Comments on the spectra of Pisot numbers

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Abstract

For $q \in (1, 2)$, Erdős, Joó and Komornik studied the spectra of $q$, defined as

$$Y^m(q) := \{ \epsilon_n q^n + \cdots + \epsilon_0 : \epsilon_i \in \{0, 1, \ldots, m\}, \ n \in \mathbb{N} \} = \{0 = y_0 < y_1 < y_2 < y_3 < \cdots\}.$$  

Feng and Wen showed that for $q$ a Pisot number, the gap sequence

$$y_1 - y_0, y_2 - y_1, \ldots, y_{k+1} - y_k, \ldots$$

is the iterative fixed point of a substitution. The second author used this substitution to determine the frequency of particular gap sizes in the spectra, and gave a detailed account when $q$ is the golden ratio. In this paper we give some remarkable properties for this substitution, and the incidence matrix associated with it. In particular, if $P(x)$ is the characteristic polynomial of the incidence matrix, and $p(x)$ the minimal polynomial of the Pisot number $q$, then $p(x) | P(x)$. Moreover, $q$ is an eigenvalue of maximal modulus. As a corollary of this, an open question of the second author’s regarding the frequencies of gap sizes is answered. We also give conditions under which the gap frequencies are guaranteed to exist. In addition, we show that $P(x)$ can be used to describe the index $i_k$ where $y_{i_k} = q^k$ in $Y^m(q)$. Lastly, substitutions and frequencies are determined precisely for two classes of Pisot numbers.

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1. Introduction and results

For a real number \( q > 1 \) the spectrum of \( q \) is the set of numbers \( p(q) \), where \( p(x) \) ranges over all polynomials whose coefficients are restricted to a finite set of integers. For an integer \( m \geq 1 \) let

\[
Y^m(q) := \{ \epsilon_n q^n + \cdots + \epsilon_1 q + \epsilon_0 : \epsilon_i \in \{0, 1, \ldots, m\}, \ n \in \mathbb{N} \}.
\]

Order the set \( Y^m(q) \) as

\[
Y^m(q) = \{ 0 = y_0 < y_1 < y_2 < y_3 < \cdots \}.
\]

There has been much recent interest in the class of spectra \( Y^m(q) \). Much of the interest has centered around the gap sequence \( \{y_{k+1} - y_k\} \). Define

\[
l^m(q) = \liminf_{k \to \infty} (y_{k+1} - y_k)
\]

and

\[
L^m(q) = \limsup_{k \to \infty} (y_{k+1} - y_k).
\]

We write \( l(q) \) and \( L(q) \) for \( l^1(q) \) and \( L^1(q) \), respectively. Yann Bugeaud proved in [5] that if \( q \in (1, 2) \), then \( q \) is a Pisot number if and only if \( l^m(q) > 0 \) for all \( m \geq 1 \). (Recall a Pisot number is a real algebraic integer \( q > 1 \), such that all of the conjugates of \( q \), are of modulus strictly less than 1.) There are many interesting results along these lines. See, for instance, [7–9,11]. A major open question in this area is whether \( l(q) \) can be positive for any non-Pisot numbers in \((1, 2)\).

For Pisot numbers satisfying \( q^r = q^{r-1} + \cdots + q + 1 \), \( r \geq 2 \), Erdős, Joó, and Joó determine that \( l(q) = 1/q \) (see [10]). For this class of Pisot numbers the actual structure of the spectrum \( Y^1(q) \) has been completely determined [4,6]. The coefficients of \( y_k \in Y^1(q) \) are just the coefficients of \( k \) in the generalized Fibonacci base representation. For example, for \( r = 3 \), let \( F_0 = 1, \ F_1 = 2, \ F_2 = 4, \) and \( F_i = F_{i-1} + F_{i-2} + F_{i-3} \) for \( i \geq 3 \). In the generalized Fibonacci base representation each \( k \in \mathbb{N} \) can be written uniquely as a sum of \( F_i \) which do not contain three terms with consecutive indices \( F_i, F_{i+1}, F_{i+2} \). It follows then, for instance, as \( 10 = F_3 + F_1 + F_0 \), that \( y_{10} = q^3 + q + 1 \) in \( Y^1(q) \).

From this characterization of \( Y^1(q) \), Bugeaud showed that the gap sequence for \( Y^1(q) \) can be generated by a substitution in the following way. Let \( a_1 = 1, \ a_2 = q - 1, \ldots, a_r = q^{r-1} - \cdots - q - 1 \), and define the substitution

\[
\sigma(a_1) = a_1 a_2, \quad \sigma(a_2) = a_1 a_3, \quad \ldots, \quad \sigma(a_{r-1}) = a_1 a_r, \quad \sigma(a_r) = a_1.
\]

This map can be extended to the set of all words over the \( a_i \) using the property \( \sigma(xy) = \sigma(x)\sigma(y) \). The fixed point \( \lim_{k \to \infty} \sigma^k(a_1) \) of \( \sigma \) then gives the gap sequence of \( Y_1(q) \). Thus, \( 1, q - 1, \ldots, q^{r-1} - \cdots - q - 1 \) are the only gap sizes for \( Y_1(q) \). Moreover, Bugeaud used this characterization of the gap sequence to determine that these gaps occur with frequency \( \frac{1}{q^2}, \frac{1}{q^3}, \ldots, \frac{1}{q^m} \), respectively. Recall that a gap of size \( \beta \) occurs with frequency \( \alpha \) if

\[
\lim_{n \to \infty} \frac{|\{k: y_{k+1} - y_k = \beta, \ k \leq n\}|}{n} = \alpha.
\]
In [12] Feng and Wen proved that for a general Pisot number $q > 1$ and $m \geq q - 1$ the gap sequence of $Y^m(q)$ is the image of a substitution sequence over a finite alphabet of symbols. They also gave an algorithm for determining this substitution. In [14] the second author modified this algorithm to determine the frequencies of the gap sizes in $Y^m(q)$. He also determined the gap frequencies for several Pisot numbers, and gave a detailed description of the gap frequencies for the golden ratio. The gap frequencies for the examples computed were all found to be in $\mathbb{Q}[q]$, but this was left as an open question. In Section 3 below we prove that this is always the case, provided the frequencies exist.

