + \int_0^\pi \theta (m \phi + \beta u_x) \, dx \end{aligned}$$

$$\leq \rho \int_0^\pi |\dot{u}|^2 dx + J \int_0^\pi |\dot{\phi}|^2 dx - \gamma_1 \int_0^\pi (u_x^2 + \phi_x^2 + \phi^2) dx + C_2 \int_0^\pi \theta_x^2 dx. \tag{5.20}$$

Here  $\gamma_1$  and  $C_2$  are two positive constants which can be estimated. Now, if we define the function

$$\mathcal{L}(t) = S(t) + N(Q(t) + R(t)) + N_1 E_1(t) + N_3 E_3(t), \tag{5.21}$$

where  $N$  and  $N_i$ ,  $i = 1, 3$ , are large enough such that  $\mathcal{L}(t)$  is positive. Using Lemma 5.1 and the above inequalities we get that

$$\frac{d\mathcal{L}}{dt} \leq -\gamma_b E_1. \tag{5.22}$$

We can finish the proof of the polynomial decay in a similar way as in Section 3.  $\square$

### 6. Conclusions

In this paper we have analyzed the behavior of porous continua. We have considered three situations where the dissipation mechanisms are not so strong to guarantee the exponential stability of the solutions. These are:

- (i) Porous viscoelastic bodies when the dissipation is produced by the deformation-rate tensor.
- (ii) Porous viscoelastic bodies when the dissipation is produced by the time-rate of the porosity function and when the motion of the voids is quasi-static.
- (iii) Porous thermoelastic bodies when the dissipation is due to the thermal effect.

In the two first cases we have seen that the decay can be controlled by a polynomial. The same happens in the third case whenever the constitutive constants satisfy the condition (5.2).

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