The main objective of this paper is to study the incidence matrix $M$ of the substitution which characterizes the gap sequence for $Y^m(q)$. In Section 3 we use this incidence matrix to construct a linear recurrence which gives the positions of the powers of $q$ in the spectrum $Y^m(q)$. We then use this to prove the surprising result that $q$ is an eigenvalue of $M$, and hence that the minimal polynomial for $q$ divides the characteristic polynomial for $M$. We will also show that $q$ is an eigenvalue of $M$ of maximal modulus, and that the frequencies of the gap sizes of $Y^m(q)$, when they exist, are given by an eigenvector associated with $q$. In addition, we will show that the existence of the gap frequencies is guaranteed when there are no other eigenvalues of modulus $q$.

In Section 4 we illustrate these results on two special classes of Pisot numbers. Specifically, we consider Pisot numbers with minimal polynomials $x^n - ax^{n-1} - \cdots - ax - b$ and $x^2 - ax + b$. We will start, however, by describing Feng and Wen’s algorithm for constructing the substitution whose fixed point is the gap sequence of $Y^m(q)$.

2. Constructing the substitution sequence

We begin by describing the algorithm of Feng and Wen [12]. For the remainder of this section let $q > 1$ be a Pisot number. For each $y_k \in Y^m(q)$ define

$$
\gamma_k = \left(Y^m(q) - y_k\right) \cap \left[\frac{-m}{q-1}, y_{k+1} - y_k\right].
$$

(1)

We can think of $\gamma_k$ as being associated with the interval $[y_k, y_{k+1}]$. Feng and Wen showed that there are only a finite number of distinct $\gamma_k$. They also show for $qy_k \leq y_i < qy_{k+1}$ that

$$
\gamma_i = (qy_k + [0, 1, \ldots, m] - y_i + qy_k) \cap \left[\frac{-m}{q-1}, y_{i+1} - y_i\right].
$$

(2)

It follows that for $qy_k \leq y_i < qy_{k+1}$, each $\gamma_i$ is determined by $\gamma_k$. Thus, we can define a substitution $\sigma$ on the $\gamma_i$’s as follows. If

$$
[qy_k, qy_{k+1}] = [y_i, y_{i+1}] \cup \cdots \cup [y_{\ell}, y_{\ell+1}],
$$

where $i < \ell$, then

$$
\sigma(\gamma_k) = \gamma_i \cdots \gamma_\ell.
$$

By (1) we have $\gamma_0 = \{0, 1\}$, and it is not hard to see that $\sigma(\gamma_0)$ begins with $\gamma_0$. Therefore $\sigma$ has a fixed point. It also follows from (1) that the only positive element of $\gamma_k$ is $y_{k+1} - y_k$. We can therefore identify each $\gamma_k$ with $y_{k+1} - y_k$, and so the fixed point of $\sigma$, namely $\lim_{k\to\infty} \sigma^k(\gamma_0)$, encodes the gap sequence of $Y^m(q)$. 
An important consequence of this construction is that \( \sigma^n(\gamma_0) \) is the gap sequence of \( Y^m(q) \) up to \( q^n \). That is, if \( y_{k_n} = q^n \), then \( \{y_{k+1} - y_k\}_{0 \leq k < k_n} \) is encoded by \( \sigma^n(\gamma_0) \) under the correspondence that identifies \( \gamma_k \) with \( y_{k+1} - y_k \). Specifically, this means that \( |\sigma^n(\gamma_0)| \), the length of \( \sigma^n(\gamma_0) \), is the index of \( q^n \) in the spectrum \( Y^m(q) \).

**Example 1.** Let \( q \) be the Pisot number with minimal polynomial \( x^3 - x^2 - x - 1 \). The first few terms of the spectrum \( Y^{1}(q) \) are

\[
0, 1, q, q + 1, q^2, q^2 + 1, q^2 + q, q^3, q^3 + 1, q^3 + q, \ldots.
\]

Using (1), we see that the first few \( \gamma_i \) produced by Feng and Wen’s algorithm are

\[
\begin{align*}
\gamma_0 &= \{0, 1\}, & \gamma_3 &= \{-1, 0, q^2 - q - 1\}, & \gamma_6 &= \{1 - q, 0, 1\}, \\
\gamma_1 &= \{-1, 0, q - 1\}, & \gamma_4 &= \{1 + q - q^2, 0, 1\}, & \gamma_7 &= \{-1, 0, 1\}, \\
\gamma_2 &= \{1 - q, 0, 1\}, & \gamma_5 &= \{-1, 0, q - 1\}.
\end{align*}
\]

It is worth observing that \( \gamma_1 = \gamma_5 \) and \( \gamma_2 = \gamma_6 \). To obtain the substitution, we start with \( \gamma_0 = \{0, 1\} \) associated with the interval \([\gamma_0, \gamma_1] = [0, 1]\). Then

\[
q[\gamma_0, \gamma_1] = q[0, 1] = [0, 1] \cup [1, q] = [\gamma_0, \gamma_1] \cup [\gamma_1, \gamma_2].
\]

Therefore \( \gamma_0 \) maps to \( \gamma_0 \gamma_1 \). The latter set \( \gamma_1 \) could have alternatively been determined from \( \gamma_0 \) and (2) as follows:

\[
\gamma_1 = q\gamma_0 + \{0, 1\} - \gamma_1 + q(0, 1) - 1 + 0 = \{-1, 0, q - 1\}.
\]

Continuing in this manner, we arrive at the following substitution:

\[
\begin{align*}
\gamma_0 &\mapsto \gamma_0 \gamma_1, & \gamma_2 &\mapsto \gamma_4 \gamma_1, & \gamma_4 &\mapsto \gamma_7 \gamma_1, \\
\gamma_1 &\mapsto \gamma_2 \gamma_3, & \gamma_3 &\mapsto \gamma_2, & \gamma_7 &\mapsto \gamma_2 \gamma_1.
\end{align*}
\]

Notice that states \( \gamma_0, \gamma_2, \gamma_4, \) and \( \gamma_7 \) can be equated, and the resulting substitution sequence will be the same. Thus, we get that the substitution generating the gap sequence is

\[
\sigma(a) = ab, \quad \sigma(b) = ac, \quad \sigma(c) = a,
\]

where \( a, b, \) and \( c \) correspond to \( 1, q - 1, \) and \( q^2 - q - 1 \), respectively. This is precisely the substitution obtained by Bugeaud [4,6].

**Example 2.** Let \( q = \frac{3 + \sqrt{5}}{2} \) be the Pisot number with minimal polynomial \( x^2 - 3x + 1 \). Using (1) and (2) we get that the first seven distinct \( \gamma_i \) are

\[
\begin{align*}
\gamma_0 &= \{0, 1\}, & \gamma_8 &= \{2 - q, 0, 3 - q\}, \\
\gamma_1 &= \gamma_4 = \{-1, 0, 1\}, & \gamma_9 &= \{-1, q - 3, 0, q - 2\}, \\
\gamma_2 &= \gamma_5 = \gamma_7 = \{-1, 0, q - 2\}, & \gamma_{10} &= \{-1, 2 - q, 0, 1\}, \\
\gamma_3 &= \gamma_6 = \{2 - q, 0, 1\}.
\end{align*}
\]
The resulting substitution is

\[
\gamma_0 \mapsto \gamma_0 \gamma_1 \gamma_2, \quad \gamma_3 \mapsto \gamma_8 \gamma_9 \gamma_1 \gamma_2, \quad \gamma_9 \mapsto \gamma_10 \gamma_1 \gamma_2, \\
\gamma_1 \mapsto \gamma_3 \gamma_1 \gamma_2, \quad \gamma_8 \mapsto \gamma_8 \gamma_9, \quad \gamma_{10} \mapsto \gamma_8 \gamma_9 \gamma_10 \gamma_1 \gamma_2. \\
\gamma_2 \mapsto \gamma_3 \gamma_1 \gamma_2.
\]

Notice that the states \(\gamma_3, \gamma_{10}\) and \(\gamma_2, \gamma_9\) can be equated and the substitution sequence will be the same. Thus, the substitution generating the gap sequence for \(Y_2(q)\) is

\[
\sigma(a) = abc, \\
\sigma(b) = dbc, \\
\sigma(c) = dc, \\
\sigma(d) = ec_{dc}, \\
\sigma(e) = ec,
\]

where \(a = b = d = 1, c = q - 2,\) and \(e = 3 - q.\)

3. Incidence matrix of the substitution

We recall a few facts about substitutions. Let \(A = \{a_1, a_2, \ldots, a_n\}\) be a finite alphabet of symbols. Denote by \(A^*\) the set of finite words over \(A.\) A substitution is a map \(\sigma : A^* \rightarrow A^*\) such that \(\sigma(xy) = \sigma(x)\sigma(y)\). Here \(\sigma\) can be extended to a map from \(A^N \rightarrow A^N\) by concatenation. The incidence matrix of \(\sigma\) is the \(n \times n\) matrix \(M = (m_{ij})_{n \times n},\) where \(m_{ij}\) is the number of occurrences of the letter \(a_i\) in \(\sigma(a_j)\). Notice that if \(w \in A^*,\) and if \(v\) is the vector whose \(i\)th coordinate is the number of occurrences of \(a_i\) in \(w,\) then the \(i\)th coordinate of \(Mv\) is the number of occurrences of \(a_i\) in \(\sigma(w).\)

For the remainder of this paper let \(\sigma\) be the substitution described in Section 2 which generates the gap sequence of \(Y_m(q),\) and let \(M\) be its associated \(n \times n\) incidence matrix. The following theorem shows how to define the sequence \(i_k\) recursively, where \(\gamma_{i_k} = q^k.\)

**Theorem 3.1.** For \(k \geq 0\) let \(i_k\) be the index of \(q^k\) in the ordered spectrum \(Y_m(q).\) Let

\[
P(x) = x^n - c_{n-1}x^{n-1} - \cdots - c_1x - c_0
\]

be the characteristic polynomial of \(M.\) Then for \(k \geq n\) we have

\[
i_k = c_{n-1}i_{k-1} + \cdots + c_1i_{k-n+1} + c_0i_{k-n}. \quad (3)
\]

**Proof.** Let \(e = [1, 0, \ldots, 0]^T,\) the first standard basis vector. By the Cayley–Hamilton Theorem, \(P(M) = 0.\) Thus, for \(k \geq n,\) \(P(M) \cdot (M^{k-n}e) = 0,\) and so

\[
M^k e = c_{n-1}M^{k-1} e + \cdots + c_1M^{k-n+1} e + c_0M^{k-n} e \quad (4)
\]

for \(k \geq n.\) By the properties of incidence matrices, the \(j\)th coordinate of \(M^k e\) is the number of occurrences of \(\gamma_j\) in \(\sigma^k(\gamma_0).\) By the remarks preceding Example 1, this is just the number of
times \( \gamma_j \) occurs in the gap sequence of \( Y^m(q) \) up to \( q^k - y_{ik-1} = y_{ik} - y_{ik-1} \). The sum of the coordinates of \( M^k e \) is therefore \( i_k \). If on the right-hand side of (4) we sum the coordinates of each of the vectors \( M^{k-n+\ell} e \), for \( 0 \leq \ell \leq n - 1 \), we get (3). \( \square \)

**Example 3.** Consider the Pisot number \( q \), the root of \( x^3 - x^2 - x - 1 \) of Example 1. The incidence matrix for the substitution generating the gap sequence of \( Y^1(q) \) is

\[
M = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]

The first few terms of \( Y^1(q) \) are \( y_0 = 0, y_1 = 1, y_2 = q, y_3 = q + 1, \) and \( y_4 = q^2 \). This gives as initial conditions \( i_0 = 1, i_1 = 2, \) and \( i_2 = 4 \). The characteristic polynomial for \( M \) is \( x^3 - x^2 - x - 1 \). Thus, for \( k \geq 3 \),

\[
i_k = i_{k-1} + i_{k-2} + i_{k-3}.
\]

The next few terms of the sequence, \( i_3 = 7, i_4 = 13, \) and \( i_5 = 24 \) are the indices of \( q^3, q^4, \) and \( q^5 \) in \( Y^1(q) \), respectively.

**Example 4.** Let \( q \) be the root of \( x^2 - 3x + 1 \), as in Example 2. The incidence matrix for the substitution is

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}.
\]

The indices of \( 1, q, q^2, q^3, \) and \( q^4 \) in \( Y^2(q) \) are \( 1, 3, 8, 23, \) and \( 64 \), respectively. Take these as the initial terms of the sequence \( \{i_k\} \). The characteristic polynomial of \( M \) is \( x^5 - 5x^4 + 8x^3 - 5x^2 + x \). By Theorem 3.1, we have that for \( k \geq 5 \),

\[
i_k = 5i_{k-1} - 8i_{k-2} + 5i_{k-3} - i_{k-4}.
\]

So, for instance, the indices of \( q^5 \) and \( q^6 \) in \( Y^2(q) \) are \( i_5 = 173 \) and \( i_6 = 460 \).

Notice that in Example 3, the characteristic polynomial for \( M \) is the minimal polynomial for \( q \). In Example 4 the characteristic polynomial for \( M \) is \( x^5 - 5x^4 + 8x^3 - 5x^2 + x = x(x - 1)^2(x^2 - 3x + 1) \), and thus the minimal polynomial of \( q \) is a factor of the characteristic polynomial of \( M \). We will prove that this is always the case. In fact, we will prove that \( q \) is an eigenvalue of \( M \) of maximal modulus. We first comment on several consequences of Theorem 3.1.

First recall that \( i_k \) is the index of \( q^k \) in \( Y^m(q) \). It is known that \( 0 < l^m(q) \), and further than \( l^m(q) = \lim \inf_{k \to \infty} (y_{k+1} - y_k) = \inf \{y_{k+1} - y_k\} \) [5], so \( c = l^m(q) \) is the minimal gap size between two consecutive terms. Similarly it is known that \( L^m(q) = \lim \sup_{k \to \infty} (y_{k+1} - y_k) \leq 1 \) [5]. Thus there exists a maximal gap size, \( C \geq L^m(q) \) between two consecutive terms. Thus,

\[
\frac{q^k}{C} \leq i_k \leq \frac{q^k}{c} \text{ for all } k.
\]

(5)
Let $P(x)$ be as in Theorem 3.1. Since $i_k$ satisfies the linear recurrence (3), it can be written in the form

$$i_k = \sum_{t=1}^{K} p_t(k)\alpha_t^k,$$

where $\alpha_1, \ldots, \alpha_K$ are the distinct roots of $P(x)$, and $p_t(x)$ is a polynomial whose degree is bounded by the multiplicity of $\alpha_t$. It follows from this and (5) that $P(x)$ has a root of modulus $q$. Moreover, the $p_t(x)$ corresponding to roots of modulus $q$ are constants, and for any $\alpha_t$ where $|\alpha_t| > q$ we have $p_t(x) = 0$.

Let $z_{k,j}$ be the $j$th coordinate of $M^k\mathbf{e}$, which is the number of occurrences of state $\gamma_j$ in the spectrum up to $q^k - y_{ik-1}$. From Eq. (4) in the proof of Theorem 3.1, we see that the $z_{k,j}$ also satisfy recurrence (3), with different initial conditions. Thus, $z_{k,j}$ can be written in the form

$$z_{k,j} = \sum_{t=1}^{K} r_{t,j}(k)\alpha_t^k.$$

Since $i_k = \sum_{j=1}^{n} z_{k,j}$, and since $z_{k,j} > 0$ for large $k$, it follows that the dominant contributing term in (7) must involve a root $\alpha$ of $P(x)$ of modulus at most $q$. The following lemma will be needed to prove that $q$ is the largest eigenvalue of $M$.

**Lemma 3.1.** Let $z_{k,j}$ be the $j$th coordinate of the vector $M^k\mathbf{e}$. Let $\epsilon > 0$. Then there are infinitely many $k$ for which $z_{(k+1),j}/z_{k,j} < q + \epsilon$.

**Proof.** In order to simplify the notation, for the remainder of this proof we write $z_k$ for $z_{k,j}$. Let $\alpha$ be the dominant root in sum (7). We know that $|\alpha| \leq q$.

We will use the following classical result (see, for instance, [18]) that for a sequence $\{u_n\}$ of positive numbers,

$$\liminf_{n \to \infty} \frac{u_{n+1}}{u_n} \leq \liminf_{n \to \infty} u_n^{1/n} \leq \limsup_{n \to \infty} u_n^{1/n} \leq \limsup_{n \to \infty} \frac{u_{n+1}}{u_n}.$$

We will proceed by considering two cases.

**Case 1.** If $|\alpha| < q$ we have $z_k < Dk^d|\alpha|^k$ where $D$ and $d$ are constants, the latter being a non-negative integer bounded by the multiplicity of $\alpha$. Hence $\limsup_{k \to \infty} \frac{1}{z_k} \leq |\alpha|$. This implies that $\liminf_{k \to \infty} \frac{1}{z_k} \leq |\alpha| < q$. Hence $\frac{u_{k+1}}{u_k} < q$ for infinitely many $k$. Furthermore, from (5) we have that $\frac{z_k}{i_k} < C Dk^d|\alpha|^k q^{-k} \to 0$ which means that $\gamma_m$ occurs with frequency zero in the fixed point of our substitution.

**Case 2.** If $|\alpha| = q$ then from (7) we have $z_k \leq Bq^k$ for some constant $B$. Hence $\limsup_{k \to \infty} \frac{1}{z_k} \leq q$. Hence $\liminf_{k \to \infty} \frac{z_{k+1}}{z_k} \leq q$. This implies that for every $\epsilon > 0$ there exists infinitely many $k$ such that $\frac{z_{k+1}}{z_k} < q + \epsilon$. \qed

**Theorem 3.2.** The Pisot number $q$ is an eigenvalue of $M$ of maximal modulus.
Proof. Let $z_k = M^k e$, and for $1 \leq j \leq n$, let $z_{k,j}$ denote the $j$th coordinate of $z_k$. For large $k$ the coordinates of $z_k$ are all strictly positive, since each state gets mapped to at least once by the substitution $\sigma$. For $k$ sufficiently large then, the largest modulus for an eigenvalue of $M$ is less than the quantity

$$\max_{1 \leq j \leq n} \frac{1}{z_{k,j}} (Mz_k)_j,$$

where $(Mz_k)_j$ is the $j$th coordinate of $Mz_k$ (see [15, Theorem 8.1.26]). Notice that for $1 \leq j \leq n$,

$$\frac{1}{z_{k,j}} (Mz_k)_j = \frac{1}{z_{k,j}} (M(M^k e))_j = \frac{1}{z_{k,j}} (M^{k+1} e)_j = \frac{z_{(k+1),j}}{z_{k,j}}.
$$

It follows from Lemma 3.1 that for every $\epsilon$ there exists $k$ such that quantity (8) is less than $q + \epsilon$. Therefore the maximal modulus of an eigenvalue of $M$ is at most $q$. In the comments preceding Lemma 3.1 we saw that $M$ must have an eigenvalue of modulus $q$. So $q$ is the maximal modulus of an eigenvalue of $M$. Since $M$ is a non-negative matrix, $q$ is an eigenvalue of $M$ (see [15, Theorem 8.3.1]).

At this point, it is worth quoting [1, Theorem 8.3.11] for example.

**Theorem 3.3.** [1, Theorem 8.3.11] Let $M$ be a non-negative $d \times d$ matrix. Then there exists a real number $r \geq 0$ (called the Perron–Frobenius eigenvalue of $M$) such that:

(a) $r$ is an eigenvalue of $M$. Furthermore, any (complex) eigenvalue of $M$ satisfies $|\lambda| \leq r$.

(b) There exists a non-negative eigenvector corresponding to the $A$ eigenvalue $r$.

(c) There exists a positive integer $h$ such that any eigenvalue of $M$ with $|\lambda| = r$ satisfies $\lambda^h = r^h$.

In the case above, $q$ is the Perron–Frobenius eigenvalue of $M$, and any other eigenvalue of $M$ with modulus $q$ is $q$ times some root of unity.

We now consider the eigenvector associated with $q$. We will show that in the event that the frequencies of the gaps exist, these frequencies are just the coordinates of the eigenvector associated with $q$. This is reminiscent of a well-known result for primitive substitutions. Recall that a matrix $M$ is primitive if all the entries of $M^k$ are strictly positive for some $k$. A substitution $\sigma$ is primitive if its incidence matrix is primitive. This means that there exists some $k$ such that for every pair $a, b \in A$ we have that $a$ is a character in the word $\sigma^k(b)$. If $\sigma$ is a primitive substitution, then [17] shows that every character in $A$ has a positive frequency in the fixed point of $\sigma$. Moreover, these frequencies are given by a positive eigenvector associated with the dominant eigenvalue of the incidence matrix, normalized so that the coordinates sum to one.

These results do not apply directly to the substitution generated by the algorithm in Section 2 since it is never primitive. To see this, notice that the generator $\gamma_0$ is only mapped to once, by $\gamma_0$ itself. It may be possible to reduce the substitution to one that is primitive, as in Example 1. This is not always possible, however, as Example 2 shows.

Thus, we are not guaranteed that the frequencies exist. When they do exist, however, a result similar to the case of primitive matrices holds. The following theorem is a rewording of [1, Theorem 8.4.6].
Theorem 3.4. The frequencies of the gap sequence of $Y_m(q)$, when they exist, are given by a non-negative eigenvector $v$ associated with the eigenvalue $q$ of $M$, normalized so that the sum of the coordinates is one. More specifically, the $j$th coordinate of $v$ is the frequency of $\gamma_j$ in the fixed point of $\sigma$ which encodes the gap sequence.

The next theorem establishes conditions which guarantee the existence of the gap frequencies.

Theorem 3.5. If $M$ has no eigenvalues of modulus $q$, other than $q$ itself, then the gap frequencies exist.

Proof. Let $z_k$ be the $j$th coordinate of $M^ke$. That is, $z_k$ is the number of times $\gamma_j$ occurs in the word $\sigma^k(\gamma_0)$. In the proof of Case 1 of Lemma 3.1 we showed that if the largest root in (7) with a non-zero coefficient is less than $q$ in modulus, then the frequency of $\gamma_j$ is zero. Suppose that the dominant contributing root in (7) has modulus $q$. Since there are no complex roots of modulus $q$, it follows from (5) and (7) that there is a constant $c$ such that

$$z_k = cq^k + o(q^k) = q^k(c + o(1)).$$

(Here we use the notation of [2] so that if $f(x) = o(g(x))$ then $f(x)/g(x) \to 0$ as $x \to \infty$.) In a similar manner we can show that there is a constant $d$ such that

$$i_k = q^k(d + o(1)).$$

Thus,

$$\lim_{k \to \infty} \frac{z_k}{i_k} = \frac{c}{d}.$$ 

Thus, the frequency of $\gamma_j$ exists. The frequency of any gap in the gap sequence is the sum of the frequencies of the $\gamma_j$ which correspond to it. Thus, all the gap frequencies exist. □

The following theorem partially solves an open question in [14].

Theorem 3.6. Let $\beta \in \{y_n - y_{n-1} : y_n \in Y^m(q)\}$. Let $\beta$ occur with frequency $\alpha$ in $Y^m(q)$. Then $\alpha \in \mathbb{Q}[q]$.

Before we prove Theorem 3.6 we need a minor lemma. Recall in Theorem 3.1 that we were interested in $i_k$, the index of $q^k$ in an ordered spectrum $Y^m(q)$. Here the $i_k$ satisfied the recurrence relationship (3). It is these recurrence relationships that this next lemma addresses. It seems like this lemma should be a well-known result, but the authors were unable to find a reference to it.

Lemma 3.2. Let $P(x) = x^n - b_{n-1}x^{n-1} - \cdots - b_0 \in \mathbb{Z}[x]$ be the recurrence polynomial for the integer recurrence

$$i_k = b_{n-1}i_{k-1} + \cdots + b_0i_{k-n} = \sum_{j=1}^{l} p_j(k)\alpha_j^k,$$

then $p_j(x) \in \mathbb{Q}[\alpha_j][x]$. 

Proof. Let us write \( p_j(k) = c_{j,0} + c_{j,1}k + \cdots + c_{j,n_j}k^{n_j} \). Then we have that

\[
i_k = \sum_{j=1}^{l} \sum_{i=0}^{n_j} c_{j,i}k^i \alpha_j^i.
\]

Let

\[
N = \left( \sum_{j=1}^{l} (n_j + 1) \right) - 1.
\]

Consider the equations

\[
i_0 = c_{1,0} + c_{2,0} + \cdots + c_{t,0},
\]

\[
i_1 = c_{1,0} \alpha_1 + \cdots + c_{1,n_1}k^{n_1} \alpha_1 + \cdots + c_{t,0} \alpha_t + \cdots + c_{t,n_t}k^{n_t} \alpha_t,
\]

\[
\vdots
\]

\[
i_N = c_{1,0} \alpha_1^N + \cdots + c_{1,n_1}k^{n_1} \alpha_1^N + \cdots + c_{t,0} \alpha_t^N + \cdots + c_{t,n_t}k^{n_t} \alpha_t^N.
\]

Let

\[
A = \begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
\alpha_1 & \alpha_1 & \cdots & \alpha_1 & \cdots & \alpha_t & \alpha_t & \cdots & \alpha_t \\
\alpha_1^2 & 2\alpha_1^2 & \cdots & 2^{n_1} \alpha_1^2 & \cdots & \alpha_t^2 & 2\alpha_t^2 & \cdots & 2^{n_t} \alpha_t^2 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha_1^N & N\alpha_1^N & \cdots & N^{n_1} \alpha_1^N & \cdots & \alpha_t^N & N\alpha_t^N & \cdots & N^{n_t} \alpha_t^N
\end{bmatrix}.
\]

In matrix form Eq. (9) becomes

\[
A \begin{bmatrix}
c_{1,0} & c_{1,1} & \cdots & c_{1,n_1} & \cdots & c_{t,0} & c_{t,1} & \cdots & c_{t,n_t}
\end{bmatrix}^T = \begin{bmatrix}
i_0 & i_1 & \cdots & i_N
\end{bmatrix}^T.
\]

We see that the determinant of \( A \) is symmetric in the roots \( \alpha_i \) (up to sign), and non-zero. Thus by Cramer’s rule, we can solve for the \( c_{0,0} \) (for example) as

\[
c_{0,0} = \frac{1}{\Delta} \begin{bmatrix}
i_0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
i_1 & \alpha_1 & \cdots & \alpha_1 & \cdots & \alpha_t & \alpha_t & \cdots & \alpha_t \\
\alpha_1^2 & 2\alpha_1^2 & \cdots & 2^{n_1} \alpha_1^2 & \cdots & \alpha_t^2 & 2\alpha_t^2 & \cdots & 2^{n_t} \alpha_t^2 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
i_N & N\alpha_1^N & \cdots & N^{n_1} \alpha_1^N & \cdots & \alpha_t^N & N\alpha_t^N & \cdots & N^{n_t} \alpha_t^N
\end{bmatrix},
\]

where \( \Delta \) is the determinant \( A \). The determinant in (10) is a symmetric function in all of the roots (up to sign), with the exception of \( \alpha_1 \). In fact, it is symmetric up to the same sign that \( \Delta \) is symmetric in its roots, hence \( c_{0,0} \) is completely symmetric in the roots \( \alpha_2, \ldots, \alpha_t \). Let \( \alpha_1, \ldots, \alpha_\ell \) be conjugates, and \( \alpha_\ell+1, \ldots, \alpha_t \) come from some other minimal polynomial. So we have that
\(c_{0,0} \in \mathbb{Q}[\alpha_2, \ldots, \alpha_\ell]\), as it is completely symmetric in the \(\alpha_{\ell+1}, \ldots, \alpha_\ell\). We also have that \(c_{0,0}\) is a symmetric function in the \(\alpha_2, \ldots, \alpha_\ell\). So this implies that \(c_{0,0} \in \mathbb{Q}[\alpha_1]\).

This technique can be generalized to \(c_{i,j}\) in the obvious way, and the result follows. \(\square\)

**Proof of Theorem 3.6.** We only need to consider the case \(\alpha \neq 0\). The frequency of \(\beta\) is the sum of the frequencies of the states \(\gamma_j\) in the substitution which correspond to \(\beta\). Let \(\gamma_j\) be such a state. We can assume that \(\gamma_j\) has a non-zero frequency.

As in the proof of Theorem 3.5, let \(z_k\) be the \(j\)th coordinate of \(M^k\). Let \(q = q_1, q_2, \ldots, q_s\) be the eigenvalues of \(M\) of modulus \(q\). Let \(h\) be as in Theorem 3.3. Since \(q\) is the largest modulus of an eigenvalue of \(M\), it follows from (5) and (7) that there are constants \(c_1, \ldots, c_s\) such that

\[
z_k = c_1q_1^k + \cdots + c_sq_s^k + o(q^k).
\]

By Theorem 3.3(c),

\[
z_{kh} = cq_{kh} + o(q_{kh}) = q_{kh}^c + o(1),
\]

where \(c = c_1 + \cdots + c_s\). We claim that \(c \in \mathbb{Q}[q]\). To see this, notice that \(c_\ell \in \mathbb{Q}[q_\ell]\) for \(1 \leq \ell \leq s\) (by Lemma 3.2 applied to recurrence (7) for \(z_k\)). Also, if \(q_i\) and \(q_\ell\) are conjugate algebraic integers, then \(c_i\) and \(c_\ell\) are conjugates, as the expression is fixed under Galois actions. Thus, the sum \(c_1 + \cdots + c_s\) is in \(\mathbb{Q}[q]\), since the conjugates of a given \(q_\ell\) of modulus \(q\) are a complete set of conjugate roots of unity times \(q\). The sum of all such conjugates will then be in \(\mathbb{Q}[q]\).

In a similar manner it can be shown that there is a \(d \in \mathbb{Q}[q]\) such that

\[
i_{kh} = q_{kh}^d + o(1).
\]

Since we are assuming the frequency of \(\gamma_j\) exists, it follows from (11) and (12) that it is given by

\[
\lim_{k \to \infty} \frac{z_{kh}}{i_{kh}} = \frac{c}{d} \in \mathbb{Q}[q].
\]

Since the frequency \(\alpha\) of \(\beta\) is the sum of the frequencies of the \(\gamma_j\) which correspond to \(\beta\), it follows that \(\alpha \in \mathbb{Q}[q]\). \(\square\)

It should be pointed out here that this vastly improves on the algorithm of [14] that was used to find the gap frequencies. In [14], numerical solutions were found, and then PSLQ was used to determine what algebraic numbers these numerical solutions corresponded to. Instead, by Theorem 3.4 we can simply find the eigenvector of the matrix in question, which although still hard, is a better understood problem. It should be pointed out that for all examples in [14], \(q\) was in fact a simple dominant eigenvalue of \(M\). In fact we would conjecture that this is always the case. If so, then it will follow from Theorem 3.5 that the gap frequencies always exist.

**Example 5.** Let \(q\) be the root of \(x^2 - 3x + 1\), and \(\sigma\) of be as in Examples 2 and 4. The eigenvector of \(M\) associated with \(q\) is

\[
v = \left[ \begin{array}{ccc} 0 & 0 & 1/2 \\ \sqrt{5} - 1/4 & 3 - \sqrt{5}/4 & 0 \\ \end{array} \right]' = \left[ \begin{array}{ccc} 0 & 0 & 1/2 \\ q - 2/2 & 3 - q/2 & \end{array} \right]' \]
The first, second, and fourth coordinates of this vector are the frequencies of the symbols in the fixed point of \( \sigma \) which correspond to 1. It follows then that the frequencies of 1, \( q-2 \), and \( 3-q \) in the gap sequence are \( \frac{q-2}{2} \), \( \frac{1}{2} \), and \( \frac{3-q}{2} \), respectively.

4. Substitutions for two classes of Pisot numbers

In this section we use the preceding results to study the spectra of two different classes of Pisot numbers. The first class of Pisot numbers has the property that the substitution is of Pisot type. A substitution of Pisot type is one for which the characteristic polynomial of the incidence matrix is the minimal polynomial of a Pisot number. Our main result is the following generalization of the result of Bugeaud’s mentioned in the introduction.

**Theorem 4.1.** Let \( q \) be the Pisot root of \( x^n - ax^{n-1} - \cdots - ax - b \), where \( a \geq b \geq 1 \) and \( n \geq 2 \), and let \( m = \lfloor q \rfloor \). Then the substitution for \( Y^m(q) \) is described as follows. For \( n \geq 2 \) we have

\[
\begin{align*}
\gamma_0 & = \{0, 1\}, \\
\gamma_1 & = \{-1, 0, 1\}, \\
\gamma_2 & = \{-1, 0, q - a\}, \\
\gamma_3 & = \{-q + a, 0, 1\}, \\
\gamma_4 & = \{-1, 0, q^2 -aq - a\}, \\
\gamma_{2n-2} & = \{-1, 0, q^{n-1} -aq^{n-2} - \cdots - a\}, \\
\gamma_{2n-1} & = \{-q^{n-1} +aq^{n-2} + \cdots + a, 0, 1\}, \\
\end{align*}
\]

with the substitutions

\[
\begin{align*}
\gamma_0 & \mapsto \gamma_0 \gamma_1^{a-1} \gamma_2, \\
\gamma_1 & \mapsto \gamma_3 \gamma_1^{a-1} \gamma_2, \\
\gamma_2 & \mapsto \gamma_3 \gamma_1^{a-1} \gamma_4, \\
\gamma_3 & \mapsto \gamma_5 \gamma_1^{a-1} \gamma_2, \\
\gamma_4 & \mapsto \gamma_3 \gamma_1^{a-1} \gamma_6, \\
\gamma_5 & \mapsto \gamma_7 \gamma_1^{a-1} \gamma_2, \\
\gamma_6 & \mapsto \gamma_3 \gamma_1^{a-1} \gamma_2, \\
\gamma_7 & \mapsto \gamma_9 \gamma_1^{a-1} \gamma_2, \\
\gamma_8 & \mapsto \gamma_3 \gamma_1^{b-1}, \\
\gamma_9 & \mapsto \gamma_1^{b-1}, \\
\gamma_{2n-2} & \mapsto \gamma_3 \gamma_1^{b-1}, \\
\gamma_{2n-3} & \mapsto \gamma_2n-1 \gamma_1^{a-1} \gamma_2, \\
\gamma_{2n-1} & \mapsto \gamma_1^{a} \gamma_2.
\end{align*}
\]

**Proof.** First notice that \( m = a \). We will do one case only. All other cases are done in a similar way. Consider \( \gamma_{2k} = \{-1, 0, q^k -aq^{k-1} - \cdots - a\} \) where \( 0 < k < n - 1 \). Here we are explicitly assuming that \( n \geq 3 \), but the same techniques can be used for the degenerate case \( n = 2 \).

Notice that

\[
q \gamma_{2k} + \{0, 1, 2, \ldots, a\} = \{-q, -q + 1, -q + 2, \ldots, -q + a, 0, 1, \ldots, a, q^k -aq - aq - 1, \ldots, q^{k+1} -aq^{k+1} - \cdots - aq + 1, \ldots, q^{k+1} -aq^{k+1} - \cdots - aq + a\}.
\]

By Feng and Wen’s algorithm, this gives us

\[
\begin{align*}
\{-q + a, 0, 1\} - 0 & = \{-q + a, 0, 1\} = \gamma_3, \\
\{0, 1, 2\} - 1 & = \{-1, 0, 1\} = \gamma_1, \\
\vdots
\end{align*}
\]
\{a - 2, a - 1, a\} - (a - 1) = \{-1, 0, 1\} = \gamma_1,
\{a - 1, a, a^{-}q^{-} - aq^{-} - \cdots - a\} - a = \{-1, 0, a^{-}q^{-} - aq^{-} - \cdots - a\},
\quad = \gamma_2 k + 2,

which gives the desired result. \square

One should also remark that it is possible to prove this result another way, by using a numeration system for the natural numbers with respect to the following recurrence relation. Let $a_1 \geq a_2 \geq \cdots \geq a_n > 0$, and define a recurrence relation by

\[ G_i = a_1 G_{i-1} + a_2 G_{i-2} + \cdots + a_n G_{i-n}. \]  

Take as initial conditions $G_0 = 1$, and for $1 \leq i \leq n - 1$, take $G_i$ which satisfy

\[ G_i > a_1(G_0 + \cdots + G_{i-1}). \]

Any natural number $n$ has a well-defined representation in the $G_i$. A detailed description of the digits of such expansions is given in [16]. When $a_1 = \cdots = a_{n-1} = a$ and $a_n = b$, the digits of $n$ are in one-to-one correspondence with the coefficients of $y_n \in Y^m(q)$. We can appeal to the methods of [4,6].

The substitution of Theorem 4.1 can actually be simplified so that the resulting substitution is primitive. Thus, the gap frequencies exist and are the coordinates of the eigenvector associated with $q$. We prove this in the next corollary.

**Corollary 4.1.** Let $q$ be the Pisot root of

\[ x^n - ax^{n-1} - \cdots - ax - b, \]

where $a \geq b \geq 1$ and $n \geq 2$. Let $m = \lfloor q \rfloor = a$. Then the gaps in $Y^m(q)$,

\[ 1, q - a, q^2 - aq - a, \ldots, q^{n-1} - aq^{n-2} - \cdots - aq - a, \]

occur with frequency

\[ \frac{q^{n-1}}{Q}, \frac{q^{n-2}}{Q}, \ldots, \frac{q}{Q}, \frac{1}{Q}, \]

respectively, where $Q = q^{n-1} + \cdots + q + 1$.

**Proof.** The substitution sequence is now defined on an alphabet of $2n$ symbols. Notice that $\gamma_0, \gamma_1, \gamma_3, \gamma_5, \ldots, \gamma_{2n-1}$, all correspond to 1 in the gap sequence. Moreover, the images of each of $\gamma_0, \gamma_1, \gamma_3, \ldots, \gamma_{2n-1}$ under the substitution all correspond to the same string of values in the gap sequence (namely a 1's followed by a $q - a$). It follows that if we equate $\gamma_0 = \gamma_1 = \cdots = \gamma_{2n-1}$ then resulting substitution will still produce the gap sequence. Relabeling the remaining symbols as $a_0 = \gamma_0 = \gamma_1 = \cdots = \gamma_{2n-1}$, and $a_k = \gamma_{2k}$ gives the substitution.
\[
\begin{align*}
\alpha_0 &\mapsto \alpha_0^a \alpha_1, \\
\alpha_1 &\mapsto \alpha_0^a \alpha_2, \\
\vdots & \\
\alpha_{n-2} &\mapsto \alpha_0^a \alpha_{n-1}, \\
\alpha_{n-1} &\mapsto \alpha_0^b.
\end{align*}
\]

We are now ready to determine the gap frequencies of the spectrum \(Y^m(q)\). The incidence matrix of the substitution generating the gap sequence of \(Y^m(q)\) is

\[
M = \begin{bmatrix}
a & a & \cdots & a & b \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix},
\]

the companion matrix for \(x^n - ax^{n-1} - \cdots - ax - b\), the minimal polynomial of \(q\). It follows from Theorem 3.5 that the gap frequencies exist. A routine exercise reveals that the eigenvector of \(M\) corresponding to \(q\), normalized so that the sum of the coordinates is 1, is

\[
\frac{1}{q^{n-1} + \cdots + q + 1} \begin{bmatrix} q^{n-1} \\ q^{n-2} \\ \vdots \\ q \\ 1 \end{bmatrix}.
\]

The result now follows from Theorem 3.4. \(\square\)

Based on the results of [3,13], one might naturally ask whether the results of Corollary 4.1 hold for general Pisot numbers of the form

\[x^n - a_{n-1}x^{n-1} - \cdots - a_1x - a_0,\]

with \(a_n \geq a_{n-1} \geq \cdots \geq a_0 > 0\). The answer is no, however, as the correspondence between the digits of \(n\) with respect to the \(G_i\) and the coefficients of \(n\) does not hold. For the Pisot number with minimal polynomial \(x^3 - 2x^2 - x - 1\), for example, a computer implementation of the algorithm in [14] shows that the characteristic polynomial of the incidence matrix is \(x^{11}(x^3 - 2x^2 - x - 1)(x^4 - x^2 - 3x - 1)(x - 1)^2(x^2 + x + 1)\). We close with a computation of the substitution and the gap frequencies for the remaining class of quadratic Pisot numbers not included in Theorem 4.1.

**Theorem 4.2.** Let \(q\) be the Pisot root of \(x^2 - ax + b\), where \(b \geq 1\) and \(a \geq b + 2\), and let \(m = \lfloor q \rfloor\). Then the substitution for \(Y^m(q)\) is described by
\[ \gamma_0 = \{0, 1\}, \quad \gamma_4 = \{-q + a - 1, 0, -q + a\}, \]
\[ \gamma_1 = \{-1, 0, 1\}, \quad \gamma_5 = \{-1, q - a, 0, q - a + 1\}, \]
\[ \gamma_2 = \{-1, 0, q - a + 1\}, \quad \gamma_6 = \{-1, -q + a - 1, 0, -q + a\}, \]
\[ \gamma_3 = \{-q + a - 1, 0, 1\}, \quad \gamma_7 = \{-1, -q + a - 1, 0, 1\}, \]

with the substitutions

\[ \gamma_0 \mapsto \gamma_0 \gamma_1^{a-2} \gamma_2, \quad \gamma_4 \mapsto \gamma_4 (\gamma_5 \gamma_6)^{b-1} \gamma_5, \]
\[ \gamma_1 \mapsto \gamma_3 \gamma_1^{a-2} \gamma_2, \quad \gamma_5 \mapsto \gamma_7 \gamma_1^{a-2-b} \gamma_2, \]
\[ \gamma_2 \mapsto \gamma_3 \gamma_1^{a-2-b} \gamma_2, \quad \gamma_6 \mapsto \gamma_4 (\gamma_5 \gamma_6)^{b-1} \gamma_5, \]
\[ \gamma_3 \mapsto \gamma_4 (\gamma_5 \gamma_6)^{b-1} \gamma_5 \gamma_1^{a-b-2} \gamma_2, \quad \gamma_7 \mapsto \gamma_4 (\gamma_5 \gamma_6)^{b-1} \gamma_5 \gamma_1^{a-b-2} \gamma_2. \]

Here \( \gamma_1^n \) where \( n = 0 \) is taken to be the empty string.

**Proof.** First observe that \( m = a - 1 \). We will prove the first map. All of the other maps are proved in a similar manner. Notice that

\[ q \{0, 1\} + \{0, 1, \ldots, a - 1\} \]
\[ = \{0, 1, 2, \ldots, a - 2, a - 1, q, q + 1, q + 2, \ldots, q + (a - 1)\}. \]

By Feng and Wen’s algorithm, this gives us the states

\[ \{0, 1\} - 0 = \{0, 1\} = \gamma_0, \]
\[ \{0, 1, 2\} - 1 = \{-1, 0, 1\} = \gamma_1, \]
\[ \{1, 2, 3\} - 2 = \{-1, 0, 1\} = \gamma_1, \]
\[ \vdots \]
\[ \{a - 3, a - 2, a - 1\} - (a - 2) = \{-1, 0, 1\} = \gamma_1, \]
\[ \{a - 2, a - 1, q\} - (a - 1) = \{-1, 0, q - a + 1\} = \gamma_2 \]

which shows that \( \gamma_0 \mapsto \gamma_0 \gamma_1^{a-2} \gamma_2 \) as required. \( \square \)

**Corollary 4.2.** Let \( q \) be the Pisot root of \( x^2 - ax + b \), and let \( m = \lfloor q \rfloor \). Then the gap frequencies in \( Y^n(q) \) are as follows:

- 1 has a frequency of \( \frac{q-b-1}{a-b} \);
- \( q - a + 1 \) has a frequency of \( \frac{1}{a-b} \);
- \( a - q \) has a frequency of \( \frac{a-q}{a-b} \).
Proof. As before, we can notice that $\gamma_2$ and $\gamma_5$ both correspond to 1 and the image of each under the substitution corresponds to the same string of values. We can say similar things about $\gamma_6$ and $\gamma_4$, as well as with $\gamma_3$ and $\gamma_7$. So relabeling to get $\alpha_i = \gamma_i$, for $0 \leq i \leq 4$ and additionally $\alpha_2 = \gamma_5$, $\alpha_3 = \gamma_7$ and $\alpha_4 = \gamma_6$.

Then it is just a matter of finding the gap frequencies corresponding to the eigenvector of

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\quad a-2 & a-2 & a-2-b & a-2-b & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}.
$$

\square

5. Conclusions. Comments

The paper gives some nice, and somewhat surprising results concerning the substitution(s) associated with the spectra of a Pisot number. There are two things that would be interesting to expand upon. First, determine if the gap frequencies always exist. As mentioned before, by Theorem 3.5 this would follow if it could be shown that $q$ is a simple dominant eigenvalue of $M$.

Secondly, we give a complete description of $Y^m(q)$ for some special cases of $m$ and $q$. This can and should be expanded upon. There are probably much larger classes of Pisot numbers that can be determined and for a larger number of $m$.

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References

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